In Part II (Chapters 3 to 11) we describe the second of the two theories that transformed twentieth-century physics — quantum theory. Just as relativity can be roughly characterized as the study of phenomena involving high speeds \((v \sim c)\), so quantum theory can be described as the study of phenomena involving small objects — generally of atomic size or smaller.

We begin our account of quantum theory in Chapter 3, with a descriptive survey of the main properties of atoms (their size, mass, constituents, and so on). In Chapters 4 and 5 we describe some puzzling properties of microscopic systems that began to emerge in the late nineteenth century — some properties of light in Chapter 4, and of atoms in Chapter 5. All of these puzzles pointed up the need for a new mechanics — quantum mechanics, as we now say — to replace classical mechanics in the treatment of microscopic systems. In Chapters 6 to 9 we describe the basic ideas of the new quantum mechanics, which began to develop around 1900 and was nearly complete by 1930. In particular, we introduce the Schrödinger equation, which is the basic equation of quantum mechanics, just as Newton’s second law is the basic equation of Newtonian mechanics.

Armed with the Schrödinger equation, we can calculate most of the important properties of the simplest of all atoms, the hydrogen atom (with its one electron), as we describe in Chapter 8. In Chapter 9 we introduce one more important idea of the new quantum mechanics — the electron’s spin angular momentum. Then in Chapters 10 and 11 we are ready to apply all these ideas to the general multielectron atom. We will see that quantum theory gives a remarkably complete account of all the 100 or so known different atoms, and that, in principle at least, it explains all of chemistry.

Although quantum mechanics was first developed to explain the properties of atoms, it has also been applied with extraordinary success to other systems, some larger than atoms, such as molecules and solids, and some smaller, such as subatomic particles. We will describe these other applications in Parts III and IV.
Chapter 3

Atoms

3.1 Introduction

Quantum physics is primarily the physics of microscopic systems. Historically, the most important such system was the atom, and even today the greatest triumph of quantum theory is the complete and accurate account that it gives us of atomic properties. For the next several chapters we will be discussing atoms, and in this chapter, therefore, we give a brief description of the atom and its constituents, the electron, proton, and neutron, together with some of the evidence for these entities.

3.2 Elements, Atoms, and Molecules

The concept of the atom arose in scholars’ search to identify the basic constituents of matter. More than 2000 years ago Greek philosophers had recognized that this search requires answers to two questions: First, among the thousands of different substances we find around us — sand, air, water, soil, gold, diamonds — how many are the basic substances, or elements, from which all the rest are formed? Second, if one took a sample of one of these elements and subdivided it over and over again, could this process of repeated subdivision go on forever, or would one eventually arrive at some smallest, indivisible unit, or atom (from a Greek word meaning “indivisible”)? With almost no experimental data, the ancient Greeks could not find satisfactory answers to these questions; but this in no way lessens their achievement in identifying the right questions to ask.
John Dalton
(1766–1844, English chemist)

The son of an English weaver, Dalton left school at age eleven. A year later he became a schoolteacher, and this led to his interest in science. In his book, New System of Chemical Philosophy, Dalton argued that the chemists’ law of definite proportions is evidence for the existence of atoms. He had many other interests, including meteorology and color blindness. (He was himself color-blind—which can’t have helped his chemical research.) He was a Quaker and extremely modest. In 1810 he declined membership in the Royal Society, although he was subsequently elected without his prior consent.

TABLE 3.1
A few of the elements with their chemical symbols. For a complete list of the elements, see the periodic table inside the back cover or the alphabetical list in Appendix C.

<table>
<thead>
<tr>
<th>Element</th>
<th>Chemical Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>H</td>
</tr>
<tr>
<td>Helium</td>
<td>He</td>
</tr>
<tr>
<td>Carbon</td>
<td>C</td>
</tr>
<tr>
<td>Copper</td>
<td>Cu</td>
</tr>
<tr>
<td>Oxygen</td>
<td>O</td>
</tr>
<tr>
<td>Sodium</td>
<td>Na</td>
</tr>
<tr>
<td>Aluminum</td>
<td>Al</td>
</tr>
<tr>
<td>Chlorine</td>
<td>Cl</td>
</tr>
<tr>
<td>Iron</td>
<td>Fe</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>N</td>
</tr>
<tr>
<td>Lead</td>
<td>Pb</td>
</tr>
<tr>
<td>Uranium</td>
<td>U</td>
</tr>
</tbody>
</table>

Serious experimental efforts to identify the elements began in the eighteenth century with the work of Lavoisier, Priestley, and other chemists. By the end of the nineteenth century, about 80 of the elements had been correctly identified, including all of the examples listed in Table 3.1. Today we know that there are 90 elements that occur naturally on earth. All of the matter on earth is made up from these 90 elements, occasionally in the form of a pure element, but usually as a chemical compound of two or more elements, or as a mixture of such compounds.

