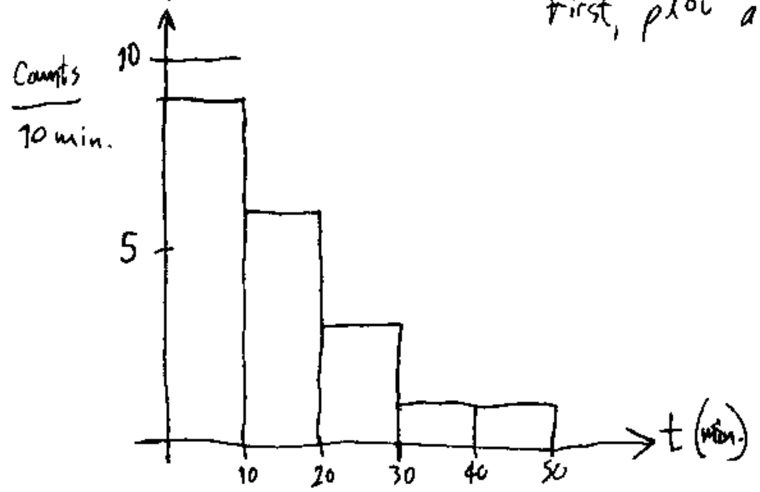
P2150 | PROBLEM SET | SOLUTIONS

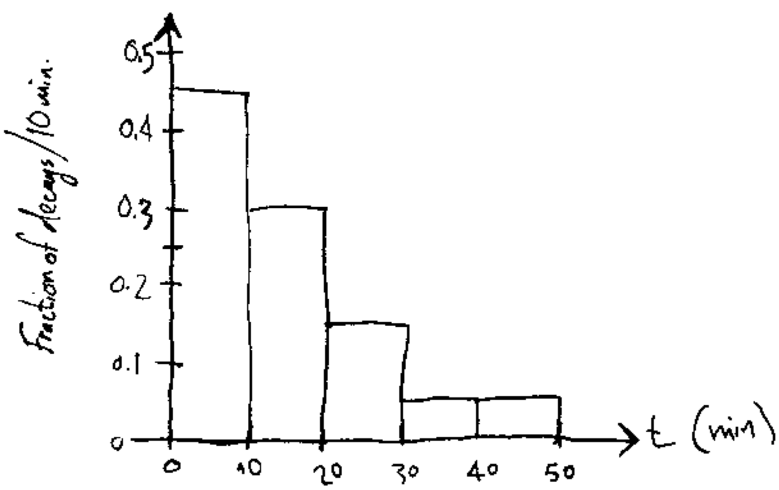
(1) Taylor 5.2

First, plot as counts:



Total counts = 20

To normalize, divide by 20:



(2) To fit, note that error on the counts N each bin is $\sqrt{N}$ (it's a Poisson process)

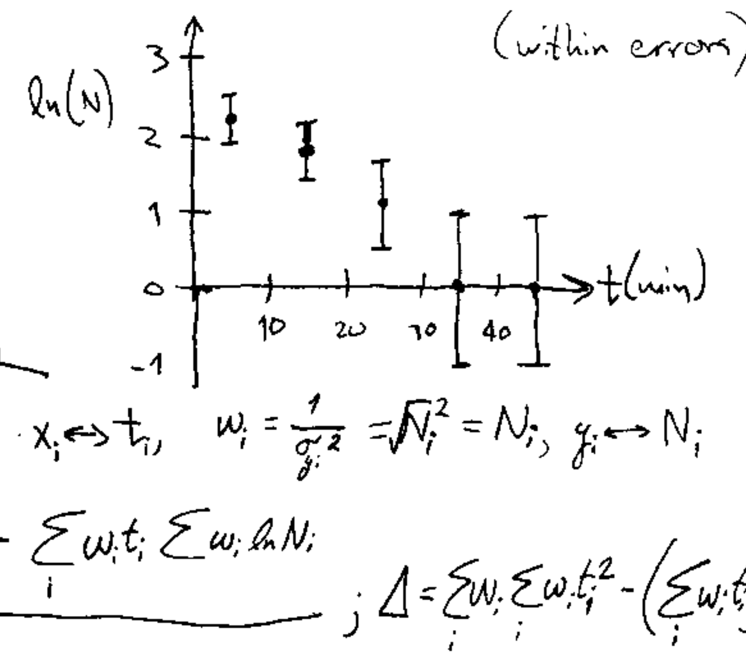
Bins:	center time (min) :	5	15	25	35	45
	Counts N :	9	6	3	1	1
	Error $\sigma_i = \sqrt{N}$:	3	$\sqrt{6}$	$\sqrt{3}$	1	1

