Getting to the Schrödinger equation

Announcements:

- Learning Assistant program informational session will be held today at 6pm in UMC 235. Fliers are available if you are interested.
- Next homework assignment will be available by tomorrow
- I will be giving a public talk about physics at the Large Hadron Collider (LHC) at 2pm on Saturday in G1B30. If you are interested in particle physics, you may find it interesting.

Erwin Schrödinger
(1887 – 1961)

Where we go from here

We will finish up classical waves

Classical waves obey the wave equation: $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Then we will go back to matter waves which obey a different wave equation called the time dependent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$

On Friday we will derive the time independent Schrödinger equation: $$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi$$

Solving the standard wave equation

The standard *wave equation* is $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

Generic prescription for solving differential equations in physics:

1. Guess the functional form(s) of the solution
2. Plug into differential equation to check for correctness, find any constraints on constants
3. Need as many independent functions as there are derivatives.
4. Apply all boundary conditions (more constraints on constants)

Step 2: Check solution and find constraints

Claim that $y = A\sin(Bx)\cos(Ct)$ is a solution to $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$

Time to check the solution and see what constraints we have

LHS: $\dfrac{\partial^2 y}{\partial x^2} = -AB^2\sin(Bx)\cos(Ct)$

RHS: $\dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2} = -\dfrac{AC^2}{v^2}\sin(Bx)\cos(Ct)$

Setting LHS = RHS: $-AB^2\sin(Bx)\cos(Ct) = -\dfrac{AC^2}{v^2}\sin(Bx)\cos(Ct)$

This works as long as $B^2 = \dfrac{C^2}{v^2}$

We normally write this as $y = A\sin(kx)\cos(\omega t)$

so this constraint just means $k^2 = \dfrac{\omega^2}{v^2}$ or $v = \dfrac{\omega}{k} = \lambda f$

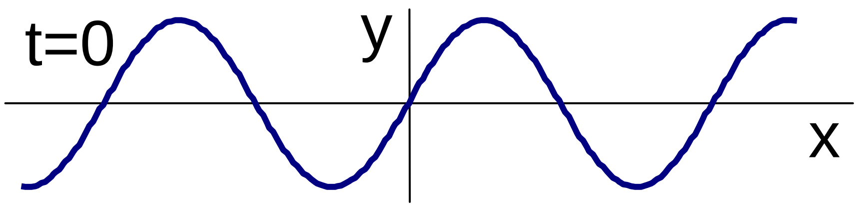
Constructing general solution from independent functions

Since the wave equation has two derivatives, there must be two independent functional forms.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$y = A\cos(kx)\sin(\omega t)$$

$$y = B\sin(kx)\cos(\omega t)$$



The general solution is $y = A\sin(kx)\cos(\omega t) + B\cos(kx)\sin(\omega t)$

Can also be written as $y = C\sin(kx - \omega t) + D\sin(kx + \omega t)$

and we have the constraint that $k^2 = \frac{\omega^2}{v^2}$

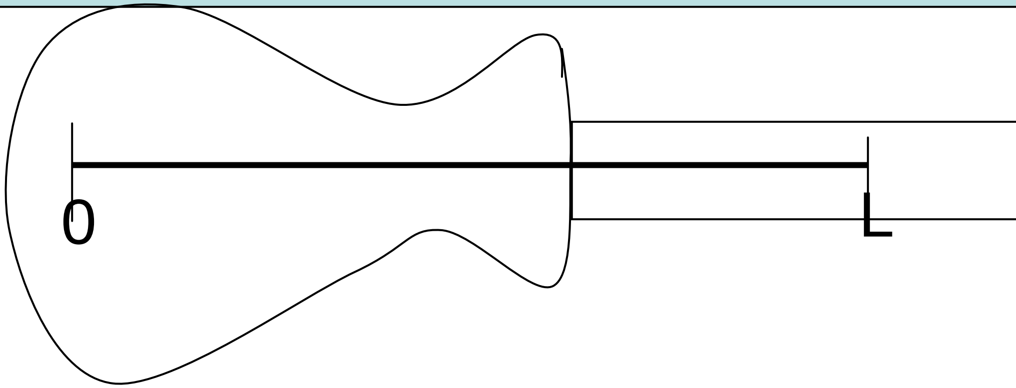
We have finished steps 1, 2, & 3 of solving the differential equation.

Last step is applying boundary conditions. This is the part that actually depends on the details of the problem.

