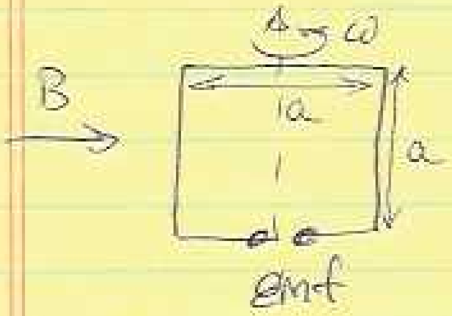
Set 11 Solutions

Phys 3310
Fall 2003

7.10 ac Electric Generator



A square loop rotates in a uniform $\vec{B}$ as shown. The emf is going to arise from the changing magnetic flux:

$$\mathcal{E}(t) = -\frac{d\Phi_m}{dt}$$

where

$$\Phi_m(t) = \int_{\text{loop area}} \vec{B}\cdot d\vec{A}$$

In the picture, lets say $\vec{B} = B\hat{x}$. Then, at $t=0$, lets take $\vec{A} = a^2\left[\hat{x}\cos\omega t + \hat{y}\sin\omega t\right]$ so that

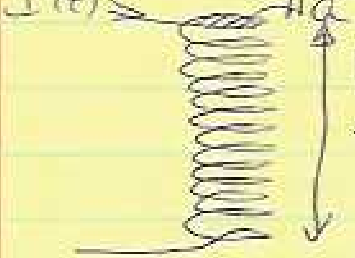
$$\Phi_m = \int \vec{B}\cdot d\vec{A} = Ba^2\cos\omega t$$

Then,

$$\boxed{\mathcal{E}(t) = -\frac{d\Phi_m}{dt} = Ba^2\omega\sin\omega t}$$

Solutions 11

7.15 Ampere-reasoning for $\vec{E}$



We know that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

or

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt}$$

Therefore, we can determine $\vec{E}$ for sufficiently symmetric situations using the same type of reasoning that we used in Ampère's Law.

In this case, the solenoid produces a uniform B inside, but B=0 outside

$$B = \mu_0 n I(t) \quad \text{parallel to axis of coil } \underline{\text{inside}}$$
$$= 0 \quad \text{outside coil}$$

Thus, there is magnetic flux in the circular region inside the coils, but none outside. Because of the cylindrical symmetry, it is clear that $\vec{E}$ follows circular paths. Thus, we can choose

Solutions 11

7.15 cont a circular path for $\oint E \cdot dl$
There are two cases:

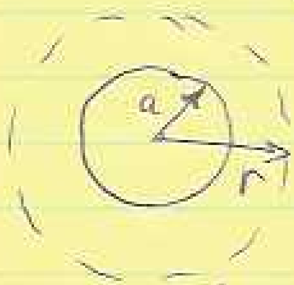
Path inside solenoid $r<a$

$$\oint E \cdot dl = E\,2\pi r$$

and $$-\frac{d\Phi_m}{dt} = -\frac{d}{dt} B\pi r^2 = \mu_0 n \pi r^2 \frac{-dI}{dt}$$

So

$$\boxed{E_{in} = \frac{1}{2}\mu_0 n r \frac{-dI}{dt}}$$



Path outside $r>0$

$$\oint E \cdot dl = E\,2\pi r$$

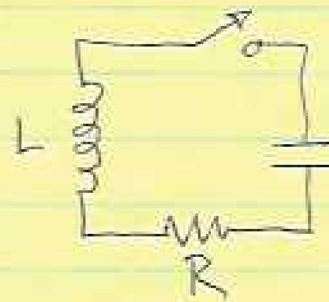
and $$-\frac{d\Phi_m}{dt} = -\frac{d}{dt} \underbrace{B\pi a^2}_{\text{Total } \Phi_m \text{ from solenoid}} = \mu_0 n \pi a^2 \frac{-dI}{dt}$$

or

$$\boxed{E_{out} = \frac{\mu_0 n a^2}{2r} \frac{-dI}{dt}}$$

Solutions 11

7.25 LC & LRC Circuits



Let's consider the general circuit with resistance. Once we solve this general case, we can take the $R \to 0$ limit to deal with the simpler LC circuit.

