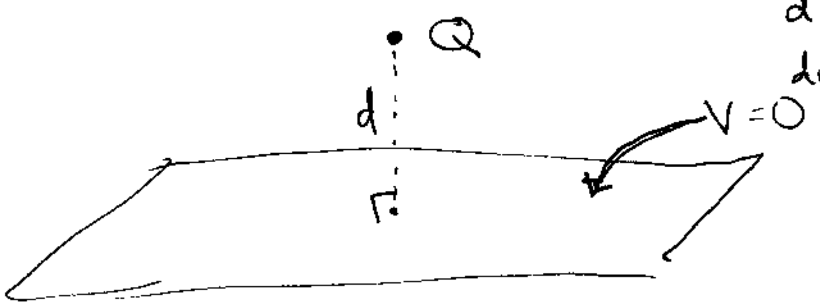
P3310 LECTURE 12

- Study session this evening: EOZ will be late (~5:30) but can stay until 6:45.

- Review session for exam: Prefer Sunday or Monday eve?

Method of Images: A way to exploit the uniqueness of solutions to Laplace eqn:

Start with the ubiquitous example: point charge distance d above a grounded conductor.



What is the potential in the region above the plane?

It is due to: ① The charge Q

② The surface charge on the conductor, $\sigma(x,y)$.

But $\sigma(x,y)$ is whatever it has to be to cancel the field in the conductor; this induced charge can be hard to calculate.

If we can find any function $V(\vec{r})$ that satisfies the boundary conditions and $\nabla^2 \rho = -\frac{\rho}{\varepsilon_0}$, we know it is the only solution.
We know $\rho = Q\,\delta^3(\vec{r}-\vec{r}_q) = Q\,\delta(x)\,\delta(y)\,\delta(z-d)$, so $\nabla^2 \rho = 0$ everywhere but $(0,0,d)$. Boundary conditions say $V=0$ at $z=0$ and $r \to \infty$.

I assert that the following potential satisfies these conditions in the $z>0$ region:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-Q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

(Define $r_+ \equiv \sqrt{x^2+y^2+(z-d)^2}$, $r_- \equiv \sqrt{x^2+y^2+(z+d)^2}$ now).

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r_+} - \frac{Q}{4\pi\varepsilon_0 r_-}$$

This satisfies appropriate conditions because:

- At $z=0$, $r_+ = r_-$ so $V(x,y,0)=0$
- At $r\to\infty$, $V\to 0$
- Has appropriate point-charge behavior at $(0,0,d)$.
- Doesn't satisfy Laplace eqn at $(0,0,-d)$ — but that doesn't matter since we only care about $z>d$ region!

So, since solutions that satisfy boundary conds are unique, it is the right solution.

But note that $\frac{-Q}{4\pi\varepsilon_0 r_-}$ is the potential of an opposite point charge at $z=-d$. Above the conductor, the potential appears as if there were an opposite charge at the position of the image of Q if the conductor were a mirror. Obviously there's nothing there, but the conductor induced charge distribution arranges itself to mimic that field above the conductor.

A nice check: Remember that $\vec{E} \perp \hat{n}$ at conductor surface.

So $E_{x,y}$ had better vanish at $z=0$:

$$\vec{E} = -\vec{\nabla} V = \frac{-Q}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left(\hat{z}\frac{\partial}{\partial z} + \hat{y}\frac{\partial}{\partial y} + \hat{x}\frac{\partial}{\partial x}\right)\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

x-component: (ignoring constants)

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

$$= -\frac{1}{2}\left\{\frac{2x}{[x^2+y^2+(z-d)]^{\frac{3}{2}}} - \frac{2x}{[x^2+y^2+(z+d)]^{\frac{3}{2}}}\right\}$$

$= 0$ at $x=0$. Similar for y. ✓

z-component: (ignoring constants)

$$\frac{\partial}{\partial z}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= -\frac{1}{2}\left\{\frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

at $z=0$:

$$= -\frac{1}{2}\left\{\frac{-2d}{(x^2+y^2+d^2)^{\frac{3}{2}}} - \frac{2d}{(x^2+y^2+d^2)^{\frac{3}{2}}}\right\}$$

$$= \frac{+2d}{(x^2+y^2+d^2)^{\frac{3}{2}}}$$

Put back consts:

$$\Rightarrow \vec{E}_{z=0} = \frac{-Qd}{2\pi\varepsilon_0(x^2+y^2+d^2)^{\frac{3}{2}}}$$

Can use this now to calculate surface charge on conductor:

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n} \implies \sigma = \varepsilon_0 E_z = \frac{-Qd}{2\pi\left(r_c^2 + d^2\right)^{\frac{3}{2}}} \quad \text{where } r_c = \sqrt{x^2+y^2}$$

Can integrate over $z=0$ surface to find <u>total</u> induced charge:

$$q_{ind} = \int dA\, \sigma(x,y) = \int_0^{2\pi} d\phi \int_0^{\infty} dr_c\, r_c \frac{-Qd}{2\pi\left(r_c^2+d^2\right)^{\frac{3}{2}}}$$

$$= -Qd \int_0^{\infty} dr_c \frac{r_c}{\left(r_c^2+d^2\right)^{\frac{3}{2}}}$$

$$= Qd\left(\frac{1}{\sqrt{r_c^2+d^2}}\right)_0^{\infty} = Qd\left(0 - \frac{1}{d}\right) = \boxed{-Q}$$

Note that this could have been ~~guessed~~ found via Gauss's Law too.