Boundary conditions for guitar string

Wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$



Functional form:

$$y = A\sin(kx)\cos(\omega t) + B\cos(kx)\sin(\omega t)$$

Guitar string is fixed at *x=0* and *x=L*.

Boundary conditions are that *y(x,t)=0* at *x=0* and *x=L.*

Requiring y=0 when x=0 means

$$0 = A\sin(0)\cos(\omega t) + B\cos(0)\sin(\omega t)$$ which is $$0 = 0 + B\sin(\omega t)$$

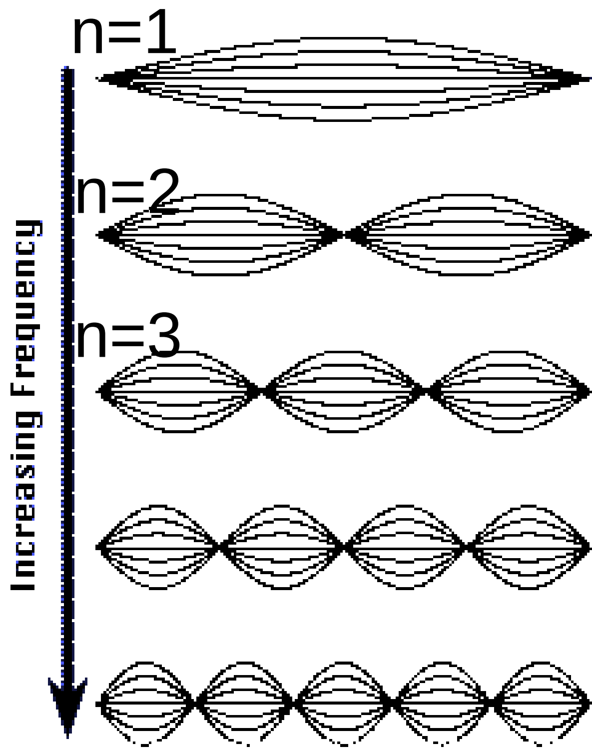
This only works if *B=0*.

So this means $y = A\sin(kx)\cos(\omega t)$

Clicker question 1 — Set frequency to DA

Boundary conditions require *y(x,t)=0* at *x=0* & *x=L.* We found for *y(x,t)=0* at *x=0* we need *B=0* so our solution is $y = A\sin(kx)\cos(\omega t)$.
By evaluating y(x,t) at x=L, derive the possible values for *k*.

A. k can have any value
B. π/(2L), π/L, 3π/(2L), 2π/L …
C. π/L
D. π/L, 2π/L, 3π/L, 4π/L …
E. 2L, 2L/2, 2L/3, 2L/4, ….



To have *y(x,t) = 0* at *x = L* we need $A\sin(kL)\cos(\omega t) = 0$

This means that we need $\sin(kL) = 0$

This is true for *kL = nπ*. That is, $k = \frac{n\pi}{L}$

So the boundary conditions quantize *k*.

This also quantizes ω because of the other constraint we have: $k^2 = \frac{\omega^2}{v^2}$

Summary of our wave equation solution

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

1. Found the general solution to the wave equation

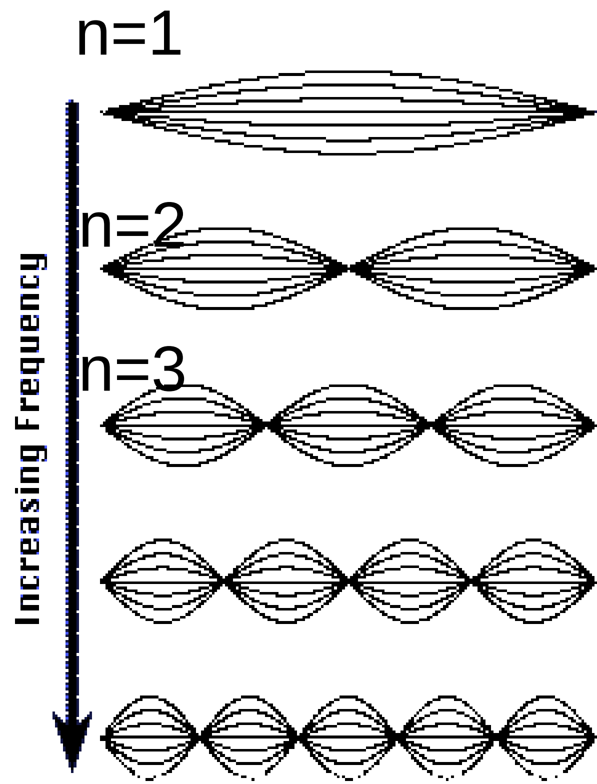
$$y = A\cos(kx)\sin(\omega t) + B\sin(kx)\cos(\omega t)$$

