Quantum tunneling and review

Announcements:

- 2nd exam is tomorrow, April 7 in MUEN 0046 from 7:30 – 9:00 pm and will be similar to last exam.
  - 9 questions but 21 total parts (1 less than last time)
  - 4/9 questions are multiple choice like
  - Don't have to show work on multiple choice but if you do it will be considered (for better or worse)
  - Need to show work on other problems
  - Bring writing utensil and calculator
- Problem solving sessions on M 3-5 and T 3-5.
- The quantum tunneling tutorial is posted on the calendar and homework page. It is worth 9 homework points.
- Homework due Wednesday is extra credit but I highly recommend understanding these problems before the exam.

Clicker question 1

How useful did you find the quantum tunneling tutorial on Friday?

A. Very useful
B. Somewhat useful
C. Not very useful
D. Not at all useful
E. Can't say since I was not in class Friday

Some observations from the tutorial

From photoelectric effect we found that it takes energy to remove an electron from a metal. The amount of energy is the work function.

This energy to overcome the work function goes into the potential energy of the electron (like overcoming a gravity well).

Energy is conserved so E = K + U

If potential energy increases, kinetic energy must decrease.

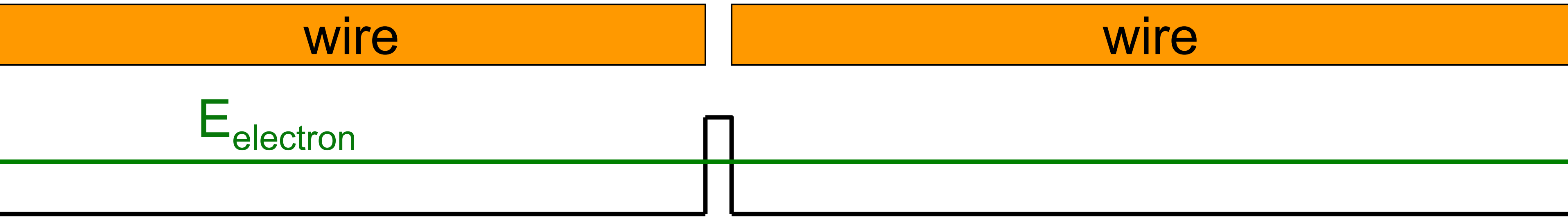
Kinetic energy determines the wavelength $K = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$

For $E>V_0$ solutions have the form ($e^{ikx} + e^{-ikx}$) which, by Euler's theorem ($e^{i\theta}=\cos\theta+i\sin\theta$), means they are sinusoidal.

For $E<V_0$ solutions have the form ($e^{\alpha x} + e^{-\alpha x}$) which leads to exponential decay (after requiring $\psi(x)\rightarrow 0$ as $x\rightarrow\pm\infty$)

Quantum tunneling through potential barrier

Consider two very long wires separated by a small gap:



This is an example of a ***potential barrier.***

Quantum tunneling occurs when a particle which does not have enough energy to go over the potential barrier somehow gets to the other side of the barrier.

This is due to the particle being able to penetrate into the *classically forbidden region*.

If it can penetrate far enough (the barrier is thin enough) it can come out the other side.

End of Chapter 4

The last exam finished with the photoelectric effect

We started thinking about X-rays which are electromagnetic waves with very short (< 1 nm) wavelengths

Crystals are used as diffraction gratings – Bragg diffraction

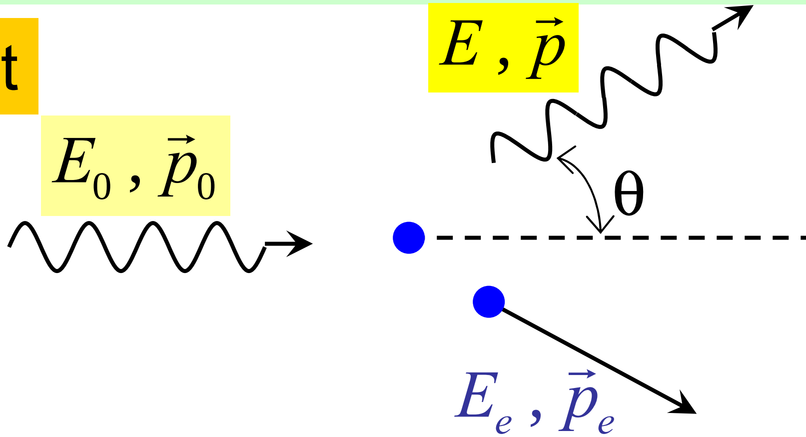
The Compton effect showed that X-rays have momentum

X-rays photons hit an atomic electron imparting momentum to the electron; the scattered photon has less energy and momentum