In addition to the 90 naturally occurring elements, there are about two dozen elements that can be created artificially in nuclear reactions. All of these artificial elements are unstable and disintegrate with half-lives much less than the age of the earth; this means that even if any of them were present when the earth was formed, they have long since decayed and are not found naturally in appreciable amounts.

The evidence that the elements are composed of characteristic smallest units, or atoms, began to emerge about the year 1800. Chemists discovered the law of definite proportions: When two elements combine to form a pure chemical compound, they always combine in a definite proportion by mass. For example, when carbon (C) and oxygen (O) combine to form carbon monoxide (CO), they do so in the proportion 3:4; three grams of C combine with four of O to form seven grams of CO.

\[ (3 \text{ g of C}) + (4 \text{ g of O}) \rightarrow (7 \text{ g of CO}) \]

If we were to add some extra carbon, we would not get any additional CO; rather, we would get the same 7 g of CO, with all the extra carbon remaining unreacted.

The law of definite proportions was correctly interpreted by the English chemist John Dalton as evidence for the existence of atoms. Dalton argued that if we assume that carbon and oxygen are composed of atoms whose masses are in the ratio \(3:4\), and if CO is the result of an exact pairing of these atoms (one atom of C paired with each atom of O), the law of definite proportions would immediately follow: The total masses of C and O that combine to form CO would be in the same ratio as the masses of the individual atoms, namely 3:4. If we add extra C (without any additional O), the extra C atoms have no O atoms with which to pair; thus we get no additional CO and the extra C remains unreacted.

* A few elements with half-lives much less than the earth’s age are observed to occur naturally, but this is because some natural nuclear process (radioactivity or cosmic-ray collisions, for example) is creating a fresh supply all the time.

† In stating Dalton’s arguments we have replaced his terminology and measured masses with their modern counterparts. In particular, Dalton had measured the ratio of the C and O masses to be 5:7, not 3:4, as we now know it.
We now know that this interpretation of the law of definite proportions was exactly correct. However, its general acceptance took 60 years or more, mainly because the situation was considerably more complicated than our single example suggests. For instance, carbon and oxygen can also combine in the ratio 3:8 to form carbon dioxide, CO$_2$. Dalton was aware of this particular complication and argued (correctly) that the carbon atom must be able to combine with one O atom (to form CO) or two O atoms (to form CO$_2$). Nevertheless, this kind of ambiguity clouded the issue for many years.

A stable group of atoms, such as CO or CO$_2$, is called a **molecule**. A few examples of molecules are listed in Table 3.2, where we follow the convention that a subscript on any symbol indicates the number of atoms of that kind (and single atoms carry no subscript — thus H$_2$O has two H atoms and one O atom). As the list shows, a molecule may contain two or more different atoms, for example, the CO and C$_6$H$_{12}$O$_6$ molecules, in which case we say that the two or more elements have combined to form a compound, of which the molecule is the smallest unit. Molecules can also contain atoms of just one kind, for example, O$_2$. In this case, we do not speak of a new compound; we say simply that the element normally occurs (or sometimes occurs) not as separate atoms, but as groups of atoms clustered together, that is, as molecules. Molecules can contain small numbers of atoms, like the examples in Table 3.2; but certain organic molecules, such as proteins, can contain tens of thousands of atoms.

<table>
<thead>
<tr>
<th>Table 3.2</th>
<th>A few simple molecules.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon monoxide, CO</td>
<td>Oxygen, O$_2$</td>
</tr>
<tr>
<td>Water, H$_2$O</td>
<td>Nitrogen, N$_2$</td>
</tr>
<tr>
<td>Ammonia, NH$_3$</td>
<td>Sulfur, S$_8$</td>
</tr>
</tbody>
</table>

Although most of the credit for establishing the existence of atoms and molecules goes to the chemists, a second strand of evidence came from the kinetic theory of gases — which we would regard today as a part of physics, and which we discuss in sections 3.7 to 3.9. This theory assumes, correctly, that a gas consists of many tiny molecules in rapid motion. By applying Newton’s laws of motion to the molecules, one could explain several important properties of gases (Boyle’s law, viscosity, Brownian motion, and more). These successes gave strong support to the atomic and molecular hypotheses. In addition, kinetic theory, unlike the chemical line of reasoning, gave information on the actual size and mass of molecules. (The law of definite proportions implied that the C and O atoms have masses in the proportion 3:4 but gave no clue as to the actual magnitude of either.) Kinetic theory gave, for example, an expression for the viscosity of a gas in terms of the size of the individual molecules. Thus, measurement of viscosity allowed one to determine the actual size of the molecules — a method first exploited by Loschmidt in 1885.

