(1a) Flux = BA is conserved. So $B_1A_1 = B_2A_2$.

$$\Rightarrow \frac{B_1}{B_2} = \frac{A_2}{A_1} < 1$$

$$\Rightarrow B_2 > B_1.$$

(1b) $H = \frac{B}{\mu}$, so taking previous relation,

$$\frac{H_1}{H_2} = \frac{B_1\mu_2}{B_2\mu_1} = \frac{A_2\mu_2}{A_1\mu_1}$$

We know $\frac{A_2}{A_1} < 1$ and $\frac{\mu_2}{\mu_1} > 1$ but we are <u>not</u> told if their product is > or < 1. So we don't have enough info to find if H_1 or H_2 is larger.

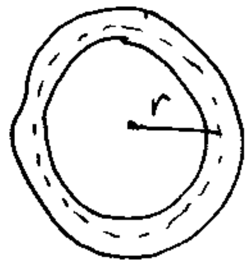
(1c) $BA = \Phi = \frac{NI}{\mathcal{R}}$ where reluctance $\mathcal{R} = \int \frac{dl}{\mu A}$

$$\mathcal{R} = \frac{\pi R}{\mu_1 A_1} + \frac{\pi R}{\mu_2 A_2} = \pi R\left(\frac{1}{\mu_1 A_1} + \frac{1}{\mu_2 A_2}\right)$$

so $$B_2A_2 = \frac{NI}{\pi R\left(\frac{1}{\mu_1 A_1} + \frac{1}{\mu_2 A_2}\right)} = \frac{NI\mu_1 A_1 \mu_2 A_2}{\pi R(\mu_2 A_2 + \mu_1 A_1)}$$

$$\Rightarrow \boxed{B_2 = \frac{NI\mu_1\mu_2 A_1}{\pi R(\mu_1 A_1 + \mu_2 A_2)}}$$

(2a) Use symmetry and "Gauss's Law for $\vec{D}$": take a Gaussian surface inside the dielectric:



Know $\vec{D} = D(r)\hat{r}$ by symmetry.

$$\oint d\vec{A}\cdot\vec{D} = Q_{free} = Q.$$

so $Q = 4\pi r^2 D(r) \implies \boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{r}}$

$\vec{E} = \frac{\vec{D}}{\varepsilon}$ so $\boxed{\vec{E} = \frac{Q}{4\pi r^2 \varepsilon}\hat{r}}$

(2b) Outside the dielectric, use Gauss's Law. The dielectric has no net charge, so total charge is Q.

$\vec{E} = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{r}$, $\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$

(2c) Could calculate polarization and use $\sigma_b = \vec{P}\cdot\hat{n}$, but given symmetry, easier to invert Gauss's Law: $Q_{encl} = \varepsilon_0 \oint \vec{E}\cdot d\vec{A}$

$$\left.\begin{array}{l} \text{Outside, } Q^{out}_{enc} = \varepsilon_0 4\pi r^2 \frac{Q}{4\pi r^2 \varepsilon_0} = Q \\ \text{In dielectric, } Q^{in}_{enc} = \varepsilon_0 4\pi r^2 \frac{Q}{4\pi r^2 \varepsilon} = Q\frac{\varepsilon_0}{\varepsilon} \end{array}\right\} \text{difference in charge must be at surface}$$

So $Q_b = Q^{out}_{enc} - Q^{in}_{enc} = \boxed{Q\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)}$

(2d) Capacitance $C = \frac{Q}{V}$ so must find V: take $V=0$ at $r=\infty$.

$$V = -\int_\infty^a dr\, E_r = -\int_\infty^b dr \frac{Q}{4\pi r^2 \varepsilon_0} - \int_b^a dr \frac{Q}{4\pi r^2 \varepsilon}$$