End result: particle-wave duality of light

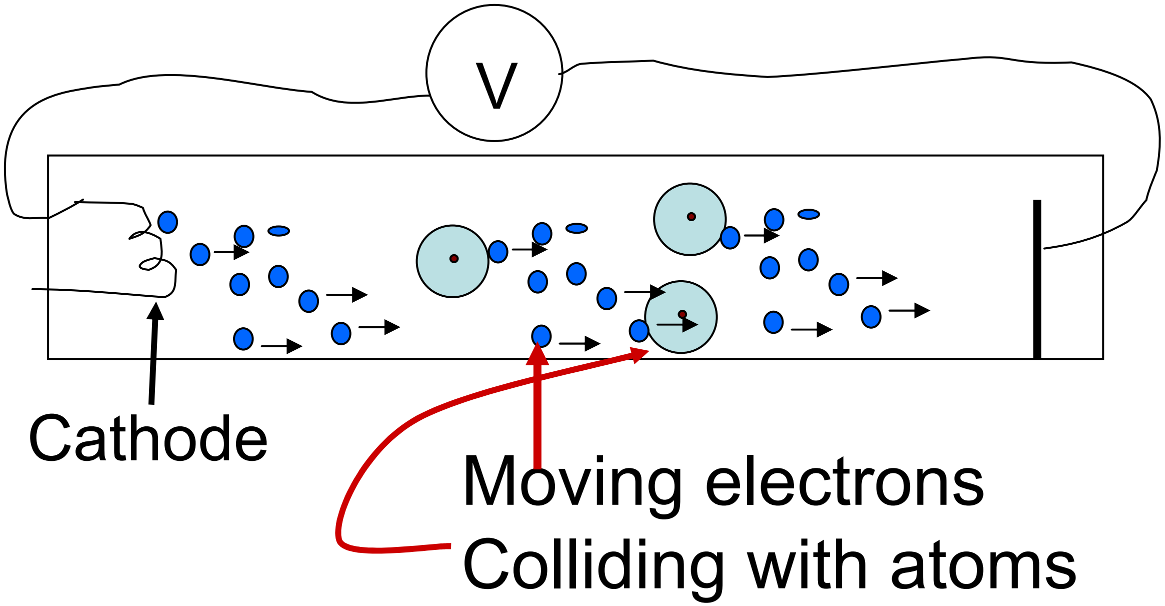
Light is both a particle and wave!

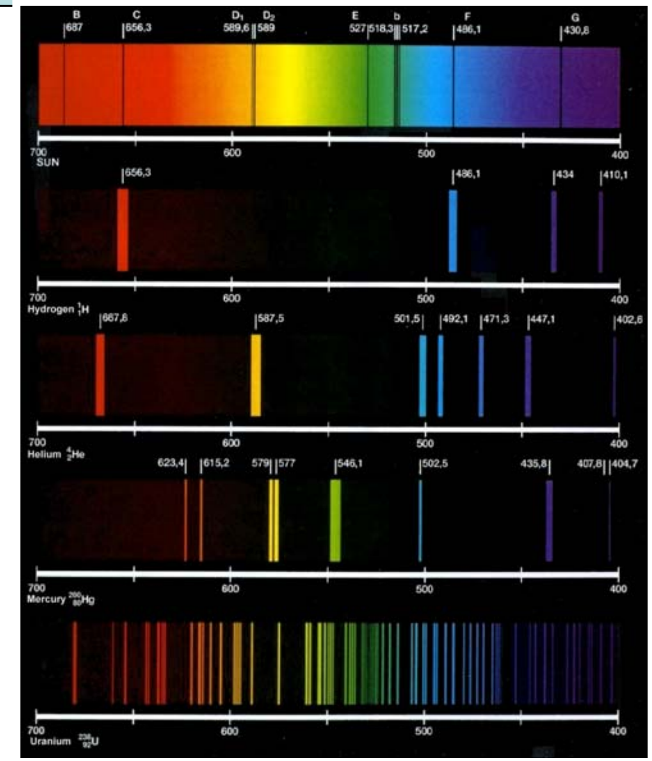
$$E = hf \qquad p = h/\lambda$$



Ch. 5: Atomic energy levels and Bohr model

Atomic discharge lamps showed us that atoms only emit certain wavelengths of light





Each wavelength corresponds to a given energy ($E = hc/\lambda$).

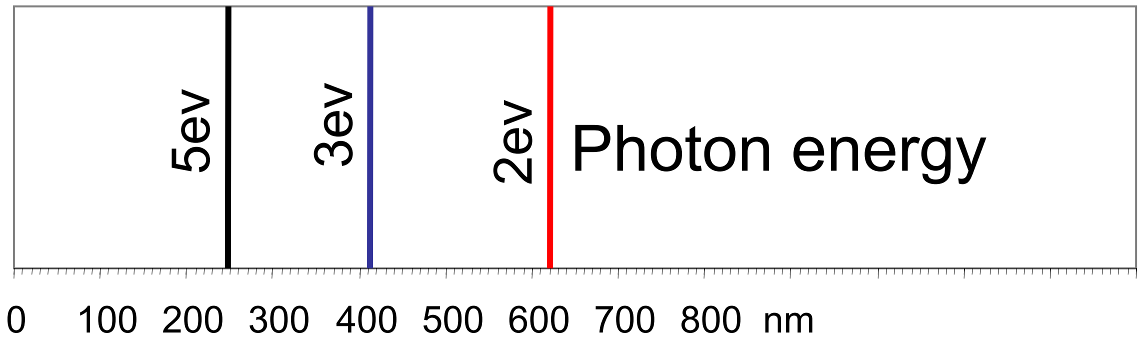
By conservation of energy, the atom must change energy by the same amount as the emitted photon.

This implies atoms can only have certain energy levels!