Now, take $y = \ln(N)$ so we get an expected straight line:

$y = \ln N$:	2.20	1.79	1.10	0.0	0.0
Error $\sigma_y = \frac{\sigma_N}{N} = \frac{1}{\sqrt{N}}$:	0.33	0.41	0.58	1.0	1.0

See Lecture 5.

Plot this: Is it line-like? Yes! (within errors)



Use weighted least sq. fit: $y = \ln(N_0) - \frac{t}{\tau}$

where τ is mean life.

Use formulae on p. 201,

where $A \leftrightarrow \ln(N_0)$, $B \leftrightarrow -\frac{1}{\tau}$, $x_i \leftrightarrow t_i$, $w_i = \frac{1}{\sigma_{y_i}^2} = \sqrt{N_i}^2 = N_i$, $y_i \leftrightarrow N_i$

$$-\frac{1}{\tau} = B = \frac{\sum_i w_i \sum_i w_i t_i \ln N_i - \sum_i w_i t_i \sum w_i \ln N_i}{\Delta} \; ; \; \Delta = \sum_i w_i \sum_i w_i t_i^2 - \left(\sum_i w_i t_i\right)^2$$

$$\sum_i w_i = 9+6+3+1+1 = 20 \; ; \; \sum_i w_i t_i = 9\cdot5 + 6\cdot15 + 3\cdot25 + 1\cdot35 + 1\cdot45 = 290$$

$$\sum_i w_i t_i \ln N_i = 9\cdot5\cdot2.2 + 6\cdot15\cdot1.8 + 3\cdot25\cdot1.1 + 1\cdot35\cdot0 + 1\cdot45\cdot0 = 342.6$$

$$\sum w_i \ln N_i = 9\cdot2.2 + 6\cdot1.8 + 3\cdot1.1 + 1\cdot0 + 1\cdot0 = 33.9 \qquad \sum w_i t_i^2 = 9\cdot5^2 + 6\cdot15^2 + 3\cdot25^2 + 1\cdot35^2 + 1\cdot45^2 = 6700$$

$$B = -\frac{1}{\tau} = \frac{20 \cdot 269.25 - 290 \cdot 33.9}{20 \cdot 6700 - (290)^2} = -0.0597$$

$$\Rightarrow \boxed{\tau = 16.8 \text{ min.}}$$

Error on B from p. 204 (prob. 8.19): $\sigma_B = \sqrt{\frac{\Sigma w}{\Delta}}$

$$\sigma_B = 0.020$$

So $-\frac{1}{\tau} = -0.60 \pm 0.020$

Error propagation: $\frac{\delta(\tau)}{\tau} = \left| \frac{\delta\left(-\frac{1}{\tau}\right)}{\left(-\frac{1}{\tau}\right)} \right| = 0.33$.

$$\rightarrow \delta\tau = 0.33 \cdot \tau = \boxed{5.6 \text{ min.}}$$

So mean lifetime of isotope is (16.8 ± 5.6) min

For χ^2: Use Taylor Eq. 12.11: $\chi^2 = \sum_i \left(\frac{\text{observed} - \text{expected}}{\sigma}\right)^2$

