Laplace's equation: $\nabla^2 V = 0$. $\Rightarrow$ equation for potential in charge-free regions. (and lots of other areas in physics)
First thing to remember is that it describes a <u>scalar</u> field!

Cartesian: $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Spherical: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Huge industries are based on solving Laplace's eqn in various cases.

Properties of solutions:

1. Solution is always the smoothest possible way to satisfy the boundary conditions (Griffiths: "most boring")

2. V can have no local maximum or minimum except at boundary. (very useful)

3. $V(\vec{r})$ is the average of V at all points equidistant from $\vec{r}$:

2D: $$V(\vec{r}) = \frac{1}{2\pi R}\oint_{\text{circle}} V\, d\ell$$

3D: $$V(\vec{r}) = \frac{1}{4\pi R^2}\oint_{\text{sphere}} V\, dA$$

<u>Method</u> <u>of</u> <u>relaxation</u>: Divide space into a grid:

- Fix $V(\vec{r})$ on edges
- Guess $V(\vec{r})$ in interior
- Set $V(\vec{r})$ = average of its neighbor
- iterate until converges well

Uniqueness and boundary conditions:

$\nabla^2 V = 0$ ← can specify V all over surface

OR

can specify $\frac{dV}{dn}$ all over surface and V at one point

Uniqueness: Assume 2 functions V_1 and V_2 both satisfy $\nabla^2 V = 0$ and have same value at a boundary:

Let $V_3 = V_1 - V_2$.

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

But $V_3 = 0$ at boundaries. So no max or min allowed

$$\Longrightarrow V_3 = 0 \Rightarrow V_1 = V_2$$