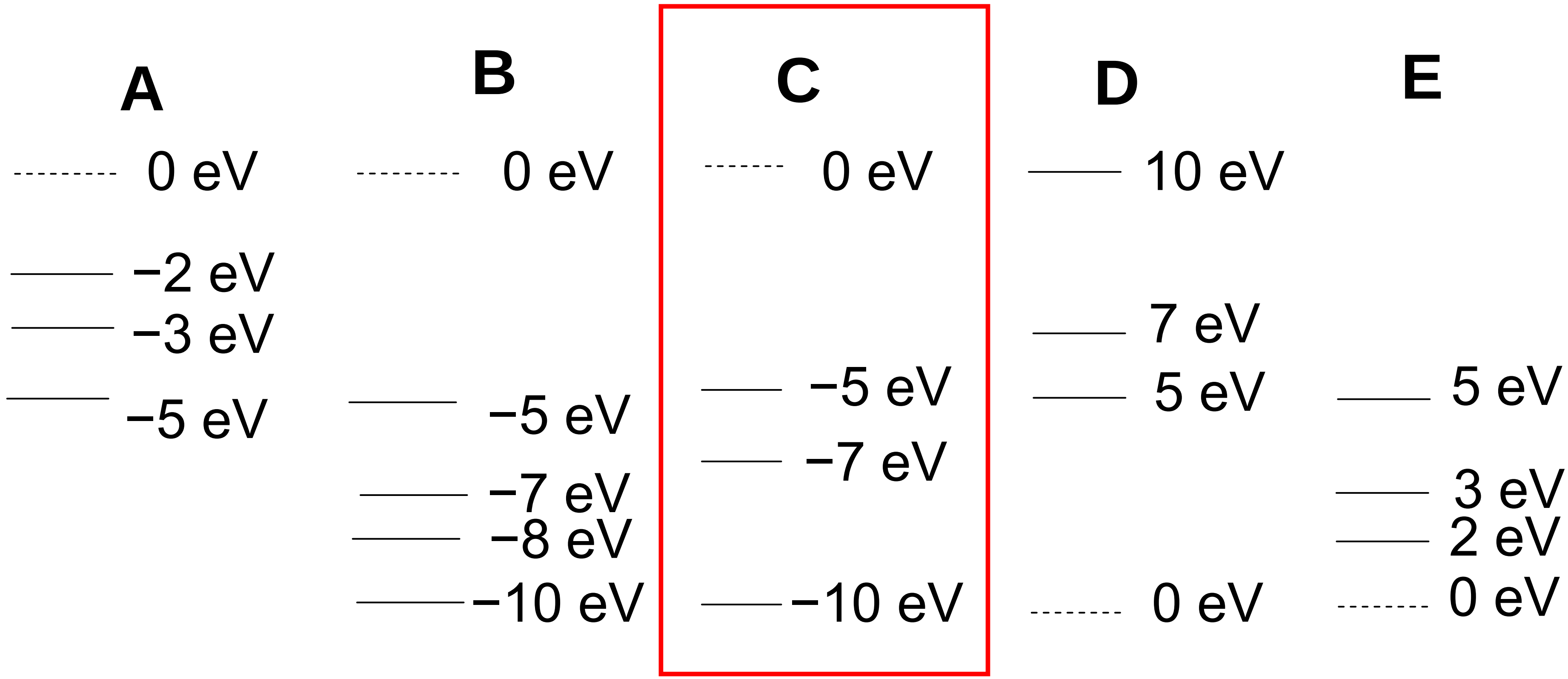
Clicker question 2

Set frequency to DA

What energy levels for electrons are consistent with this spectrum?

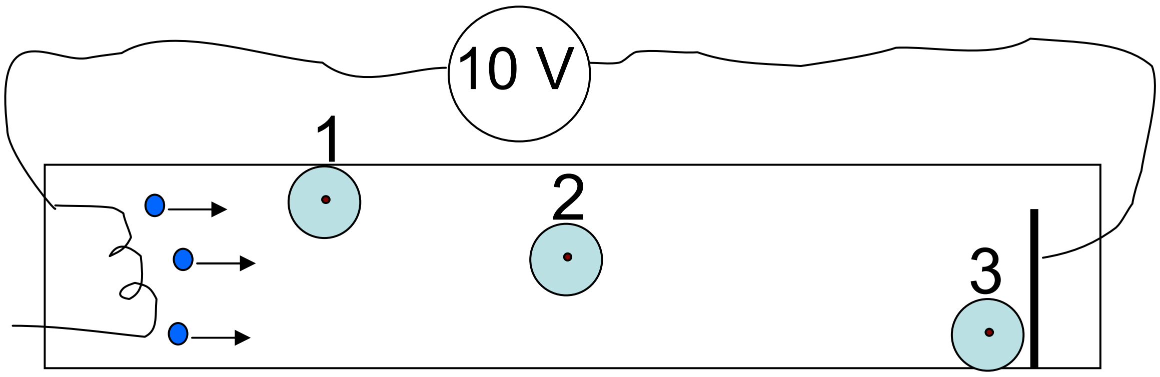


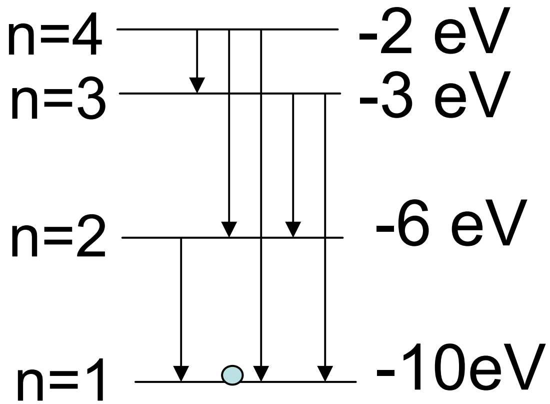
Electron Energy levels:



Clicker question 3

Set frequency to DA





Consider atoms at three points in an atomic discharge lamp with the energy levels shown. What would we see from each position?