By the beginning of the twentieth century it was fairly generally accepted that all matter was made up of elements, the smallest units of which were

*This possibility was probably the source of greatest confusion for the nineteenth-century chemists; if one wrongly identified the O$_2$ molecule as an atom, then one’s interpretation of all other molecules containing oxygen — CO, CO$_2$, and so on — would also be incorrect.
atoms. Atoms could group together into molecules, the formation of which explained the chemical compounds. The relative masses of many atoms and molecules were known quite accurately, and there were already reasonably reliable estimates of their actual masses.

3.3 Electrons, Protons, and Neutrons

Our story so far has the atom — true to its name — as the indivisible smallest unit of matter, and until the late nineteenth century there was, in fact, no direct evidence that atoms could be subdivided. The discovery of the first subatomic particle, the negatively charged electron, is generally attributed to J. J. Thomson (1897), whose experiments we describe briefly in Section 3.10. On the basis of his experiments, Thomson argued that electrons must be contained inside atoms, and hence that the atom is in fact divisible.

Fourteen years later (1911) Ernest Rutherford argued convincingly for the now familiar picture of the atom as a tiny planetary system, in which the negative electrons move far outside a positive nucleus. Rutherford’s conclusion was based on an experiment that was the forerunner of many modern experiments in atomic and subatomic physics. In these experiments, called scattering experiments, one fires a subatomic projectile, such as an electron, at an atom or nucleus. By observing how the projectile is deflected, or scattered, one can deduce the properties of the target atom or nucleus. In the Rutherford experiment, the projectiles were alpha (α) particles — positively charged, subatomic particles ejected by certain radioactive substances. When these were directed at a thin metal foil, Rutherford found that almost all of them passed straight through, but that a few were deflected through large angles. In terms of his planetary model, Rutherford argued that the great majority of alpha particles never came close to any nuclei and encountered only a few electrons, which were too light to deflect them appreciably. On the other hand, a few of the alpha particles would pass close to a nucleus and would be deflected by the strong electrostatic force between the alpha particle and the nucleus. These, Rutherford argued, were the alpha particles that scattered through large angles. As we will describe in Section 3.12, Rutherford used his model to predict the number of alpha particles that should be scattered as a function of scattering angle and energy. The beautiful agreement between Rutherford’s predictions and the experimental observations was strong evidence for the nuclear model of the atom, with its electrons far outside a tiny heavy nucleus.

Within another eight years (1919) Rutherford had shown that the atomic nucleus can itself be subdivided, by establishing that nuclear collisions can break up a nitrogen nucleus. He identified one of the ejected particles as a hydrogen nucleus, for which he proposed the new name proton (“first one”) in honor of its role in other nuclei. In 1932 Chadwick showed that nuclei contain a second kind of particle, the neutral neutron. (The experiments that identified the proton and neutron as constituents of nuclei will be described in Chapter 17.) With this discovery, the modern picture of the constituents of atoms was complete. Every atom contains a definite number of electrons, each with charge \(-e\), in orbit around a central nucleus; and nuclei consist of two kinds of nuclear particles, or nucleons, the proton, with charge \(+e\), and the uncharged neutron.

The constituents of five common atoms are shown in Table 3.3. In every case the number of electrons is equal to the number of protons, reflecting that the atom is neutral (in its normal state). Hydrogen is the only atom that has no
TABLE 3.3
The constituents of five representative atoms.

<table>
<thead>
<tr>
<th>Atom</th>
<th>Electrons</th>
<th>Protons</th>
<th>Neutrons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen, H</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Helium, He</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Carbon, C</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Iron, Fe</td>
<td>26</td>
<td>26</td>
<td>30</td>
</tr>
<tr>
<td>Uranium, U</td>
<td>92</td>
<td>92</td>
<td>146</td>
</tr>
</tbody>
</table>

neutrons. In all other atoms the numbers of neutrons and protons are roughly equal. (We will see the reason for this in Chapter 16.) In many lighter atoms (helium or carbon, for example) the two numbers are exactly the same, while in most medium atoms the number of neutrons is a little larger; in the heaviest atoms there are about 50% more neutrons than protons.

In addition to their role as the building blocks of atoms, the electron and proton also define the smallest observed unit of charge. Their charges are exactly equal and opposite, with magnitude $e$,

$$q_p = -q_e = e = 1.60 \times 10^{-19} \text{ C}$$

where “C” stands for coulomb. No charge smaller than $e$ has been detected, and all known observed charges are integral multiples of $e$,*

$$0, \pm e, \pm 2e, \pm 3e, \ldots$$

Because $e$ is so small, typical macroscopic charges are very large multiples of $e$, and the restriction to integral multiples is usually unimportant. On the atomic level, the existence of a smallest unit of charge is obviously very important.

