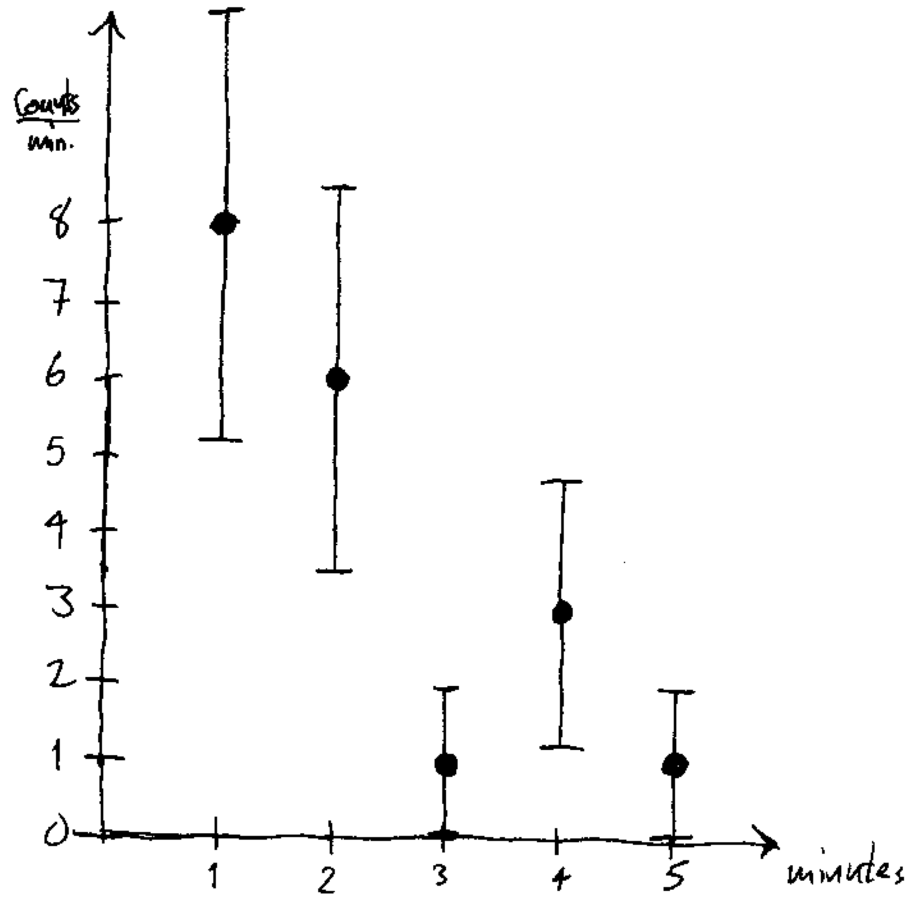
P2150 PROBLEM SET SPRING 2008

1) Histogram with errors: Error is $\sqrt{N}$ (Poisson):



(2) To fit, note that error on each bin is $\sqrt{\text{counts}}$: (Poisson process).

Bin center time (min)	1	2	3	4	5	6
Counts N_i	8	6	1	3	1	0
error $\sqrt{N_i}$	$\sqrt{8}$	$\sqrt{6}$	1	$\sqrt{3}$	1	0

Take $y_i = \ln N_i$ so exponential $\longrightarrow$ line.

$y = \ln N$	2.08	1.79	0	1.10	0	$-\infty$
Error $\sigma_y = \frac{1}{\sqrt{n}}$	0.35	0.41	1.0	0.58	1.0	∞

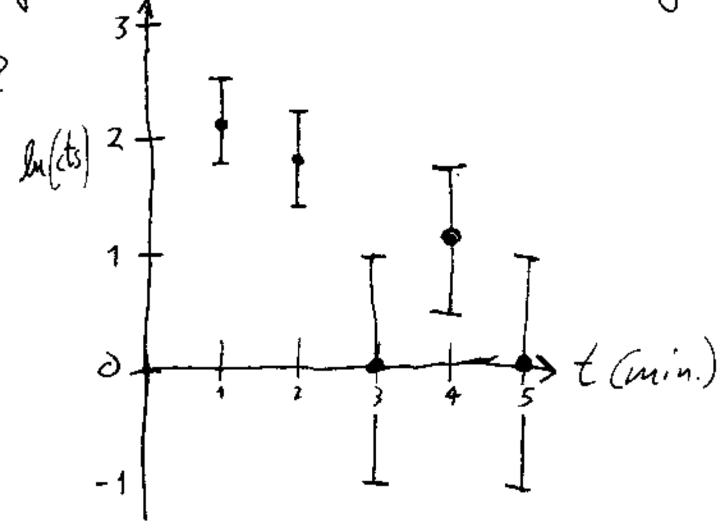
See lecture 6

What can we do here?