A. All three positions will emit the same colors
B. 1 will emit more colors than 2 which will emit more than 3
C. 3 will emit more colors than 2 which will emit more than 1
D. 3 will emit more colors than 2 while 1 will emit no light
E. Impossible to tell

Electrons at 1 have K ≈ 2 eV so cannot excite atoms

Electrons at 2 have K ≈ 5 eV so can only excite to n = 2 (1 color)

Electrons at 3 have K ≈ 9 eV so can excite to n = 2,3,4 (6 colors)

Ch. 5: Atomic energy levels and Bohr model

Balmer-Rydberg formula describes the hydrogen energy levels:

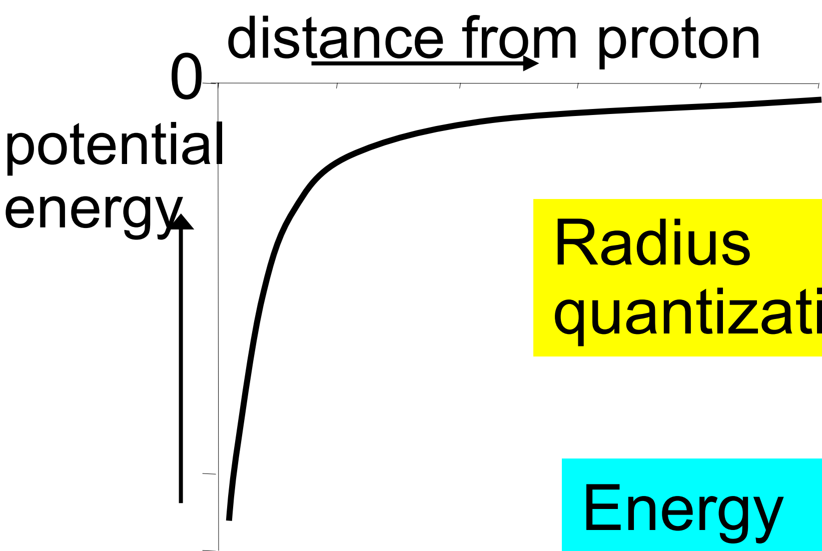
$$\frac{1}{\lambda} = R\left(\frac{1}{n'^2} - \frac{1}{n^2}\right)$$

$$R = 0.0110\,\text{nm}^{-1}$$

Bohr was able to derive this formula with his model of the atom:

Anything in orbit has $K = -½U$ and $E = K + U = ½U$

Combined with Bohr's assumption of angular momentum quantization, this gives quantized radii and quantized energy.



Angular momentum quantization:

$$L = m_e v r = n\hbar \qquad \hbar = h/2\pi$$

Radius quantization:

$$r_n = n^2 a_B \qquad a_B = \frac{\hbar^2}{m_e k e^2} = 0.053\,\text{nm}$$

Energy quantization:

$$E_n = -\frac{ke^2}{2a_B}\frac{1}{n^2} = -\frac{E_R}{n^2} = -\frac{13.6\,\text{eV}}{n^2}$$

$$E_R = hcR = \frac{ke^2}{2a_B} = \frac{m(ke^2)^2}{2\hbar^2} = 13.6\,\text{eV}$$

Hydrogen like ions

Atoms which have only one electron can be analyzed much like the hydrogen atom.

An atom with atomic number *Z* with *Z-1* electrons removed is a hydrogen like ion

The (Coulomb) force on the electron is $F_C = \frac{kq_1q_2}{r^2} = \frac{kZe^2}{r^2}$

The increase in the force results in tighter orbits and a deeper potential well, reducing the energy (more negative).

$$r_n = \frac{n^2 a_B}{Z}$$

$$E_n = -Z^2 \frac{E_R}{n^2} = -Z^2 \frac{13.6\,\text{eV}}{n^2}$$

Summary and implications of Bohr model

Electrons orbit the nucleus at particular radii corresponding to particular energies. These energies are called energy levels or states.

The only allowed electron energy transitions are between these energy levels.

There always exists one lowest energy state called the ground state to which the electron will always return.

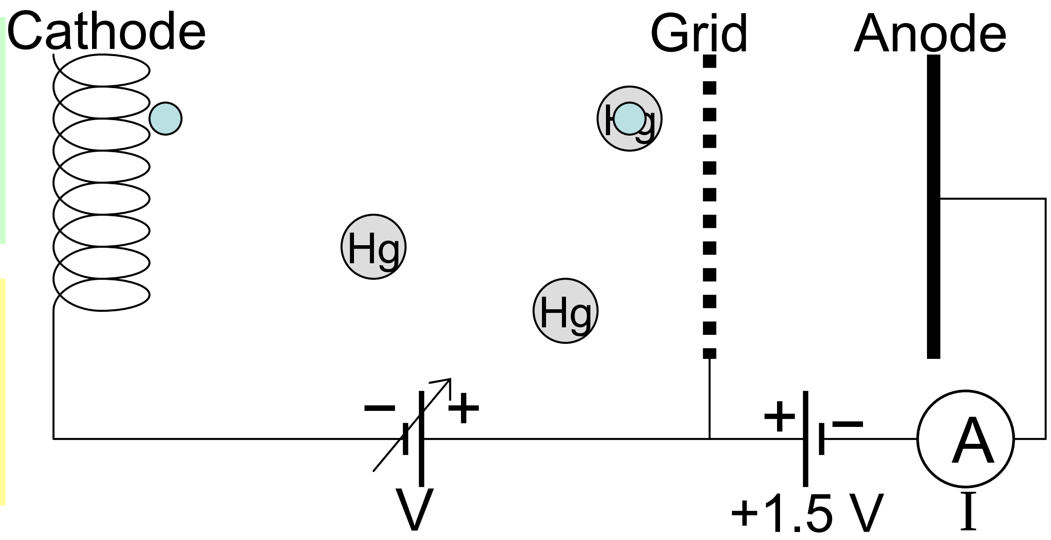
Free electrons with enough kinetic energy can excite atomic electrons. From conservation of energy, the free electron loses the same amount of kinetic energy as the atomic electron gains.