By 1932 it appeared that all matter was made from just three subatomic particles, the electron, proton, and neutron. This picture of matter was a distinct simplification compared with its predecessor, with its 100 or so elements, each with a characteristic atom. As we will describe in Chapter 18, we now know that at least some of the subatomic particles themselves have an internal structure, being made of sub subatomic particles called quarks. However, for the purposes of atomic physics, and much of nuclear physics, the picture of matter as made of electrons, protons, and neutrons seems to be quite sufficient, and this is where we will stop the story for now.

3.4 Some Atomic Parameters

The distribution of electrons in their atomic orbits, and of protons and neutrons inside the nucleus, are two of the major concerns of quantum theory, as we will describe in several later chapters. Indeed, they are still the subjects of

*As we will discuss in Chapter 18, it has been established that certain “sub-subatomic” particles, called quarks, have charge $\pm e/3$ or $\pm 2e/3$. However, quarks are never found in isolation; they always occur as a member of a bound collection of quarks with total charge 0 or an integer multiple of $e$. 
current research. Nevertheless, a surprising number of atomic and nuclear properties can be understood just from an approximate knowledge of a few parameters, such as the size of the electron orbits and the masses of the electron, proton, and neutron. In this section we discuss some of these parameters, with which you should become familiar.

The size of an atom is not a precisely defined quantity, but can be characterized roughly as the radius of the outermost electron’s orbit. This radius varies surprisingly little among the atoms, ranging from about 0.05 nm (helium) to about 0.3 nm (cesium, for example). In the majority of atoms it is between 0.1 and 0.2 nm. Thus for all atoms we can say that

\[ \text{atomic radius} \approx \text{radius of outer electron orbits} \approx 0.1 \text{ nm} = 10^{-10} \text{ m} \]  (3.1)

The radii of nuclei are all much smaller than atomic radii and are usually measured in terms of the femtometer (fm) or fermi, defined as

\[ 1 \text{ fm} = 10^{-15} \text{ m} \]  (3.2)

Nuclear radii increase steadily from about 1 fm for the lightest nucleus, hydrogen (which is a single proton, of course), to about 8 fm for the heaviest nuclei (for example, uranium). Thus we can say that

\[ \text{nuclear radius} \approx \text{a few fm} = \text{a few} \times 10^{-15} \text{ m} \]  (3.3)

The single most important thing about the two sizes (3.1) and (3.3) is their great difference. The atomic radius is at least 10⁴ times larger than the nuclear radius. Thus, if we made a scale model in which the nucleus was represented by a pea, the atom would be the size of a football stadium. Now, it turns out that in chemical reactions atoms approach one another only close enough for their outer electron orbits to overlap. This means that the chemical properties of atoms are determined almost entirely by the distribution of electrons, and are substantially independent of what is happening inside the nucleus. Further, quantum mechanics predicts (as we will find in Chapter 10) that the distribution of electrons is almost completely determined just by the number of electrons. Thus the chemical properties of an atom are determined almost entirely by the number of its electrons. This important number is called the atomic number and is denoted by the letter \( Z \).

\[ \text{atomic number, } Z, \text{ of an atom} \]
\[ = \text{number of electrons in neutral atom} \]
\[ = \text{number of protons in nucleus.} \]  (3.4)

That \( Z \) is also the number of protons in the nucleus follows because the numbers of electrons and protons are equal in a neutral atom.

We should mention that many atoms can gain or lose a few electrons fairly easily. When this happens, we say that the atom is ionized and refer to the charged atom as an ion. (For example, the \( \text{Ca}^{2+} \) ion is a calcium atom that has lost two electrons; the \( \text{Cl}^{-} \) ion is a chlorine atom that has gained one electron.) Since the number of electrons in an atom can vary in this way, one must specify that \( Z \) is the number of electrons in the neutral atom, as was done in (3.4).
Since chemical properties are determined mainly by the atomic number \( Z \), one might expect that each chemical element could be identified by the atomic number of its atoms, and this proves to be so. All the atoms of a given element have the same atomic number \( Z \), and, conversely, for every number \( Z \) between 1 and 109 (the highest value to be officially recognized with a name), there is exactly one element. Table 3.4 lists a few elements with their atomic numbers. We see that the hydrogen atom has one electron \( (Z = 1) \), helium two \( (Z = 2) \), carbon six, oxygen eight, and so on through uranium with 92, and on to the artificial Meitnerium with 109. An alphabetical list of all the elements can be found in Appendix C; a table of their properties — the periodic table — is inside the back cover.