Ignore bin since σ of ∞ has no weight in fit.

Plot: is it line-like?

Looks <u>reasonable</u>.



Now, use <u>weighted</u> least-squares line fit: $y = \ln(N_0) - \frac{t}{\tau}$ where τ is mean life. Formula is on p.201 of Taylor, with $A \leftrightarrow \ln(N_0)$ and $B \leftrightarrow \frac{-1}{\tau}$, $x_i \leftrightarrow t_i$, $w_i \leftrightarrow \frac{1}{\sigma_i^2} = \sqrt{N_i}^2 = N_i$, $y_i \leftrightarrow \ln N_i$.

$$-\frac{1}{\tau} = B = \frac{\sum_i w_i \sum_i w_i t_i \ln N_i \ - \ \sum_i w_i t_i \sum_i w_i \ln N_i}{\Delta}$$

where $\Delta = \sum_i w_i \sum_i w_i t_i^2 - \left(\sum_i w_i t_i\right)^2$

$\sum_i w_i = 8 + 6 + 1 + 3 + 1 = 19$

$\sum_i w_i t_i = 8 \cdot 1 + 6 \cdot 2 + 1 \cdot 3 + 3 \cdot 4 + 1 \cdot 5 = 40$

$\sum_i w_i t_i \ln N_i = 8 \cdot 2.08 + 12 \cdot 1.79 + 3 \cdot 0 + 12 \cdot 1.1 + 5 \cdot 0 = 51.3$

$\sum_i w_i \ln N_i = 8 \cdot 2.08 + 6 \cdot 1.79 + 1 \cdot 0 + 3 \cdot 1.1 + 1 \cdot 0 = 30.7$

$\sum_i w_i t_i^2 = 8 \cdot 1^2 + 6 \cdot 2^2 + 1 \cdot 3^2 + 3 \cdot 4^2 + 1 \cdot 5^2 = 114$

$$B = \frac{19 \cdot 51.3 - 40 \cdot 30.7}{19 \cdot 114 - 40^2} = \frac{-253.3}{556} = 0.456$$

$$\Rightarrow \tau = \frac{-1}{B} = \boxed{2.20 \text{ s}}$$

Error on B from p. 204: $\sigma_B = \sqrt{\frac{\sum_i w_i}{\Delta}} = \sqrt{\frac{19}{556}} = 0.185$

Now, $\frac{\delta\tau}{\tau} = \frac{\delta B}{B} = \frac{0.185}{0.456} = 0.41$

so $\delta\tau = (0.41 \cdot 2.20)\text{ s} = 0.89$

$$\Rightarrow \boxed{\tau = (2.2 \pm 0.9)\text{ s}}$$

To find χ^2, use basic definition (Taylor Eq 12.11):

$$\chi^2 = \sum_i \left(\frac{\text{observed} - \text{expected}}{\sigma_i}\right)^2 .$$

Can use log or original counts; either way, need "expected" — i.e. fit function.

$$N = N_o e^{-t/\tau} \quad \text{or} \quad \ln N = \ln N_o - \frac{t}{\tau} .$$

Use formula for "A" on p. 201 for $\ln N_o$:

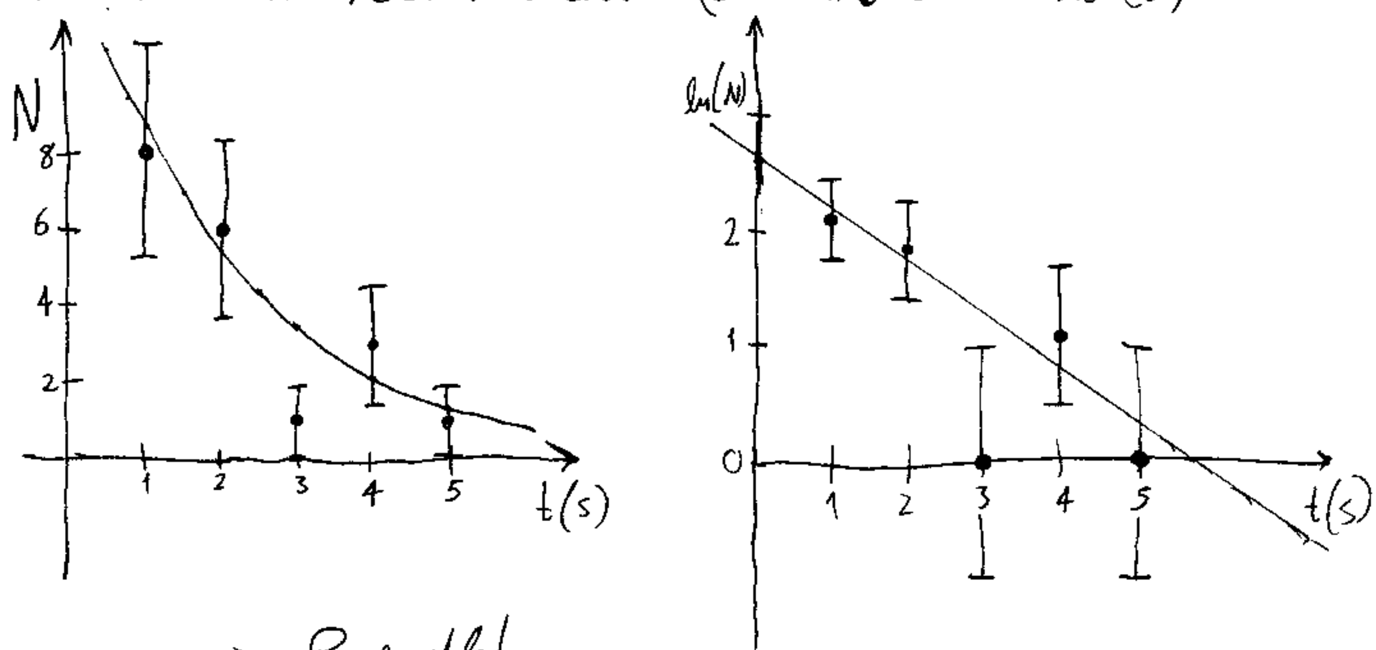
$$\ln N_o = A = \frac{\sum_i w_i t_i^2 \sum_i w_i \ln N_i - \sum_i w_i t_i \sum_i w_i t_i \ln N_i}{\Delta}$$

$$= \frac{114 \cdot 30.7 - 40 \cdot 51.3}{556} = 2.60$$

so $N_o = e^A = 13.5$

$$\Longrightarrow N(t) = 13.5 \cdot e^{-t/2.2s} \quad \underline{\text{or}} \quad \ln(N) = 2.6 - \frac{t}{2.2s}$$

Plot to check reasonableness: (Curves are the fit result.)



→ Reasonable!

Now, find χ^2:

Bin	$N_{expected}$	N_{obs}	$\sigma = \sqrt{N_{exp}}$	χ^2_i
1	8.73	8	2.95	0.06
2	5.54	6	2.35	0.04
3	3.51	1	1.87	1.80
4	2.23	3	1.49	0.27
5	1.42	1	1.19	0.12

Total $\chi^2 = 2.29$ for 3 d.o.f. $(=5-2)$

so $\tilde{\chi}_0^2 = \frac{\chi^2}{3} = 0.76$

By Table D, probability for 0.8 is 49% !

Very good fit.

3a) The results are consistent if $(\varepsilon_1' - \varepsilon_2')$ is consistent with zero:

$$\varepsilon_1' - \varepsilon_2' = [0.115 \pm 0.033] - [0.037 \pm 0.030]$$

$$= 0.078 \pm \sqrt{0.037^2 + 0.030^2}$$

$$= 0.078 \pm 0.048$$

... or a $1.63\,\sigma$ discrepancy.

Agreement this poor is expected $(100 - 89.7) = 10.3\%$ of the time (from Taylor, Appendix A). So the results are not incompatible, but not comfortably consistent either.

