Physics 3210 Spring 2008

Problem Set 7

1. A mass $M$ is fixed at the right-angled vertex where a massless rod of length $l$ is connected to a very long massless rod. A mass $m$ is free to move without friction along the entire length of the long rod. The rod of length $l$ is hinged at a support and at its end opposite to the long rod, and the whole system is free to rotate about the support and in the plane of the rods. See the figure below.

(a) Let $x$ be the distance between $M$ and $m$, and take the origin of the coordinate system to be the support point with the positive direction down. Find the Lagrange equations of motion.

(b) Find the solution to the equations of motion when both $x$ and $\theta$ are small (i.e., $\theta \ll 1$ and $x \ll 1$). Describe your results in physical terms.

[Hint: The equation for $x$ is simple and can be used to eliminate the terms $l \ddot{\theta} + g \theta$ and $l \ddot{\theta} + \ddot{x}$ that you should have in the $\theta$ equation. This will leave a simple $\theta$ equation that is now only a function of $x$. Solve these by assuming solutions of the form $x = Ae^{\omega t}$ and $\theta = Be^{\omega t}$. And don’t forget that $x(t) = 0$ could be a solution if the equation allows it.]

Additional remark: The harmonic oscillator equation $\ddot{x} + \omega^2 x = 0$ with $\omega^2 > 0$ has solutions of the equivalent forms

$$x(t) = Ae^{\omega t} + Be^{-\omega t}$$
$$x(t) = C \cos \omega t + D \sin \omega t$$
$$x(t) = E \cos (\omega t + \phi_1)$$
$$x(t) = F \sin (\omega t + \phi_2)$$

where the various constants are related to each other. And if the equation is $\ddot{x} - \omega^2 x = 0$, then the solutions are of the equivalent forms

$$x(t) = Ae^{\omega t} + Be^{-\omega t}$$
$$x(t) = C \cosh \omega t + D \sinh \omega t$$
$$x(t) = E \cosh (\omega t + \phi_1)$$
$$x(t) = F \sinh (\omega t + \phi_2)$$

For our purposes, the cos and cosh forms are generally the most useful.
2. (Taylor, Problem 7.49). A particle of mass $m$ and charge $q$ moves in a uniform constant magnetic field $\mathbf{B}$.
   (a) Prove that $\mathbf{B}$ can be written as $\mathbf{B} = \nabla \times \mathbf{A}$ with $\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r}$.
   (b) If $\mathbf{B} = B\hat{z}$, show that in cylindrical coordinates $(\rho, \phi, z)$ we have $\mathbf{A} = \frac{1}{2} B \rho \hat{\phi}$.
   (c) Write the Lagrangian (equation (35) on page 66 of the notes, or Taylor equation (7.103)) in cylindrical coordinates and find the three equations of motion.
   (d) Describe in detail the solutions of the equations of motion in the case where $\rho$ is a constant.

3. (Thornton & Marion, Problem 7-24). Consider a simple plane pendulum of mass $m$ attached to string of length $l$. After the pendulum is set in motion, the string is shortened at a constant rate $\dot{l} = -\alpha$ with the suspension point unchanged. (Note that the usual formulation for a simple pendulum has only one generalized coordinate. This problem is no exception. Here the length of the string is just a time-dependent parameter, not a dynamical variable the value of which is determined by the motion of the system.)

   (a) Find the Lagrangian.
   (b) Find the Hamiltonian.
   (c) Is the Hamiltonian conserved? Is it equal to the total energy? Is energy conserved? Explain your answers.

4. (Thornton & Marion, Problem 7-29). Consider the spring-pendulum shown below. Let the rest length of the spring be $l_0$ and the extended length be $l$. The support point of the pendulum moves up at a constant acceleration $a$. (Assume $y(0) = \dot{y}(0) = 0$.)

   (a) Find the Lagrange equations of motion for $l$ and $\theta$.
   (b) Find the Hamiltonian.
   (c) Find Hamilton’s equations of motion.
(d) Using the results of part (a), find the period of small oscillations for both $\theta$ and $l$. ("Small" means to keep only terms up to first order in $\theta$ and $l$ or their products.)