### TABLE 3.4

<table>
<thead>
<tr>
<th>Element</th>
<th>Atomic Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>1</td>
</tr>
<tr>
<td>Helium</td>
<td>2</td>
</tr>
<tr>
<td>Lithium</td>
<td>3</td>
</tr>
<tr>
<td>Beryllium</td>
<td>4</td>
</tr>
<tr>
<td>Boron</td>
<td>5</td>
</tr>
<tr>
<td>Carbon</td>
<td>6</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>7</td>
</tr>
<tr>
<td>Oxygen</td>
<td>8</td>
</tr>
<tr>
<td>Iron</td>
<td>26</td>
</tr>
<tr>
<td>Lead</td>
<td>82</td>
</tr>
<tr>
<td>Uranium</td>
<td>92</td>
</tr>
<tr>
<td>Meitnerium</td>
<td>109</td>
</tr>
</tbody>
</table>

The mass of an atom depends on the masses of the electron, proton, and neutron, which are as follows:

\[
\begin{align*}
\text{electron: } m_e &= 0.511 \text{ MeV}/c^2 = 9.11 \times 10^{-31} \text{ kg} \\
\text{proton: } m_p &= 938.3 \text{ MeV}/c^2 = 1.673 \times 10^{-27} \text{ kg} \\
\text{neutron: } m_n &= 939.6 \text{ MeV}/c^2 = 1.675 \times 10^{-27} \text{ kg}
\end{align*}
\]

(3.5)

Within 1 part in 1000, the proton and neutron masses are equal, and, by comparison, the electron mass is negligible (one part in 2000). Thus we can say that

\[m_p \approx m_n \approx m_{\text{H}} \approx 940 \text{ MeV}/c^2\]

(3.6)

where \( m_{\text{H}} \) denotes the mass of the hydrogen atom (a proton plus an electron). For many purposes it is sufficient to approximate the mass (3.6) as roughly 1 GeV/c\(^2\).

The result (3.6) makes the approximate calculation of atomic masses extremely simple, since one has only to count the total number of nucleons (that is, the number of protons plus the number of neutrons). For example, the helium atom has two protons and two neutrons and so is four times as massive as hydrogen:

\[m_{\text{He}} \approx 4m_{\text{H}}\]

the carbon atom has six protons and six neutrons and so

\[m_{\text{C}} \approx 12m_{\text{H}}\]
Since its number of nucleons determines an atom’s mass, this number is called the **mass number** of the atom. It is denoted by \( A \):

\[
\text{mass number, } A, \text{ of an atom} = \text{number of nucleons in atom} = (\text{number of protons}) + (\text{number of neutrons}) \quad (3.7)
\]

An atom with mass number \( A \) has mass approximately equal to \( A m_H \).

It often happens that two atoms with the same atomic number \( Z \) (and hence the same numbers of electrons and protons) have different numbers of neutrons in their nuclei. Such atoms are said to be **isotopes** of one another. Since the two isotopes have the same number of electrons, they have almost identical chemical properties and so belong to the same chemical element. But since they have different numbers of neutrons, their mass numbers, and hence masses, are different. For example, the commonest carbon atom has 6 protons and 6 neutrons in its nucleus, but there is also an isotope with 6 protons but 7 neutrons. These two atoms have the same chemical properties, but have different masses, about 12 and 13, respectively. To distinguish isotopes, we sometimes write the mass number \( A \) as a superscript before the chemical symbol. Thus the two isotopes of carbon just mentioned are denoted \(^{12}\text{C} \) and \(^{13}\text{C} \) (usually read as “carbon 12” and “carbon 13”).

While some elements have only one type of stable atom, the majority have two or more stable isotopes. The maximum number belongs to tin with 10 stable isotopes, and on average there are about 2.5 stable isotopes for each element. Table 3.5 lists all of the stable isotopes of four representative elements.

### Table 3.5

<table>
<thead>
<tr>
<th>Element</th>
<th>Atomic Number</th>
<th>Isotopes</th>
</tr>
</thead>
</table>
| Carbon    | 6             | \(^{12}\text{C}(98.9\%), \^{13}\text{C}(1.1\%)\) |}
| Magnesium | 12            | \(^{14}\text{Mg}(79.0\%), \^{25}\text{Mg}(10.0\%), \^{26}\text{Mg}(11.0\%)\) |}
| Chlorine  | 17            | \(^{35}\text{Cl}(75.8\%), \^{37}\text{Cl}(24.2\%)\) |}
| Iron      | 26            | \(^{54}\text{Fe}(5.8\%), \^{56}\text{Fe}(91.7\%), \^{57}\text{Fe}(2.2\%), \^{58}\text{Fe}(0.3\%)\) |}