Photons are emitted *and* absorbed only with energies corresponding to transitions between energy levels.

Franck-Hertz experiment gave more proof of atomic energy levels

Franck-Hertz experiment

Mercury atoms need 4.9 eV to go from ground state to the next higher energy level.



Electrons with K < 4.9 eV elastically scatter off Hg, with no energy loss.

Electrons with K > 4.9 eV can inelastically collide with a Hg atom, transferring 4.9 eV and losing 4.9 eV of kinetic energy in the process.

After the collision, the electron may not gain enough kinetic energy to reach the anode so the current will drop.

As voltage increases, electron can excite multiple mercury atoms.

This experiment gives more proof for the existence of atomic energy levels. Performed in 1914; Nobel prize in 1925.

Ch. 6: Matter waves

Louis de Broglie postulated that electrons have wave properties.

All matter has wave properties with the particle & wave quantities related by de Broglie relations:

$$p = h/\lambda = \hbar k$$
$$E = hf = \hbar\omega$$

Davisson-Germer scattered electrons of varying energies off a nickel crystal (similar to X-ray diffraction) and found diffraction like X-ray diffraction showing electrons behave like a wave.

The function describing how light waves propagate is the electromagnetic wave function $E(x,t) = E_{\max}\sin(ax - bt)$ with $\text{Intensity} \propto E_{\text{avg}}^2 \propto E_{\max}^2$

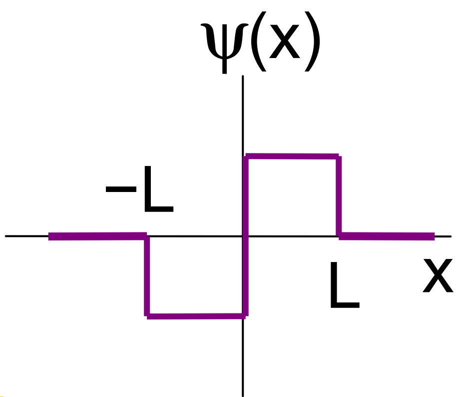
Particles also have a wave function ψ(x). In this case, $|\psi(x)|^2$ is the probability density which indicates how likely the particle is to be found at *x* (or what fraction of the particles will be found at *x*).

Properties of wave functions

Wave functions are complex valued and are ***not*** directly observable.

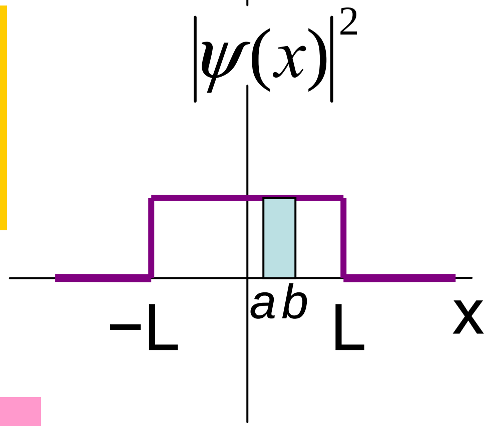
Probability density ***is*** observable: $|\psi(x)|^2 = \psi^*(x)\,\psi(x) = \psi^2_{\text{real}}(x) + \psi^2_{\text{imag}}(x)$

The probability of finding the particle at some infinitesimal distance δx around *x* is: $|\psi(x)|^2\,\delta x$



Probability of particle being between *a* and *b* is $\int_a^b |\psi(x)|^2\,dx$

The particle must be *somewhere* with a probability of 100%. This is the *normalization condition*: $\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = 1$



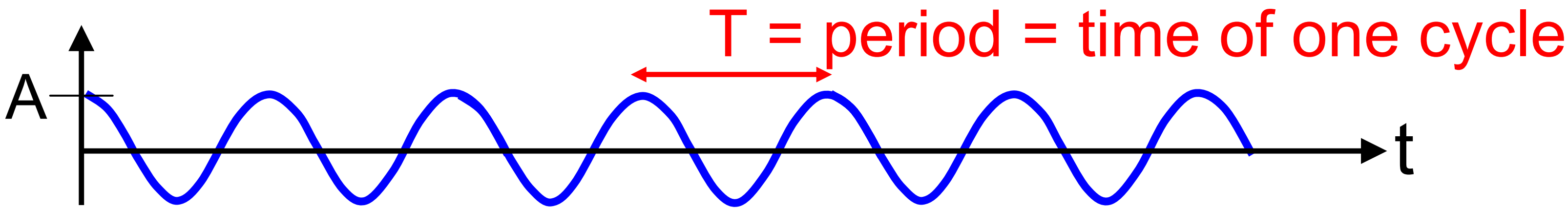
In 1D, $|\psi(x)|^2$ has dimension of 1/Length

ψ(x) is just the *spatial* part of the wave function.

Ψ(x,t) is the full wave function

Review of waves

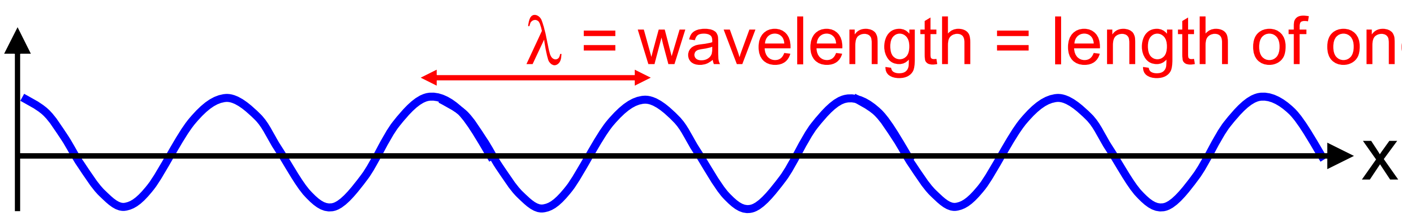
Waves in time: $y(t) = A\cos(\omega t) = A\cos(2\pi f t) = A\cos(2\pi t/T)$