We start with a charged capacitor and no current. Then, at $t=0$ we close the switch. Very generally, we have Kirchoff:

$$\sum_{loop} V_i = 0$$

or

$$V_C + V_R + V_L = 0$$

We also know that the voltages are related to charges & currents:

$$CV_C = Q$$
$$IR = V_R$$
$$V_L = -L\frac{dI}{dt}$$

Solutions 11

7.25 cont Therefore, we find

$$\frac{Q}{C} + IR + L\frac{dI}{dt} = 0$$

We want $I(t)$, so if we differentiate the equation, we get a second order equation for current:

$$\frac{1}{C}I + R\frac{dI}{dt} + L\frac{d^2I}{dt^2} = 0$$

Trying to solve equations like this with sin, cos, or $e^{\pm\alpha t}$ can be done in one shot by looking at complex exponentials:

$$I(t) \stackrel{?}{=} Ae^{i\omega t} + Be^{-i\omega t}$$

Try just $Ae^{+i\omega_+ t}$ & you get:

$$\frac{A}{C}e^{+i\omega_+ t} + Ri\omega_+ Ae^{+i\omega_+ t} + L\omega_+^2 Ae^{+i\omega_+ t} = 0$$

This equation reduces to:

$$-LC\omega_+^2 + i\omega_+ RC + 1 = 0$$

Solutions II

7.25 cont In other words, the complex exponential does solve the equation as long as we choose the correct value of ω! We need the ω that satisfy the second order equation.

$$\omega_+ = \frac{-iRC \pm \sqrt{-R^2C^2 + 4LC}}{2LC}$$

Similarly, for $e^{-i\omega_- t}$ we get

$$\omega_- = \frac{+iRC \pm \sqrt{-R^2C^2 + 4LC}}{2LC}$$

Case I: $R=0$ Then,

$$\left.\begin{aligned} \omega_+ &= \pm \frac{1}{\sqrt{LC}} \\ \omega_- &= \mp \frac{1}{\sqrt{LC}} \end{aligned}\right\}$$ Really the same two roots

$$I(t) = A e^{\pm i\omega_\pm t} + B e^{\pm i\omega_\pm t}$$

Subject to the boundary condition that... what? What is $I(t=0)$?

Solutions 11

7.25 cont. Well, we know at $t \leq 0$ that the charge is on the capacitor and at $t = 0_-$, no current flows. Just after the switch is closed, could $I(t)$ jump discontinuously to some initial value?

To do so would require having current in the inductor and that means quickly moving energy into the stored $\vec{B}$ field. The power needed is:

$$P = \mathcal{E}(t) I = -L \frac{dI}{dt} I$$

If $I(t)$ jumps discontinuously at $t=0$, then dI/dt is infinite & we'd need infinite power. Therefore, we can see that $I(t=0) = 0$.

The only way to do it is for

$$A = -B = ?$$

Then, $$I(t) = A \sin \omega_{\pm} t$$

Solutions 11

7.25 cont Finally, we need A, and we can find its value via Kirchoff at $t=0$, where

$$+L\frac{dI}{dt} + Q/C = 0$$

or

$$+L\omega A + V_0 = 0$$

or

$$A = \frac{-V_0}{L\omega}$$

Thus,

$$\boxed{\begin{gathered} I(t) = \frac{-V_0}{L\omega}\sin\omega t \\ \omega = \frac{1}{\sqrt{LC}} \end{gathered}}$$

R=0 result

What you see here is oscillation! The natural frequency or "LC frequency" is:

$$\omega = \frac{1}{\sqrt{LC}}$$

Energy originally stored in the E-field in the capacitor oscillates over to B-field in the inductor & then back.

Solutions 11

7.25 cont Next, let's discuss $R \neq 0$
Of course, we have the complete solution:

$$I(t) = A e^{+i\omega_+ t} + B e^{-i\omega_- t}$$

$$\omega_I = \frac{\mp iRC \pm \sqrt{-R^2C^2 + 4LC}}{2LC}$$

But, what does it mean?