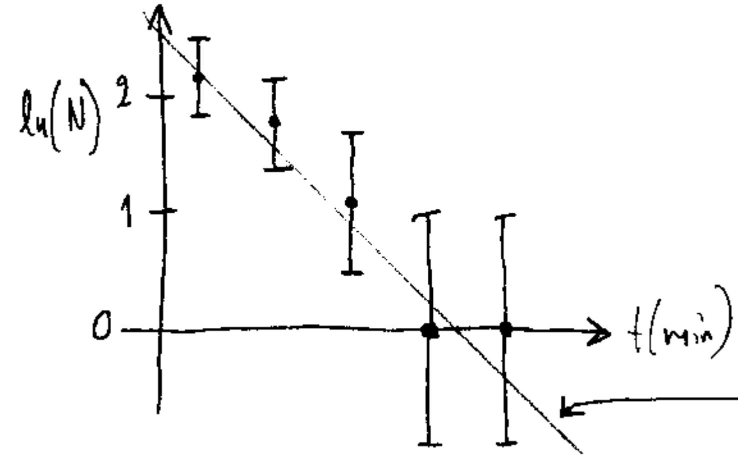
Can do this for the log or the original data points. Need "expected" function: $N = N_0 e^{-t/\tau}$ or $\ln N = \ln N_0 - \frac{t}{\tau}$. Either way, need N_0. Use "A" formula on p. 201, $A = \ln N_0$:

$$A = \frac{\Sigma w_i t_i^2 \Sigma w_i \ln N_i - \Sigma w_i t_i \Sigma w_i t_i \ln N_i}{\Delta} = \frac{6700 \cdot 33.9 - 290 \cdot 342.6}{20 \cdot 6700 - (290)^2}$$

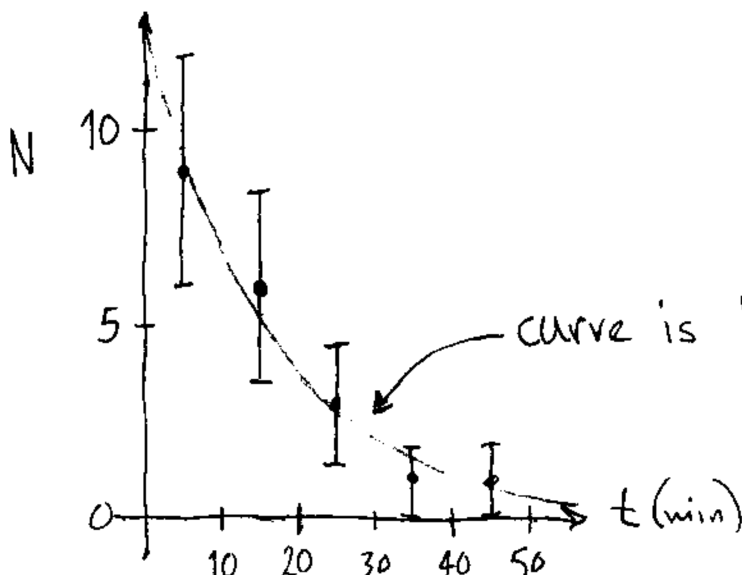
$$= 2.56 \quad \Rightarrow N_0 = e^{2.56} = 12.9$$

So $N = 12.9 \cdot \exp(-t/16.8\text{ min.})$.

To check fit quality roughly, plot it:



line is $\ln(N) = 2.56 - \frac{t}{16.8\,\text{min}}$



curve is $N = 12.8\,e^{-t/16.8\,\text{min}}$

These fits look great!