f = 1/T = ω/2π = frequency = number of cycles per second

ω = 2πf = angular frequency = number of radians per second

Waves in space: $y(x) = A\cos(2\pi x/\lambda) = A\cos(kx)$

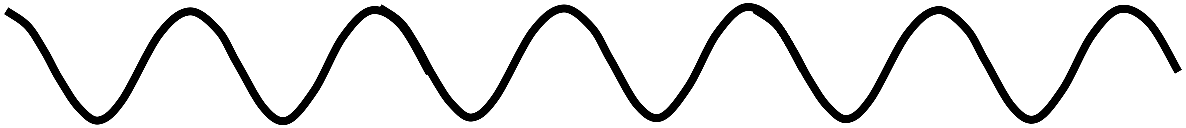


k = 2π/λ = wave number = number of radians per meter

Clicker question 4 — Set frequency to DA

Plane wave:

$$\Psi(x,t) = Ae^{i(kx-\omega t)}$$



Wave packet:

$$\Psi(x,t) = \sum_{n=0}^{\infty} A_n e^{i(k_n x-\omega_n t)}$$

For which type of wave is the momentum and position most well defined?

A. *p* is well defined for plane wave, *x* is well defined for wave packet

B. *p* is well defined for wave packet, *x* is well defined for plane wave

C. *p* is well defined for one but *x* is equally well defined for both

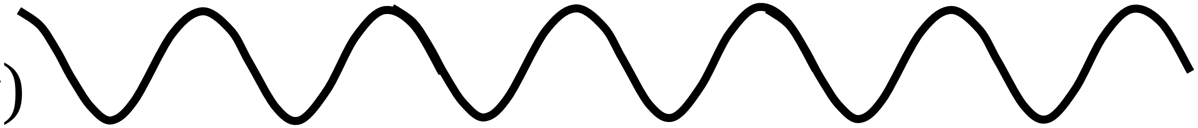
D. *p* is equally well defined for both but *x* is well defined for one

E. Both *p* and *x* are well defined for both

Plane Waves vs. Wave Packets

Plane wave:

$$\Psi(x,t) = Ae^{i(kx-\omega t)}$$



This wave represents a single *k* and ω. Therefore energy, momentum, and wavelength are well defined.

The amplitude is the same over all space and time so *position* and *time* are undefined.

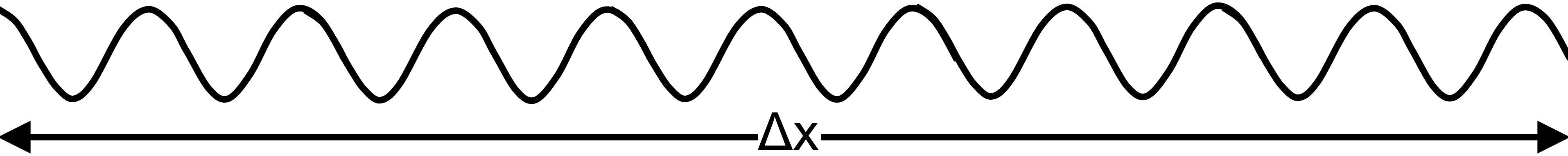
Wave packet:

$$\Psi(x,t) = \sum_{n=0}^{\infty} A_n e^{i(k_n x - \omega_n t)}$$

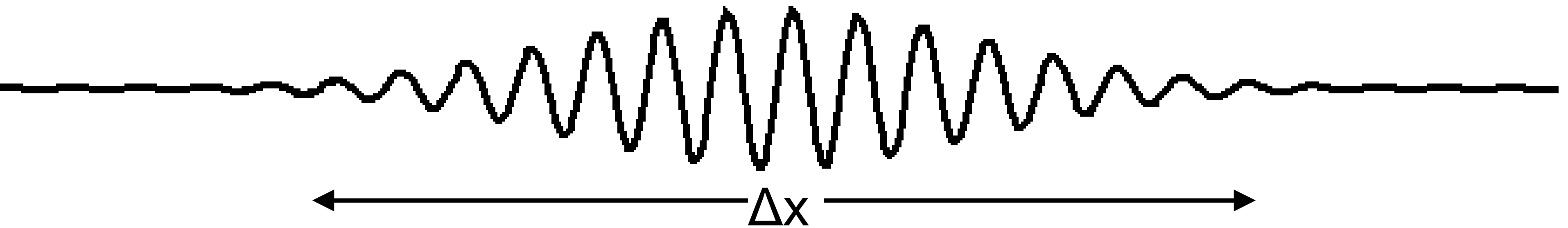
This wave is composed of many different *k* and ω waves. Thus, it is composed of many different energies, momenta, and wavelengths and so these quantities are not well defined.

The amplitude is non-zero in a small region of space and time so the position and time is constrained to be in that region.