Since the chemical properties of isotopes are so similar, the proportion of isotopes that occur in nature does not change in normal chemical processes. This means that the atomic mass of an element, as measured by chemists, is the weighted average of the masses of its various natural isotopes. This explains why chemical atomic masses are not always close to integer multiples of \( m_H \). For example, we see from Table 3.5 that natural chlorine is \( \frac{3}{4} \) the isotope \(^{35}\text{Cl}\) and \( \frac{1}{4} \) the isotope \(^{37}\text{Cl}\). Thus the chemical atomic mass of Cl is the weighted average \( 35.5 m_H \). This complication caused some confusion in the historical development of atomic theory. The English physician Prout had pointed out as early as 1815 that the masses of atoms appeared to be integer multiples of \( m_H \), suggesting that all atoms were made from hydrogen atoms. As more atomic masses were measured, examples of nonintegral masses (such as chlorine) were found, and Prout’s hypothesis was rejected. Not until a hundred years later was it seen to be very nearly correct.
3.5 The Atomic Mass Unit

Since all atomic masses are approximately integer multiples of the hydrogen atom’s mass, \( m_{\text{H}} \), it would be natural to use a mass scale with \( m_{\text{H}} \) as the unit of mass, so that all atomic masses would be close to integers. In fact, this is approximately (although not exactly) how the atomic mass scale is defined, and the so-called atomic mass unit, or \( u \), is to a good approximation just \( m_{\text{H}} \).

To understand the exact definition of the atomic mass unit, we must examine why atomic masses are only approximately integral multiples of \( m_{\text{H}} \). Let us consider an atom with atomic number \( Z \) and mass number \( A \). The number of neutrons we denote by \( N \), so that

\[
A = Z + N
\]

If we ignore for a moment the requirements of relativity, the mass of our atom would be just the sum of the masses of \( Z \) electrons, \( Z \) protons, and \( N \) neutrons.

\[
m = Zm_e + Zm_p +Nm_n = Zm_{\text{H}} +Nm_n
\]

To the extent that \( m_n \approx m_{\text{H}} \) we can write this as

\[
m \approx Zm_{\text{H}} +Nm_{\text{H}}
\]

or

\[
m \approx Am_{\text{H}}
\]

In fact, of course, \( m_{\text{H}} \) and \( m_n \) are not exactly the same, the difference being of order 1 part in 1000. Thus the step from (3.8) to (3.9) is only an approximation, and our conclusion (3.10) can be in error by about 1 part in 1000.

There is a second, and more important, reason why (3.10) is only an approximation. We have learned from relativity that when bodies come together to form a stable bound system, the bound system has less mass than its separate constituents by an amount

\[
\Delta m = \frac{B}{c^2}
\]

where \( B \) is the binding energy, or energy required to pull the bound system apart into its separate pieces. For example, the helium nucleus (2 protons + 2 neutrons) has a binding energy of about 28 MeV; thus the helium atom has 28 MeV/\( c^2 \) less mass than the nearly 4000 MeV/\( c^2 \) predicted by (3.10). This is a correction of about 7 parts in 1000 and is appreciably more important than the correction discussed previously.*

For all atoms (except hydrogen itself) there is a similar adjustment due to the nuclear binding energy. Further, it turns out that in almost all atoms the correction is in the same proportion, namely 7 or 8 parts in 1000. (We will see

*There is also a correction for the binding energy of the electrons, but this is much smaller (1 part in 10,000 at the most) and is almost always completely negligible.
why the correction is always about the same in Chapter 16.) The only significant exception to this statement is the hydrogen atom itself, which has no nuclear binding energy and hence no correction. Thus if we want a mass scale on which most atomic masses are as close as possible to integers, the mass of the H atom is certainly not the best choice for the unit. If we chose instead $^4$He or $^{12}$C and defined the unit as $\frac{1}{12}$ the mass of $^4$He or $\frac{1}{16}$ the mass of $^{12}$C, all atoms (except hydrogen) would have masses closer to integral multiples of the atomic mass unit. Which atom we actually choose is a matter of convenience, and by international agreement we use $^{12}$C, defining the u as follows:

$$1 \text{ atomic mass unit} = 1 \text{ u} = \frac{1}{12} \text{(mass of one neutral } ^{12}\text{C atom)} = 1.661 \times 10^{-12} \text{ kg} = 931.5 \text{ MeV}/c^2 \quad (3.12)$$

With the definition (3.12) we can write the mass of an atom whose mass number is $A$ as

$$\text{(mass of atom with mass number } A) \approx A \text{ u} \quad (3.13)$$

For most atoms this approximation is good to 1 part in 1000, and even in the worst case, hydrogen, it is better than 1 part in 100. Both of these claims are illustrated by the examples in Table 3.6.