5. (Taylor, Problem 13.5). A bead of mass $m$ moves on a frictionless wire bent into a helix with cylindrical coordinates $(\rho, \phi, z)$ satisfying $z = c\phi$ and $\rho = R$ where $c$ and $R$ are constants. Let the $z$-axis be up.

(a) Using $\phi$ as the generalized coordinate, write the kinetic and potential energies of the bead.

(b) Find the generalized momentum conjugate to $\phi$ and write down the Hamiltonian.

(c) Write down Hamilton’s equations and solve for $\ddot{\phi}$ (not $\phi$) and hence also $\ddot{z}$. Write your answer for $\ddot{z}$ in terms of the angle $\alpha$ defined by $\tan \alpha = c/R$. (This is the amount the helix rises divided by the horizontal distance around for each $2\pi$ revolution. Since $z = c\phi$, a change in $\phi$ of $2\pi$ means that the bead rises a distance $z = 2\pi c$, and hence $\tan \alpha = 2\pi c/2\pi R = c/R$.)

(d) Explain this result in terms of Newtonian mechanics. What happens in the particular case that $R = 0$?

6. (Taylor, Example 13.4 and Problem 13.17) Consider a mass $m$ constrained to move inside the surface of a frictionless vertical cone of radius $\rho = cz$ where $c$ is a constant. (This is in cylindrical coordinates $(\rho, \phi, z)$ with $z > 0$. So the vertex is at the origin and the cone opens upwards.) Let the generalized coordinates be $\rho$ and $z$.

(a) What is the Lagrangian of the particle?

(b) Write down the generalized momenta and the Hamiltonian $H(p, q)$.

(c) Write down the four Hamilton equations of motion.

(d) If the mass remains at a fixed height $z_0$, what is the value of $z_0$ in terms of a given initial $p_\phi$ and $m, c, g$?

(e) Let $z = z_0 + \varepsilon$ with $\varepsilon$ small. Show that $\ddot{\varepsilon} + \omega^2\varepsilon = 0$ where $\omega = \sqrt{3} \dot{\phi}_0 \sin \alpha$. Here $\dot{\phi}_0$ is defined by the equation for the generalized momenta $p_\phi$, and $\alpha$ is the half-angle of the cone defined by $\tan \alpha = \rho/\varepsilon = c/z = c$. (Since $\rho = cz$ we see that $\tan \alpha = \rho/\varepsilon = c/z = c$.) You will also need the binomial expansion $(1 + \varepsilon/z_0)^{-3} \approx 1 - 3\varepsilon/z_0$.

(f) For what value of $\alpha$ will $\omega = \dot{\phi}_0$? What does this tell you about the orbit?

7. (Taylor, Problem 13.23). Consider the contraption shown below where, as usual, the frictionless pulley and spring are massless. The symbol $l_r$ stands for the total length of the rope (neglecting the part that goes over the pulley); $l_e$ is the equilibrium length of the spring with just the mass $m$ hanging from it; $x$ is the extension of the spring from this equilibrium length (so the overall length of the spring at any instant is $l_e + x$); let $l_o$ be the natural (unextended, unloaded) length of the spring; and let $x'$ be the spring’s total extension from its unloaded length.
(a) Show that the total potential energy of the system is just \( V = \frac{1}{2}kx^2 \) plus a constant.

(b) Find the momenta conjugate to \( x \) and \( y \), and write down the Hamiltonian. Show that \( y \) is cyclic (or ignorable).

(c) Write down the four Hamiltonian equations.

(d) Solve the equations of motion subject to the following initial conditions: Hold \( M \) fixed, and with the system in equilibrium, let \( y = y_0 \).

With \( M \) still fixed, pull down the lower mass \( m \) a distance \( x = x_0 \).

At \( t = 0 \) you let go of everything, so \( x(0) = x_0, y(0) = y_0 \) and \( \dot{x}(0) = \dot{y}(0) = 0 \). [Hint: Given the initial conditions for \( x, y, \dot{x} \) and \( \dot{y} \), the equations of motion give the initial values for \( p_x \) and \( p_y \). Combine the two first order \( x \) equations into a single second order equation for \( \ddot{x} \) and solve it. Then, given \( x(t) \), you can solve for \( p_x \), then \( \dot{y} \), and finally \( y \).]