or $$y = C\sin(kx - \omega t) + D\sin(kx + \omega t)$$

2. Put solution into wave equation to get constraint $k^2 = \frac{\omega^2}{v^2}$

3. Have two independent functional forms for two derivatives

4. Applied boundary conditions for guitar string. y(x,t) = 0 at x=0 and x=L. Found that B=0 and k=nπ/L.



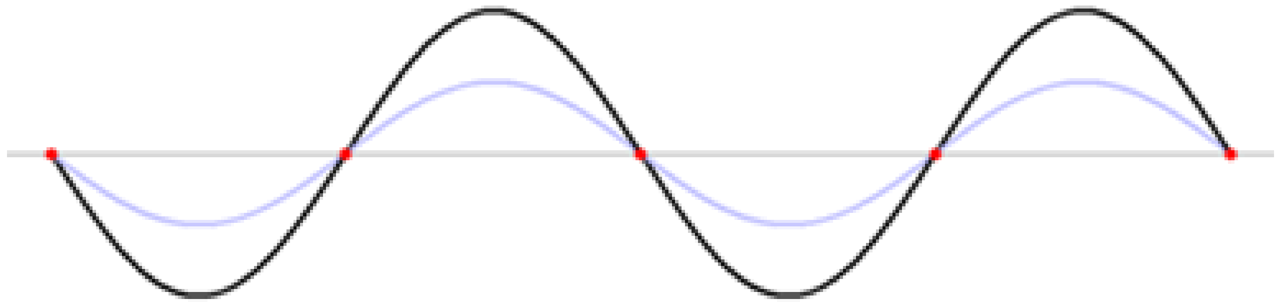
<u>Our final result:</u>

$$y = A\cos(kx)\sin(\omega t)$$

with $k^2 = \frac{\omega^2}{v^2}$ and $k = \frac{n\pi}{L}$

Standing waves

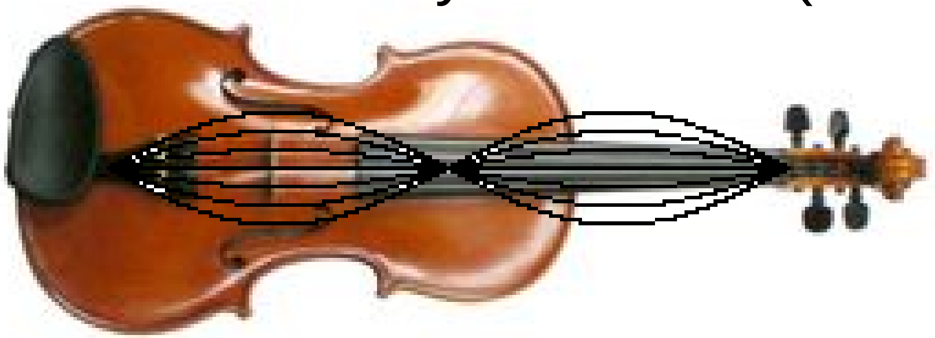
Standing wave



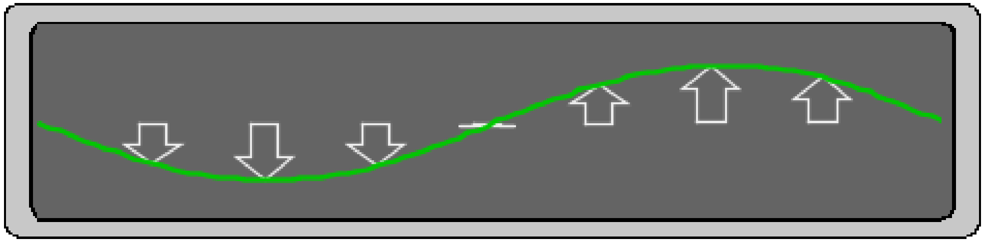
Standing wave constructed from two traveling waves moving in opposite directions

Examples of standing waves

For standing waves on violin string, only certain values of k and ω are allowed due to boundary conditions (location of nodes).



Same is true for electromagnetic waves in a microwave oven:



We also get only certain waves for electrons in an atom.

We will find that this is due to boundary conditions applied to solutions of Schrödinger equation.