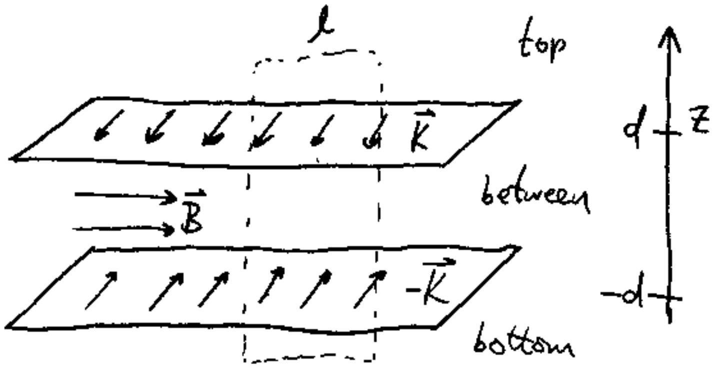
$$= \left.\frac{Q}{4\pi r \varepsilon_0}\right|_\infty^b + \left.\frac{Q}{4\pi r \varepsilon}\right|_b^a$$

$$= \frac{Q}{4\pi}\left[\frac{1}{b\varepsilon_0} + \frac{1}{a\varepsilon} - \frac{1}{b\varepsilon}\right]$$

So $C = \frac{Q}{V} = \boxed{\dfrac{4\pi}{\frac{1}{b\varepsilon_0} + \frac{1}{a\varepsilon} - \frac{1}{b\varepsilon}}}$

(Note – becomes $4\pi a \varepsilon_0$ in $\varepsilon \to \varepsilon_0$ limit.)

(3a)



Use Ampere's Law and symmetry:

Amp's Law around loop: $\quad l\left(\vec{B}_{top} - \vec{B}_{bottom}\right) = \mu_0 l\left(\vec{K}_0 + -\vec{K}_0\right)$

$$\Rightarrow \vec{B}_{top} = \vec{B}_{bottom}$$

Boundary cond. on $\vec{B}$: $\quad \vec{B}_{top} - \vec{B}_{between} = \mu_0 \vec{K}_0 \times \hat{n}$ where $\hat{n} = \hat{z}$:

$$\vec{B}_{between} = -\mu_0 \vec{K}_0 \times \hat{z} + \vec{B}_{top}$$

on lower plate: $\quad \vec{B}_{between} - \vec{B}_{bottom} = \mu_0\left(-\vec{K}_0\right) \times \hat{z}$

$$\vec{B}_{between} = \vec{B}_{bottom} - \mu_0 \vec{K}_0 \times \hat{z}$$

From top or bottom, see amperian loops with no net current.
Assuming no additional external field,

$\vec{B}_{top} = 0 \quad \vec{B}_{bottom} = 0 \qquad \vec{B}_{between} = -\mu_0 \vec{K}_0 \times \hat{z}$.

(3b) Must keep $\text{curl}\,\vec{A} = \vec{B} = 0 \quad |z| \geq d$

$$= -\mu_0 \vec{K}_0 \times \hat{z} \quad |z| < d$$

for calculational simplicity,

assume $\vec{K}_0 = K_0 \hat{y}$ for now: $\text{curl}\,\vec{A}_{\text{between}} = -\mu_0 K_0 \hat{y} \times \hat{z} = -\mu_0 K_0 \hat{x}$

$$(\text{curl}\,\vec{A})_x = -\mu_0 K_0 = -\frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial y}$$

$$(\text{curl}\,\vec{A})_y = 0 = -\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}$$

$$(\text{curl}\,\vec{A})_z = 0 = -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}$$

Can be solved if $\frac{\partial A_y}{\partial z} = \mu_0 K_0$, all other derivatives vanish.

Take $\vec{A} = \mu_0 K_0 z \hat{y} = \mu_0 z \vec{K}_0$ for $|z| < d$

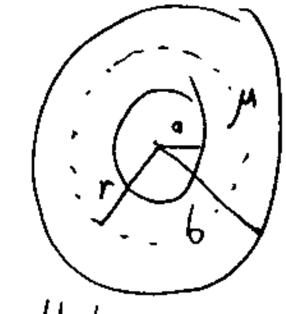
Since $\vec{B} = 0$ for $|z| > d$, can take $\vec{A}$ constant.