Notice that the frequencies are now complex numbers. Notice also that we have two situations:

i) $R^2C^2 < 4LC$ Then, the $\sqrt{}$ has positive argument and the frequencies have real and imaginary parts. In this case, the exponential $e^{i\omega t}$ has the form of

$$e^{\pm t/\tau} \, e^{i\omega_{Real} t}$$

$$\tau = \mathrm{Im}[\omega]$$

We get damped oscillator

Solutions 11

7.25 cont

ii) $R^2C^2 > 4LC$ Then the root arg is negative and the freqs are purely imaginary. Then, we get only growing or decaying exponentials. No more oscillation. The condition

$$R^2C^2 = 4LC$$

or

$$R = 2\sqrt{\frac{L}{C}}$$

$$= 2L\omega_{LC} = 2\frac{1}{C\omega_{LC}}$$

Is referred to as "critical damping"

What Fun!

7.54 Transformer Circuit

Power distribution today is done with ac power. When Edison first wired lower Manhatten, the generators and lights, etc. worked with dc

Solutions 11

7.54 cont currents. The dc systems were easy to back up with battery power and were generally thought to be safer. However, they require very low resistance wiring between generators and where the power is used, otherwise lots of power goes into useless heat on the way to the home.

Transformers provide a way to reduce the current and boost voltage at fixed power. With lower current, dissipation is reduced. The typically voltage on a transmission line is much larger than the 120V ac in the house, but is stepped down at transformers to 480V before entering neighborhoods and from 480V to 120V near your house.

This problem shows how a transformer couples power to a load.

Solutions 11

a) Ideal Transformer By definition, an ideal transformer is a pair of coils, linked only by magnetic flux, where all turns, whether in one coil or the other, have the same flux.

An example is the ideal solenoid transformer, where two solenoids are wound on the same cylinder, have the same area per loop, the same length, but different # of turns. $\vec{B}$ is zero outside, uniform inside, and thus, each turn of each solenoid has the same uniform B, the same area and the same flux.

In practice, nearly ideal transformers are made by using iron alloys with large μ, so most of the flux is from $\vec{M}$, which is only non zero inside the material. Therefore the flux follows the region where the alloy exists. Wrap your coils around the alloy & you get most of the flux.

OK . .

Solutions 11

7.54 cont So, what we know on general principles is that flux through some loop is a linear function of the current in that loop. We can say:

$$\Phi_{\substack{\text{one}\\ \text{loop}\\ \text{coil}}} = L_{\substack{\text{one}\\ \text{loop}}} I_{\text{one loop}}$$

Then, for the ideal case, where the flux in any one loop also goes through all the loops, for any coil made of N_i loops, the flux at one of them is $N_i \times \Phi_{\substack{\text{one loop}\\ \text{coil}}}$ and because there are N_i total loops

$$\Phi_{\substack{\text{total}\\ \text{self}}} = \sum_{\#\text{ of loops}} \left(N_i \times \Phi_{\substack{\text{one}\\ \text{loop}\\ \text{coil}}} \right)$$

or

$$= N_i^2 \Phi_{\substack{\text{one loop}\\ \text{coil}}}$$

$$\Phi_i = L_i I_i \Leftarrow \text{Self flux}$$

$$= N_i^2 L_{\text{one loop}} I_i$$

Similarly, for mutual inductance,

Solutions 11

7.54 cont — we must again insist that the flux from one of the coils also appears in all the loops of the other:

$$\Phi_{\substack{\text{total}\\\text{mutual}}} = \sum_{\substack{\#\text{ loops}\\\text{of one}}} \Phi_{\substack{\text{from}\\\text{the other}}}$$

$$= \sum_{N_1} \left(N_{other} \times \Phi_{\substack{\text{one}\\\text{loop}\\\text{coil}}}\right)$$

$$= N_1 N_{other} L_{\substack{\text{one}\\\text{loop}\\\text{coil}}} I_{other}$$

