P3310 LECTURE 10

EXAM 1: Thurs Feb 22, G125: 6pm / 6:30 pm / 7 pm

90 min., closed book, 8½x11 crib sheet (handwritten) allowed

No calculators/electronics. Material through next week.

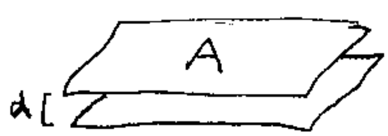
Review session. Sun Feb 18 G131 3-5 pm

Capacitance: Take two conductors, and place $+Q$ on one and $-Q$ on the other. This will cause a potential difference $V = V_+ - V_-$ between them.

Sometimes it's easy to calculate $\vec{E}$ by symmetry; sometimes it's not. Regardless, $V \propto Q$. Call the constant $C \equiv \frac{Q}{V}$ capacitance. Units (SI): $\frac{\text{Coulomb}}{\text{Volt}}$ = farad. This is huge — micro- and pico-farads are more common.

← pF, "puff"

The classic calculation: parallel plate capacitor: two conducting plates, each of area A, and separated by distance d.



We are doing calculation assuming:

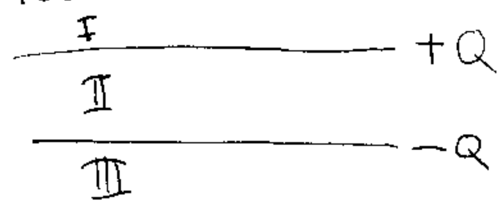
- $d \ll \sqrt{A}$ (actually $d \ll$ short dimension of plate)
- plates are aligned, same shape

Assume that charge will spread out uniformly on each plate, so $\sigma = \frac{\pm Q}{A}$. Field is thus:

$E_z =$:

Region I: $+\frac{Q}{2A\epsilon_0} - \frac{Q}{2A\epsilon_0}$ II: $\frac{-Q}{2A\epsilon_0} - \frac{Q}{2A\epsilon_0}$

III: $-\frac{Q}{2A\epsilon_0} + \frac{Q}{2A\epsilon_0}$



... so field is nonzero only between the plates. Find V by integrating $\vec{E}$:

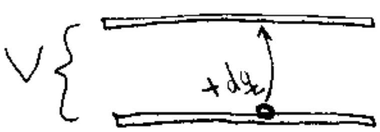
$$V = \int_0^d dz\, E_z = \frac{Q d}{A \varepsilon_0} \implies C = \frac{A \varepsilon_0}{d}$$

Aside: in Gaussian units, remember $\varepsilon_0 \to \frac{1}{4\pi}$, so $C = \frac{A}{4\pi d}$. The Gaussian unit of capacitance is centimeters!

This is actually completely sensible, since the capacitance (absent material effects) is purely a function of geometry.

1 pF $\approx$ 1 cm.

Energy in a capacitor: There are (at least) two ways to calculate the energy in a capacitor. First, consider the work necessary to charge it up:



Charge dq gains/loses energy $V dq$ when transported from one plate to the other.

So the work to move total charge Q from one plate to the other, starting with uncharged plates, is:

$$W = \int_0^Q dq\, V \quad \rightarrow \text{ but for a capacitor, } V = \frac{q}{C}$$

$$\text{so } W = \int_0^Q dq\, \frac{q}{C} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2 = \frac{1}{2} Q V.$$

Note: $\int dq$ may not be familiar, since q isn't a spatial coordinate. But it works!

But the energy in the capacitor is stored in the field, so we should get the same answer by calculating the energy in the field:

For parallel plates, we assume no fringe field (as we did to calculate capacitance anyway): $E = \frac{Q}{A\varepsilon_0}$ between plates, and $E = 0$ elsewhere.

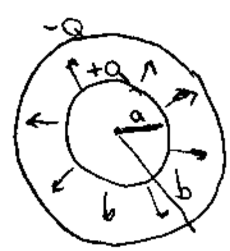
$$W = \frac{\varepsilon_0}{2}\int d^3r\, E^2(\vec{r})$$

$$= \frac{\varepsilon_0}{2}\int_{\text{surf}} dA \int_0^d dz\, E^2$$

$$= \frac{\varepsilon_0}{2} Ad\left(\frac{Q}{A\varepsilon_0}\right)^2 = \frac{Q^2 d}{2A\varepsilon_0}$$

$$= \frac{Q^2}{2C} \quad \checkmark$$

Other capacitance geometries: concentric spheres, radii a & b:



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

so $V = -\int_a^b \vec{E}\cdot d\vec{l} = -\int_a^b E\,dr$

$$= \frac{-Q}{4\pi\varepsilon_0}\int_a^b r^{-2}\,dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

So $C = \frac{Q}{V} = 4\pi\varepsilon_0 \frac{ab}{b-a}$

Do we need a second conductor at all? If $b \to \infty$, then $C = 4\pi\varepsilon_0 a$.

Some authors start by defining capacitance this way, then adding second conductor to terminate field lines.