Now, calculate χ^2: $\chi^2 = \sum\left[(N_{expect} - N_{observed})/\sigma\right]^2$. Can use N or $\ln N$:

time (min) $\rightarrow$	5	15	25	35	45
$N_{exp} = 12.8\,e^{-t/16.8\,\text{min}} \rightarrow$	9.6	5.2	2.9	1.6	0.8
$N_{obs} \rightarrow$	9	6	3	1	1
$\sigma_{N_{exp}} = \sqrt{N_{exp}} \rightarrow$	3.1	2.3	1.7	1.3	0.9
$\chi_i^2 \rightarrow$	0.04	0.12	0.003	0.21	0.05
$\ln(N_{exp}) \rightarrow$	2.27	1.65	1.06	0.47	-0.22
$\ln(N_{obs}) \rightarrow$	2.20	1.79	1.10	0	0
$\sigma_{\ln N} = \frac{1}{\sqrt{N_{exp}}} \rightarrow$	0.33	0.44	0.59	0.79	1.12
$\chi_i^2 \rightarrow$	0.04	0.10	0.005	0.35	0.04

(Note: 5 datapoints − 2 fit parameters = 3 degrees of freedom.)

$\rightarrow \chi^2 = \sum_i \chi_i^2 = 0.42$

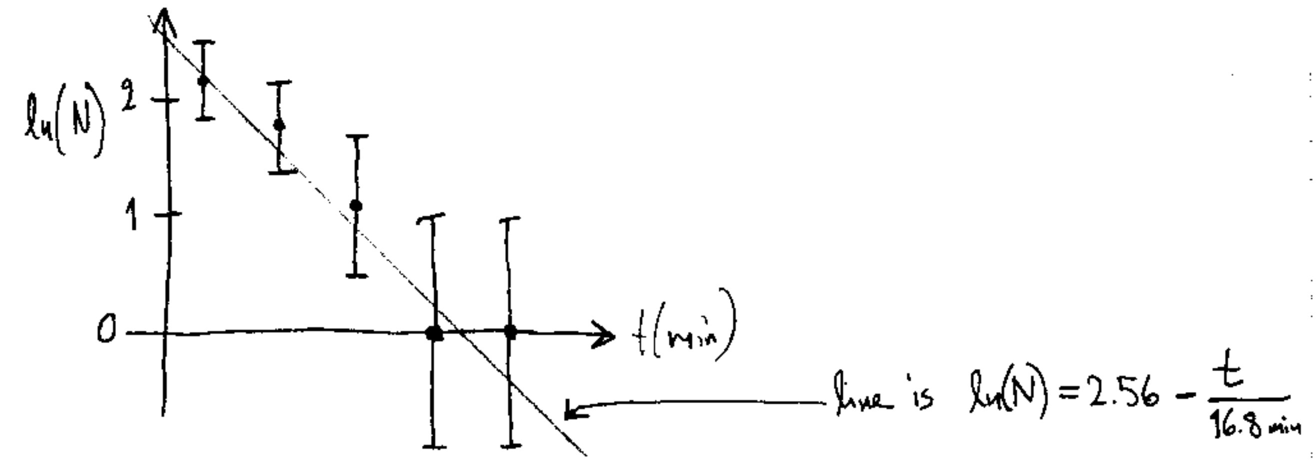
These may differ slightly due to non-Gaussianness of errors.

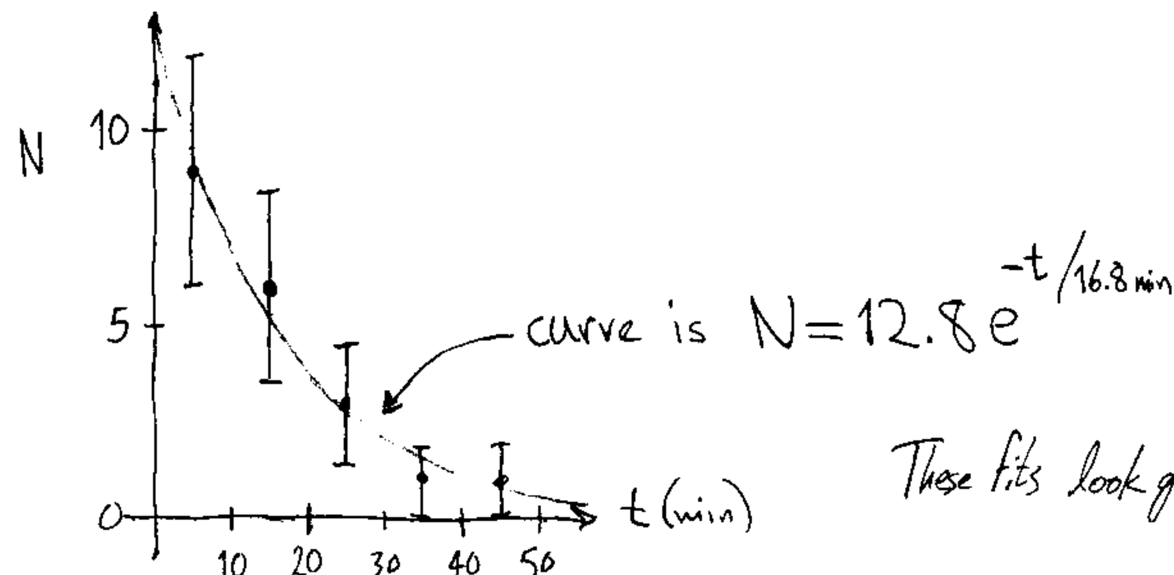
$\rightarrow \chi^2 = \sum_i \chi_i^2 = 0.53$