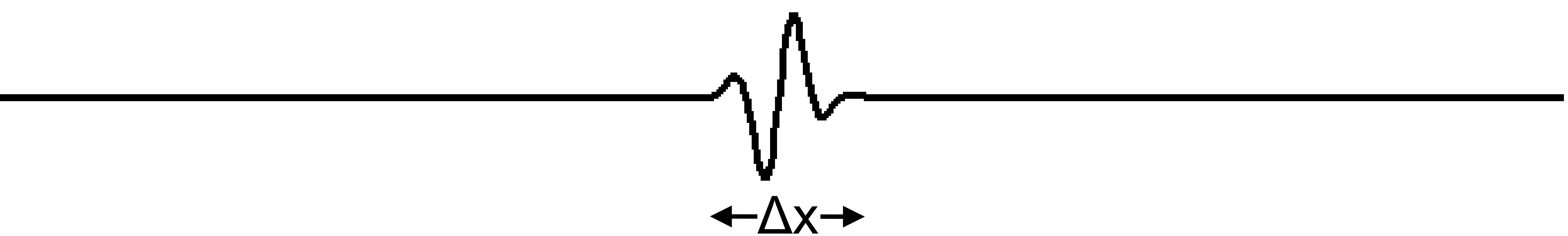
Heisenberg Uncertainty Principle: $\Delta x \, \Delta p \geq \hbar / 2$



small Δp – only one wavelength

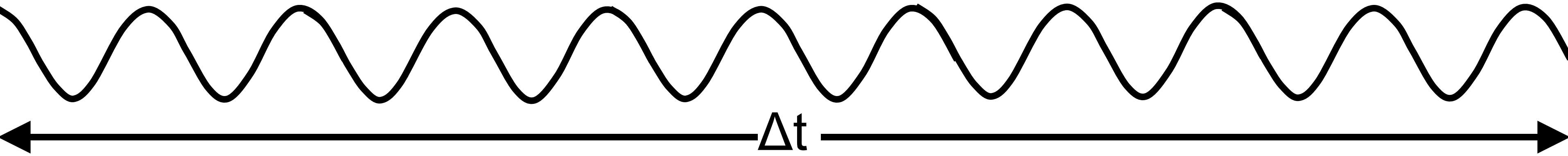


medium Δp – wave packet made of several waves

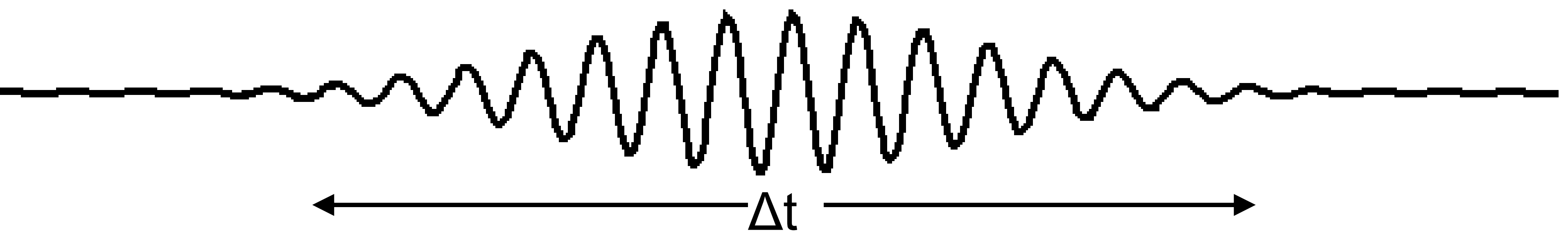


large Δp – wave packet made of lots of waves

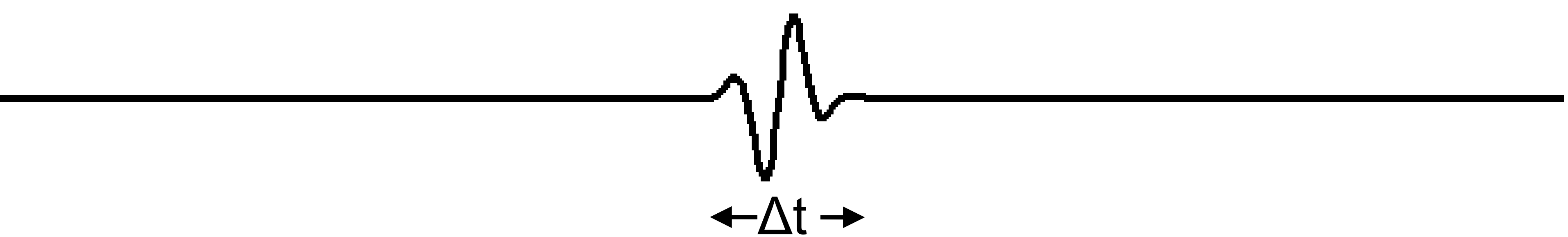
Heisenberg Uncertainty Principle: $\Delta t \, \Delta E \geq \hbar / 2$



small ΔE – only one period



medium ΔE – wave packet made of several waves



large ΔE – wave packet made of lots of waves

Ch. 7: 1D Schrödinger equation

The wave equation for electromagnetic waves is $\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$

Only works for massless particles with *v*=*c*.

For massive particles, need the time dependent Schrödinger equation (TDSE):

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

Kinetic energy + Potential energy = Total energy

For time independent potentials, V(x): $\Psi(x,t) = \psi(x)\phi(t)$

In this case, the time part of the wave function is: $\phi(t) = e^{-iEt/\hbar}$

The spatial part ψ(x) can be found from the time independent Schrödinger equation (TISE):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Free particle

When E>V (everywhere) you have a free particle. We deal with the case of V=0 but it can be applied to other cases as well.