or

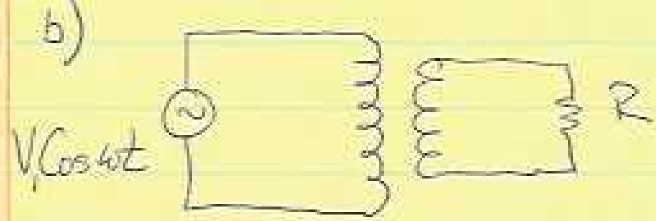
$$M = N_1 N_2 L_{\substack{\text{one}\\\text{loop}\\\text{coil}}}$$

$$L_{1,2} = N_{1,2}^2 L_{\substack{\text{one}\\\text{loop}\\\text{coil}}}$$

So

$$M^2 = L_1 L_2$$

b)



For this circuit, we have two sides. Apply Kirchoff to each:

$$V_1 \cos\omega t + \mathcal{E}_1(t) = 0 \ ; \ \mathcal{E}_2(t) + V_R = 0$$

Solutions 11

7.54 cont Here, the EMFs in the coils arise from the change in flux in each coil. The major insight lies that flux in either coil has two sources: 1) Self inductance and 2) Mutual inductance. So,

$$\mathcal{E}_1(t) = -\frac{d\Phi_1}{dt} = -\frac{d}{dt}(LI_1 + MI_2)$$
$$= -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt}$$

then,

$$\boxed{\begin{aligned} L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} &= V_1 \cos(\omega t) \\ L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} &= -I_2 R \end{aligned}}$$

and

c) Solve the equations. Lets ~~differentiate~~ assume solutions of the form

$$I_1 = I_1 e^{i\omega t}$$ $V_1 \cos\omega t \to V_1 e^{i\omega t}$

$$I_2 = I_2 e^{i\omega t}$$ Later we'll take the real part!

then,

$$i\omega L_1 I_1 + i\omega M I_2 = V_1$$
$$i\omega L_2 I_2 + i\omega M I_1 = -I_2 R$$

Solutions 11

7.54 cont Solve these algebraically to get:

$$I_1 = V_1 \frac{i\omega L_2 + R}{-\omega^2(L_1L_2 - M^2) + i\omega L_1 R}$$

$$I_2 = V_1 \frac{i\omega M}{-\omega^2(L_1L_2 - M^2) + i\omega L_1 R}$$

For the case of an ideal trans. $M^2 = L_1L_2$ and we find

$$I_1 = V_1\left[\frac{L_2}{L_1R} + \frac{1}{i\omega L_1}\right]$$

$$= \frac{V_1}{R}\left[\frac{L_2}{L_1} + \frac{R}{i\omega L_1}\right]$$

$$I_2 = V_1 \frac{M}{L_1R}$$

d) The output voltage is I_2R, so

$$\frac{I_2R}{V_1} = \frac{M}{L_1} = \frac{N_2N_1 L_{\text{one loop}}}{N_1^2 L_{\text{one loop}}}$$

$$= \frac{N_2}{N_1}$$

Solutions 11

7.54 cont

e) Power in and out. Finally, let's look at the power. We need to look at the product of current and voltage and look at the parts that are in phase. In other words, its only the parts that are in phase the cause real dissipation.

In the second coil, I_2 & $V_2 = I_2 R$ are in phase, so

$$P_2 = I_2 V_2 = I_2^2 R$$
$$= \frac{V_1^2}{R} \frac{M^2}{L_1^2}$$

In the primary, V_1 is real ($\cos\omega t$) and

$$I_1 = V_1 \left[\frac{L_2}{L_1 R} + \frac{1}{i\omega L_1} \right]$$

So

$$P_1 = V_1 \left(\mathrm{Re}[I_1] \right) = \frac{V_1^2}{R} \frac{L_2}{L_1}$$

Solutions 11

7.54 cont   We are asked

$$P_1 \overset{?}{=} P_2$$

i.e.

$$\frac{V_1^2}{R}\frac{L_2}{L_1} \overset{?}{=} \frac{V_1^2}{R}\frac{M^2}{L_1^2}$$

or   to get equality, we need

$$L_1 L_2 = M^2$$

If the transformer is ideal then we get perfect power transfer.

Ta da