3b) Weighted average: weight by $\frac{1}{\sigma^2}$:

$$\bar{\varepsilon}' = \frac{\varepsilon_1'/\sigma_1^2 + \varepsilon_2'/\sigma_2^2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{105.6 + 41.1}{1841} = 0.080$$

Error on $\bar{\varepsilon}'$ is $\frac{1}{\sqrt{\Sigma w}} = \frac{1}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} = \frac{1}{\sqrt{524}} = 0.023$

(Taylor 7.12)

so $\boxed{\bar{\varepsilon}' = 0.080 \pm 0.023}$ Which is 3.5σ from zero!

3c) The result 0.115 ± 0.033 is $3.5\,\sigma$ from zero, which is nonzero at 99.95% confidence. So, (if it weren't for the presence of the other measurement) they would be justified in claiming an observation. This exact question was one of the most important quandaries in particle physics in the 1990s. A clear observation of $\varepsilon' \neq 0$ was the goal of a set of experiments searching for a new form of "CP violation" (matter-antimatter asymmetry). Given that one measurement was comfortably consistent with zero, many physicists were uncomfortable accepting the discovery even though the results were consistent and the weighted average was far from zero!

More precise measurements in the early 2000's confirmed that $\varepsilon' \neq 0$, and the currently accepted value is 0.083 ± 0.013, confirming that both measurements had been "right" and the true value was very close to the weighted average all along. (Note — numbers have been scaled by factor of 50 for this problem set.)

4 Taylor 9.4

Data:

i	1	2	3	4	5	
t	14	12	13	15	16	mean $\bar{t} = 14$
T	20	18	18	22	22	mean $\bar{T} = 20$
$T-t$	6	6	5	7	6	mean $\overline{(T-t)} = 6$

(a)

$$\sigma_t^2 = \frac{1}{N}\sum_i (t_i - \bar{t})^2 = \frac{1}{5}\left(0^2+2^2+1^2+1^2+2^2\right) = \frac{10}{5} = 2$$

$$\sigma_T^2 = \frac{1}{N}\sum_i (T_i - \bar{T})^2 = \frac{1}{5}\left(0^2+2^2+2^2+2^2+2^2\right) = \frac{16}{5} = 3.2$$

$$\sigma_{Tt} = \frac{1}{N}\sum_i (t_i-\bar{t})(T_i-\bar{T}) = \frac{1}{5}\left[0\cdot 0 + (-2)(-2) + (-1)(-2) + (1)(2) + (2)(2)\right]$$

$$= \frac{1}{5}(12) = 2.4$$

(b) let $q = T - t$. $\frac{\partial q}{\partial T} = 1$, $\frac{\partial q}{\partial t} = -1$.

$$\sigma_{T-t}^2 = \left(\frac{\partial q}{\partial t}\right)^2 \sigma_t^2 + \left(\frac{\partial q}{\partial T}\right)^2 \sigma_T^2 + 2\frac{\partial q}{\partial t}\frac{\partial q}{\partial T}\sigma_{Tt}$$

$$= \sigma_t^2 + \sigma_T^2 - 2\sigma_{Tt} = 2 + 3.2 - 2\cdot 2.4 = \boxed{0.4}$$

(c) $\sigma_{T-t} = \sigma_t^2 + \sigma_T^2 = \boxed{5.2}$ much bigger, obviously.

(d) $\sigma_{(T-t)}^2 = \frac{1}{N}\sum_i \left[T_i - t_i - \left(\overline{T-t}\right)^2\right] = \frac{1}{5}\left(0+0+1^2+1^2+0\right) = \frac{2}{5} = \boxed{0.4}$

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Taylor 11.20

With rock:

$$\text{Mean } \mu_{rock} = \frac{225 \pm \sqrt{225}}{10 \text{ min}} = (22.5 \pm 1.5)\,\text{min}^{-1}.$$

Without rock:

$$\text{Mean } \mu_{bkg} = \frac{90 \pm \sqrt{90}}{6 \text{ min}} = (15 \pm 1.6)\,\text{min}^{-1}$$

$$\text{Signal} = \mu_{rock} - \mu_{bkg} = \left[(22.5 \pm 1.5) - (15 \pm 1.6)\right]\text{min}^{-1}$$

$$= \left(7.5 \pm \sqrt{1.5^2 + 1.6^2}\right)\text{min}^{-1}$$

$$= (7.5 \pm 2.2)\,\text{min}^{-1}$$ which is $3.4\,\sigma$ from zero

This is nonzero at >99.9% confidence level.

The rock is almost certainly radioactive.