Free particles have oscillating solutions

$$\psi(x) = A\cos(kx) + B\sin(kx) \quad \text{or} \quad \psi(x) = Ce^{ikx} + De^{-ikx}$$

To get the full wave function we multiply by the time dependence:

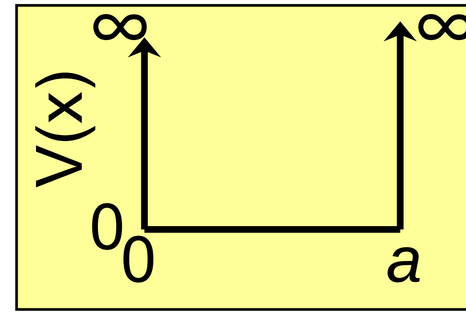
$$\Psi(x,t) = \psi(x)e^{-i\omega t} = Ce^{i(kx-\omega t)} + De^{-i(kx+\omega t)}$$

This is the sum of two waves with momentum ħk and energy ħω.

The *C* wave is moving right and the *D* wave is moving left.

Infinite square well

Like the free particle, E>V but only in region 0<x<a.



Functional form of solution is also oscillating:

$$\psi(x) = A\cos(kx) + B\sin(kx)$$

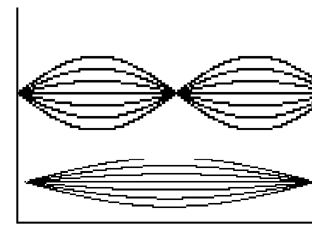
But this time have boundary conditions

Putting in *x* = 0 gives $\psi(0) = A$ so *A* = 0

Putting in *x* = *a* gives $\psi(a) = B\sin(ka)$

To get $\sin(ka) = 0$ requires $ka = n\pi$

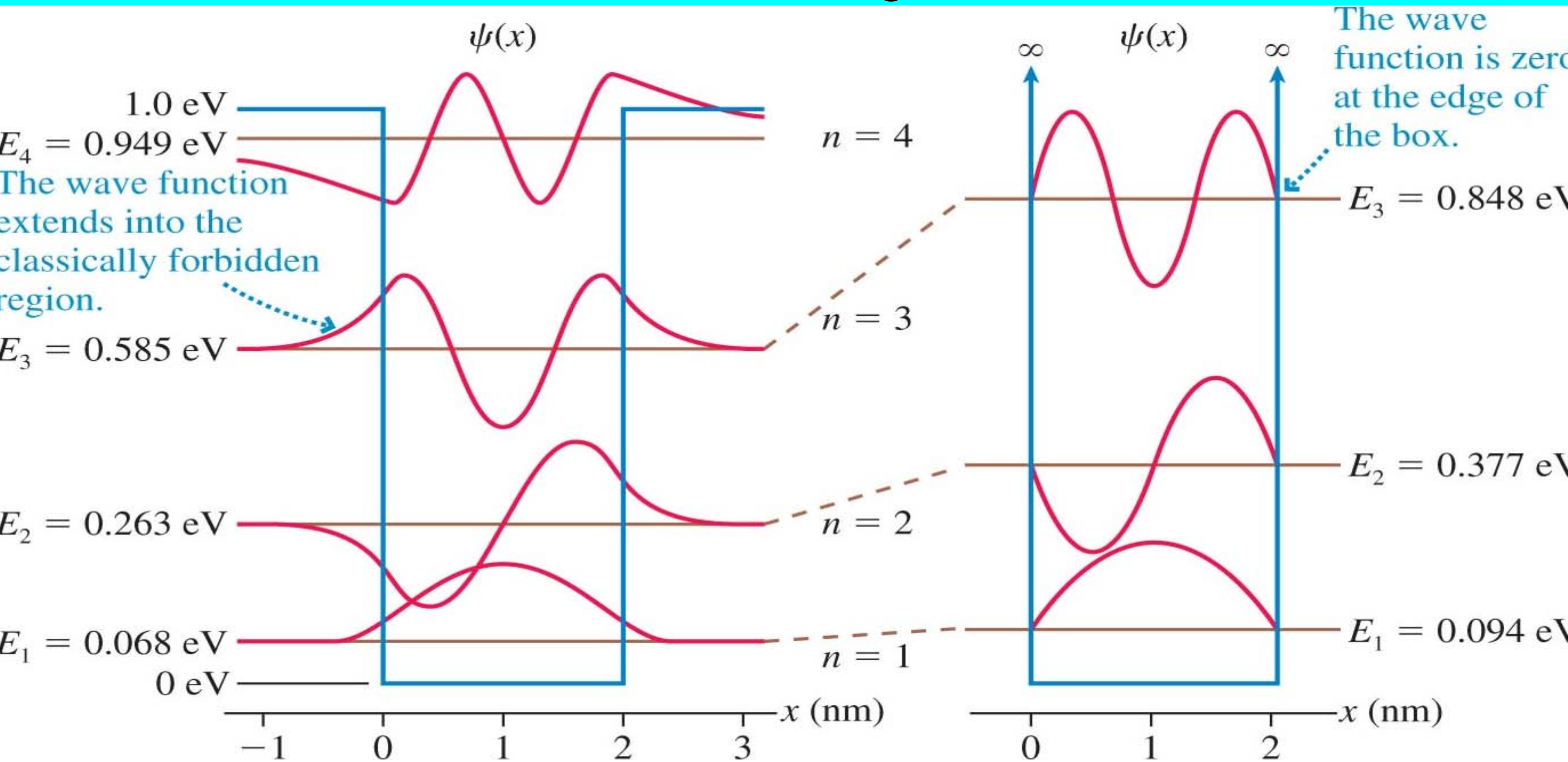
Get the condition $k = \frac{n\pi}{a}$ or $\lambda = \frac{2a}{n}$



Find that energies are quantized: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

After applying normalization condition we get $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

Particle in finite square well can be found in classically forbidden regions



Note that energy level *n* has *n* antinodes

Penetration depth $\lambda = 1/\alpha$ measures how far particle can be found in the classically forbidden region.

$$\alpha = \frac{\sqrt{2m(V-E)}}{\hbar}$$