**TABLE 3.6**
Atomic masses in u are always close to an integer.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1$H</td>
<td>1.008</td>
<td>$^{35}$Cl</td>
</tr>
<tr>
<td>$^2$He</td>
<td>4.003</td>
<td>$^{56}$Fe</td>
</tr>
<tr>
<td>$^{12}$C</td>
<td>12 exactly</td>
<td>$^{208}$Pb</td>
</tr>
<tr>
<td>$^{16}$O</td>
<td>15.995</td>
<td>$^{238}$U</td>
</tr>
</tbody>
</table>

**Example 3.1**

Using the natural abundances in Table 3.5, find the chemical atomic mass of magnesium in u to three significant figures.

The required mass is the weighted average of the masses of the isotopes $^{24}$Mg, $^{25}$Mg, $^{26}$Mg, as listed in Table 3.5. To three significant figures we can use the approximation (3.13) to give

$$\text{average atomic mass of Mg } = \frac{0.79 \times 24 + 0.10 \times 25 + 0.11 \times 26}{1} \text{ u} = 24.3 \text{ u} \quad (3.14)$$

in agreement with the observed value given inside the back cover. On those rare occasions when one needs greater accuracy, the more precise atomic masses given in Appendix D can be used.

*The atomic mass unit defined here, sometimes called the unified mass unit, replaces two older definitions, one based on $^{16}$O and the other on the natural mixture of the three oxygen isotopes.*
3.6 Avogadro’s Number and the Mole

From a fundamental point of view the natural way to measure a quantity of matter is to count the number of molecules (or atoms). In practice, a convenient amount of matter usually has a very large number of molecules, and it is sometimes better to have a larger unit than the individual molecule. The usual choice for that larger unit is the mole, which is defined as follows.

We first define **Avogadro’s number**, \( N_A \), as the number of atoms in 12 grams of \( ^{12}\text{C} \).

\[
\text{Avogadro's number, } N_A = (\text{number of atoms in 12 grams of } ^{12}\text{C}) = 6.022 \times 10^{23}
\]

(3.15)

In this definition there is nothing sacred or fundamental about 12 grams; it is simply a reasonable macroscopic amount. (There is something convenient about the number 12 when used in conjunction with \( ^{12}\text{C} \), as we will see directly.) Since \( N_A \) atoms of \( ^{12}\text{C} \) (each of mass 12 u) have total mass 12 g we see that

\[
N_A \times (12 \text{ u}) = 12 \text{ grams}
\]

Canceling the 12 and dividing, we see that

\[
N_A = \frac{1 \text{ gram}}{1 \text{ u}}
\]

(3.16)

That is, \( N_A \) is just the number of u in a gram.

We now define a mole of objects (carbon atoms, water molecules, physicists) as \( N_A \) objects. Thus a mole of carbon is \( N_A \) carbon atoms, a mole of water is \( N_A \) water molecules (and a mole of physicists is \( N_A \) physicists). Since a mole always contains the same number of objects, the mass of a mole is proportional to the mass of the object concerned. From its definition, a mole of \( ^{12}\text{C} \) has mass exactly 12 grams. Thus a mole of \(^4\text{He}\) has mass approximately 4 grams (more exactly 4.003 grams); a mole of hydrogen atoms, about 1 gram; a mole of \( \text{H}_2 \) molecules, about 2 grams; a mole of water, about 18 grams (since the \( \text{H}_2\text{O} \) molecule has mass \( 2 + 16 = 18 \text{ u} \)); and so on. The official abbreviation for the mole is “mol.”

In chemistry the masses of atoms and molecules are usually specified by giving, not the mass of an individual atom or molecule, but the mass of a mole. Thus, instead of saying that

\[
(\text{mass of } ^{12}\text{C atom}) = 12 \text{ u}
\]

one says that

\[
(\text{mass of 1 mole of } ^{12}\text{C}) = 12 \text{ grams}
\]

Masses expressed in this way, in grams per mole, are often misleadingly called atomic or molecular weights. Evidently, the mass of an atom or molecule in u has the same numerical value as its atomic or molecular weight in grams per mole.
The mole is not a fundamental unit. It is an arbitrary unit, which is convenient for people who deal with large numbers of atoms and molecules. (It can be compared with the “dozen,” another arbitrary unit, which is convenient for people who deal with large numbers of eggs or doughnuts.) A purely theoretical physicist has little reason to work with either moles or Avogadro’s number, but for most of us, it is important to be able to translate between the fundamental language of molecules and the practical language of moles.