$\boxed{\frac{\chi^2}{dof} = 0.14 \text{ (one way) or } 0.18} \Rightarrow$ conf. level $> 90\%$ (Taylor Appendix D).

Fit may be too good! Did Taylor manipulate/fudge the data?

To check fit quality roughly, plot it:





These fits look great!

Now, calculate χ^2: $\chi^2 = \sum\left[(N_{expect} - N_{observed})/\sigma\right]^2$. Can use N or $\ln N$:

(Note: 5 datapoints − 2 fit parameters = 3 degrees of freedom.)

time (min) →	5	15	25	35	45	
$N_{exp} = 12.8\, e^{-t/16.8\,\text{min}}$ →	9.6	5.2	2.9	1.6	0.8	
N_{obs} →	9	6	3	1	1	
$\sigma_{N_{exp}} = \sqrt{N_{exp}}$ →	3.1	2.3	1.7	1.3	0.9	
χ_i^2 →	0.04	0.12	0.003	0.21	0.05	→ $\chi^2 = \sum_i \chi_i^2 = 0.42$
$\ln(N_{exp})$ →	2.27	1.65	1.06	0.47	-0.22	
$\ln(N_{obs})$ →	2.20	1.79	1.10	0	0	
$\sigma_{\ln N} = \frac{1}{\sqrt{N_{exp}}}$ →	0.33	0.44	0.59	0.79	1.12	
χ_i^2 →	0.04	0.10	0.005	0.35	0.04	→ $\chi^2 = \sum_i \chi_i^2 = 0.53$

These may differ slightly due to non-Gaussianness of errors.

$\boxed{\frac{\chi^2}{dof} = 0.14 \text{ (one way) or } 0.18} \Rightarrow$ conf. level > 90% (Taylor Appendix D).

Fit may be too good! Did Taylor manipulate/fudge the data?

3a The results are consistent if $(\varepsilon_1' - \varepsilon_2')$ is consistent with zero:

$$\varepsilon_1' - \varepsilon_2' = [2.30 \pm 0.65] - [0.74 \pm 0.59]$$

$$= 1.56 \pm \sqrt{0.65^2 + 0.59^2}$$

$$= 1.56 \pm 0.88$$

... or a $1.77\,\sigma$ discrepancy.

Agreement this poor is expected $(100-92.3) = 7.7\%$ of the time (from Taylor, Appendix A). So the results are not incompatible, but not comfortably consistent either.

3b Weighted average: weight by $\frac{1}{\sigma^2}$:

$$\bar{\varepsilon}' = \frac{\varepsilon_1'/\sigma_1^2 + \varepsilon_2'/\sigma_2^2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{5.44 + 2.13}{5.24} = 1.44$$

Error on $\bar{\varepsilon}'$ is $\frac{1}{\sqrt{\Sigma w}} = \frac{1}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} = \frac{1}{\sqrt{5.24}} = 0.44$
(Taylor 7.12)

so $\boxed{\bar{\varepsilon}' = 1.44 \pm 0.44}$ Which is 3.3σ from zero!

