1. [Total: 10 pts]
For the periodic function \( f(t) = A(t^2 - 1/3) \) for \(-1 < t \leq 1\) (with \( A \) a given constant)

a) [4 pts] Compute Fourier coefficients \( a_n \) and \( b_n \). (Some vanish - predict which ones before doing any integrals)

b) [6 pts] Suppose this force drives a weakly damped oscillator with damping parameter \( \beta = 0.05 \) and a natural period \( T = 2 \). Find the long time motion \( x(t) \) of the oscillator.

2. [Total: 14 pts]

a) [8 pts] Demonstrate that the Fourier transform of the following function:

\[
f(t) = \sin(t), \quad \text{for} \quad -\frac{\pi}{2} < t < \frac{\pi}{2}
\]

\[
= 0, \quad \text{for} \quad |t| > \frac{\pi}{2}
\]

is

\[
g(\omega) = -i\omega \frac{\cos \left(\frac{\omega \pi}{2}\right)}{\pi (1 - \omega^2)}.
\]

(Hint: It might help to express \( e^{-i\omega t} \) in terms of sines and cosines. Recalling the properties of integrals of odd and even functions can also save you some work.)

b) [6 pts] The function \( e^{-c|x|} \) (with \( c \) a positive, known constant) is associated with bound states in quantum mechanics, and (pretty much as always in quantum mechanics) its Fourier transform is important to know. Find the Fourier transform of \( e^{-c|x|} \) and sketch it.

3. [Total: 8 pts]
For the function shown in the Figure

\[
f(t)
\]

a) [2 pts] Qualitatively, how do you expect \( g(\omega) \), the Fourier transform of \( f(t) \), to change if you make \( a \) bigger or smaller?

b) [4 pts] Find the Fourier transform \( g(\omega) \).

c) [2 pts] Sketch \( g(\omega) \) (or plot it in Mathematica if you prefer). Was your prediction in part (a) correct? Explain.
4. [Total: 10 pts]
Evaluate the following integrals: [2 pts each]

a) \( \int_0^\pi \sin(x)\delta(x - \pi/2) \, dx \)

b) \( \int_0^3 (5t - 2)\delta(2 - t) \, dt \)

c) \( \int_0^5 (t^2 + 1)\delta(t + 3) \, dt \)

d) \( \int_{-\infty}^{\infty} e^x\delta(3x) \, dx \) (Hint: Can you try a substitution?)

e) If a particle feels a force \( F(t) \) of the form \( F = A\delta(t) \), with \( t \) the time, what are the units of \( A \)? Give a physical interpretation to this formula - what sort of force are we trying to represent here?

5. [Total: 8 pts]
Prove the following properties of the Dirac \( \delta \)-function [4 pts each]:

a) \( \delta(x) = \delta(-x) \)

b) \( x\delta(x) = 0 \)