Continuity of $\vec{A}$ requires:

$$\boxed{\begin{aligned} z \leq -d:&\quad \vec{A} = -\mu_0 d \vec{K}_0 \\ -d \leq z \leq d:&\quad \vec{A} = \mu_0 z \vec{K}_0 \\ z \geq d:&\quad \vec{A} = \mu_0 d \vec{K}_0 \end{aligned}}$$

Note – can add any constant vector, and still be correct & in Coulomb gauge.

(4a) Inside $r<a$, $\vec{B}=0$ obviously.



In material, $a<r<b$. Take an Amperian loop at r:

$$\oint d\vec{l}\cdot\vec{H} = I_{free} = I_0 = 2\pi r H_\phi = 2\pi r H \text{ by symmetry.}$$

$$\text{So } \vec{H} = H_\phi = \frac{I_0}{2\pi r}$$

$$\vec{B} = \mu\vec{H} = \frac{I_0\mu}{2\pi r}\hat{\phi}$$

Outside material, $r>b$. Take Amperian loop:

$$\oint d\vec{l}\cdot\vec{H} = I_{free} = I_0 - I_0 = 0$$

by symmetry, $\vec{B}=\vec{H}=0$ here.

Summarizing:

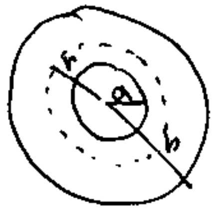
$r<a$: $\vec{B}=0$

$a<r<b$: $\vec{B} = \dfrac{I_0\mu}{2\pi r}\hat{\phi}$

$r>b$: $\vec{B}=0$

(4b) Use Amperian loop inside material to find I_{encl}:

For inner surface:



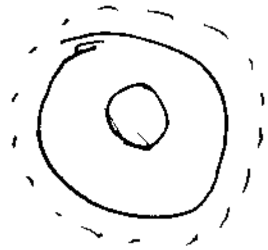
$$I_{enc} = I_b^{in} + I_{free} = \frac{1}{\mu_0}\oint d\vec{l}\cdot\vec{B}$$

$$= \frac{I\mu}{\mu_0}$$

$$\Rightarrow I_b^{in} + I_0 = \frac{I\mu}{\mu_0}$$

$$\boxed{I_b^{in} = I_0\left(\frac{\mu}{\mu_0} - 1\right)}$$

For outer surface:



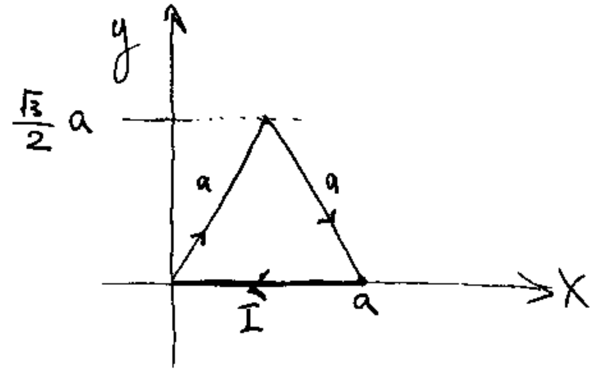
$$I_{enc} = I_b + I_{free} = 0$$

$$I_{free} = I_0 - I_0 = 0$$

$$I_{b\,total} = I_b^{in} + I_b^{out} = 0 \Rightarrow I_b^{out} = -I_b^{in}$$

$$\boxed{I_b^{out} = I_0\left(1 - \frac{\mu}{\mu_0}\right)}$$

(5)



I flows clockwise $\Rightarrow \vec{m} = -IA\hat{z}$

where A = area of triangle $= \frac{1}{2}(a)\left(\frac{\sqrt{3}}{2}a\right) = \frac{\sqrt{3}}{4}a^2$

$$\Rightarrow \vec{m} = \frac{-\sqrt{3}}{4}Ia^2\hat{z}.$$

Field is uniform $\Rightarrow$ no net force on triangle.

Torque : $\vec{N} = \vec{m} \times \vec{B}$

$$= \frac{-\sqrt{3}}{4}Ia^2B_0(\hat{z}\times\hat{y})$$

$$= +\frac{\sqrt{3}}{4}Ia^2B_0\,\hat{x}.$$