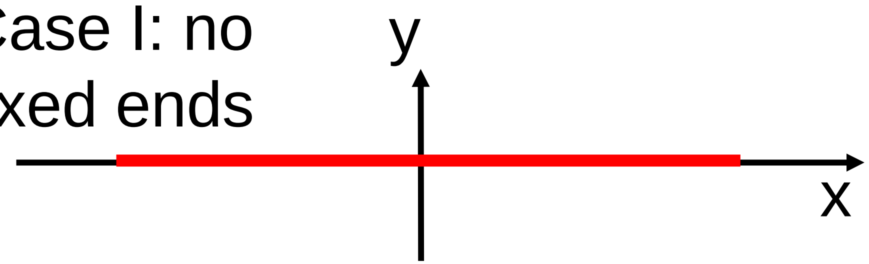
Clicker question 2

Set frequency to DA

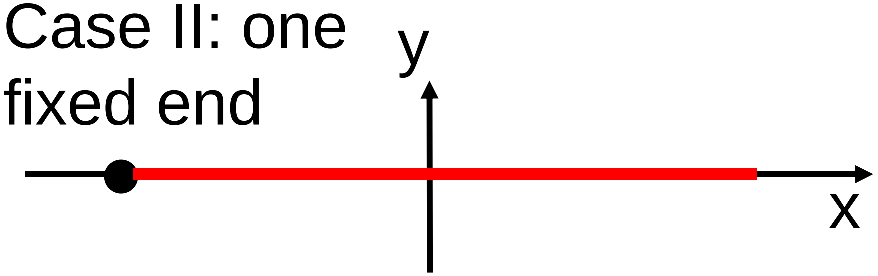
For which of the three cases do you expect to have only certain frequencies and wavelengths allowed? That is, in which cases will the allowed frequencies be quantized?

A. Case I
B. Case II
C. Case III
D. More than one case

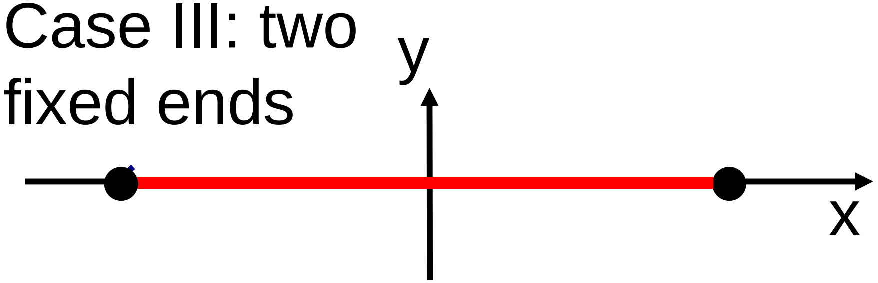
Case I: no fixed ends



Case II: one fixed end



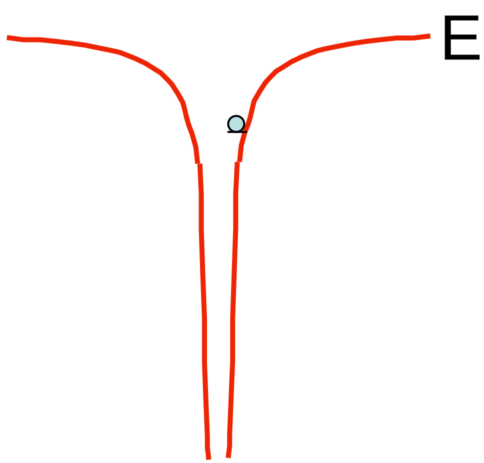
Case III: two fixed ends



After applying the 1st boundary condition we found B=0 but we did not have quantization. After the 2nd boundary condition we found $k=n\pi/L$. This is the quantization.

Boundary conditions cause the quantization

Electron bound in atom



Boundary Conditions
⇒ standing waves

Only certain energies allowed
Quantized energies

Free electron

No Boundary Conditions
⇒ traveling waves

Any energy allowed

Getting to Schrödinger's wave equation

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Works for light (photons), why doesn't it work for electrons?

Clicker question 3

Set frequency to DA

The equation $E = hc/\lambda$ is…

A. true for photons and electrons
B. true for photons but not electrons
C. true for electrons but not photons
D. not true for either electrons or photons

$$E = hf = \hbar\omega$$

works for photons and electrons

$$p = \frac{h}{\lambda} = \hbar k$$

works for photons and electrons

$$E = pc = \frac{hc}{\lambda}$$

only works for massless particles (photons)