3c The result 2.30 ± 0.65 is $3.5\,\sigma$ from zero, which is nonzero at 99.95% confidence. So, (if it weren't for the presence of the other measurement) they would be justified in claiming an observation. This exact question was one of the most important quandaries in particle physics in the 1990s. A clear observation of $\varepsilon' \neq 0$ was the goal of a set of experiments searching for a new form of "CP violation" (matter-antimatter asymmetry). Given that one measurement was comfortably consistent with zero, many physicists were uncomfortable accepting the discovery even though the results were consistent and the weighted average was far from zero!

More precise measurements in the early 2000's confirmed that $\varepsilon' \neq 0$, and the currently accepted value is 16.6 ± 2.6, confirming that both measurements had been "right" and the true value was very close to the weighted average all along.

④ Data: Taylor 9.4

i	1	2	3	4	5	
t	14	12	13	15	16	mean $\bar{t} = 14$
T	20	18	18	22	22	mean $\bar{T} = 20$
$T-t$	6	6	5	7	6	mean $(\overline{T-t}) = 6$

ⓐ $\sigma_t^2 = \frac{1}{N}\sum_i (t_i - \bar{t})^2 = \frac{1}{5}(0^2+2^2+1^2+1^2+2^2) = \frac{10}{5} = 2$

$\sigma_T^2 = \frac{1}{N}\sum_i (T_i - \bar{T})^2 = \frac{1}{5}(0^2+2^2+2^2+2^2+2^2) = \frac{16}{5} = 3.2$

$\sigma_{Tt} = \frac{1}{N}\sum_i (t_i - \bar{t})(T_i - \bar{T}) = \frac{1}{5}\left[0\cdot 0 + (-2)(-2) + (-1)(-2) + (1)(2) + (2)(2)\right]$

$= \frac{1}{5}(12) = 2.4$

ⓑ let $q = T - t$. $\frac{\partial q}{\partial T} = 1$, $\frac{\partial q}{\partial t} = -1$.

$\sigma_{T-t}^2 = \left(\frac{\partial q}{\partial t}\right)^2 \sigma_t^2 + \left(\frac{\partial q}{\partial T}\right)^2 \sigma_T^2 + 2\frac{\partial q}{\partial t}\frac{\partial q}{\partial T}\sigma_{Tt}$

$= \sigma_t^2 + \sigma_T^2 - 2\sigma_{Tt} = 2 + 3.2 - 2\cdot 2.4 = \boxed{0.4}$

ⓒ $\sigma_{T-t} = \sigma_t^2 + \sigma_T^2 = \boxed{5.2}$ much bigger, obviously.

ⓓ $\sigma_{(T-t)}^2 = \frac{1}{N}\sum_i \left[T_i - t_i - (\overline{T-t})^2\right] = \frac{1}{5}(0+0+1^2+1^2+0) = \frac{2}{5} = \boxed{0.4}$

5

Taylor 11.20

With rock:

$$\text{Mean } \mu_{rock} = \frac{225 \pm \sqrt{225}}{10 \text{ min}} = (22.5 \pm 1.5)\,\text{min}^{-1}.$$

Without rock:

$$\text{Mean } \mu_{bkg} = \frac{90 \pm \sqrt{90}}{6 \text{ min}} = (15 \pm 1.6)\,\text{min}^{-1}$$

$$\text{Signal} = \mu_{rock} - \mu_{bkg} = \left[(22.5 \pm 1.5) - (15 \pm 1.6)\right]\text{min}^{-1}$$

$$= \left(7.5 \pm \sqrt{1.5^2 + 1.6^2}\right)\,\text{min}^{-1}$$

$$= (7.5 \pm 2.2)\,\text{min}^{-1}$$ which is $3.4\,\sigma$ from zero

This is nonzero at >99.9% confidence level.

The rock is almost certainly radioactive.