To conclude this brief discussion of the mole, we should mention that the mole is officially considered to have an independent dimension called the “amount of substance” (in the same way that meters have the dimension called “length”). For this reason Avogadro’s number,

\[ N_A = 6.022 \times 10^{23} \text{ objects/mole} \]

is often called Avogadro’s constant, since it is viewed as a dimensional quantity with the units mole\(^{-1}\) (or mol\(^{-1}\)).

### 3.7 Kinetic Theory

The description of the macroscopic properties of gases in terms of the microscopic behavior of atoms is called **kinetic theory**. In 1900 the notion that the world is made of very tiny particles, atoms, was widely, but not fully, accepted by the scientific community. As the twentieth century dawned, a vocal minority of competent scientists refused to accept the concept of atoms as fact because there was no direct, irrefutable evidence of their existence. Atoms were much too small to see directly; it was not until 1953 that individual atoms were first imaged with the newly invented field-ion microscope.* The evidence for the existence of atoms from chemistry, though viewed as compelling by many scientists, was indirect and not considered conclusive by all. Only when the increasingly precise predictions of kinetic theory were borne out by experiment did the scientific community completely embrace the atomic hypothesis. The last skeptics were finally convinced when experiment verified the theory of Brownian motion (described in Sec. 3.9)—a theory developed by a young, unknown, scientist named Albert Einstein.

As our first application of kinetic theory, we show that the ideal gas law \( pV = nRT \), an experimentally discovered relation, can be derived from considerations of the motion of atoms. We will also find that the microscopic view of the world afforded by kinetic theory provides a very intuitive and satisfactory explanation of the meaning of temperature, a concept that was poorly understood before kinetic theory. Thus, the kinetic theory derivation of the ideal gas law was an important step in the path to the modern atomic view of matter.

Chemists usually write the ideal gas law as

\[ pV = nRT \]  \hspace{1cm} (3.17)

where \( n \) is the number of moles of the gas, the constant \( R = 8.314 \text{ J/K} \cdot \text{mole} \) is called the **universal gas constant**, and \( p, V, \text{ and } T \) are the pressure, volume, and

---

*This kind of microscope and others capable of resolving atoms are further described in Chapter 14.
temperature of the gas. Physicists usually think in terms of the number \( N \) of molecules, rather than the number \( n \) of moles, so they replace \( n \) by \( n = N/N_A \) and rewrite (3.17) as \( pV = (N/N_A)RT \) or

\[
pV = Nk_BT \tag{3.18}
\]

where the constant \( k_B \) is called **Boltzmann’s constant**.

\[
k_B = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J/K} = 8.62 \times 10^{-5} \text{ eV/K} \tag{3.19}
\]

In 1900 the value of the universal gas constant was well established by comparing (3.17) with experiment. However, Avogadro’s number \( N_A \) was not well known. Estimates of its value varied by a factor of 100 or more. Basically, there was no good way known to count atoms. The inability of experimentalists to determine Avogadro’s number contributed to skeptics’ arguments against the atomic hypothesis.

We now derive the ideal gas law in the form \( pV = NkT \) by considering the motion of atoms. We consider a gas of \( N \) atoms or molecules, each with mass \( m \), in thermal equilibrium enclosed in a rectangular container of length \( L \) and cross-sectional area \( A \). A system is in thermal equilibrium when all parts of the system have the same average energy. We assume that the gas is so dilute and the interactions among the molecules are so weak that the gas can be considered ideal, that is, the molecules are noninteracting. Our goal is to derive an expression for the pressure that the gas exerts on the walls of the container, in terms of the microscopic properties of the molecules.

We orient a coordinate system so that the right and left walls of the container are perpendicular to the \( x \) axis as shown in Fig. 3.1. Consider now a particular molecule labeled \( i \) moving toward the right wall with an \( x \) component of velocity \( v_{xi} \). If the wall is smooth and rigid, the molecule will bounce off the wall like a light ray reflecting from a mirror, and after the collision, the new velocity will have \( x \) component \(-v_{xi}\) while the \( y \) and \( z \) components will be unchanged. The change in momentum of the molecule is then \( \Delta p_{xi} = p_{x{\text{final}}} - p_{x{\text{initial}}} = -2mv_{xi} \). The time \( \Delta t_i \) between collisions of molecule \( i \) with the right side of the container is the time for the molecule to bounce back and forth once between the right and left walls, so \( \Delta t_i = 2L/v_{xi} \). Recall that we are assuming a gas of non-interacting atoms, so that each molecule bounces between the right and left walls without interacting with the other molecules. Except for extremely dilute gases, this assumption of complete non-interaction is unfounded, but, as we will discuss below, collisions among the atoms do not affect our final results. The force from this one molecule on the right wall is a series of brief impulses, but the time-averaged force due to this molecule is the total change in momentum between collisions divided by the total time, (recall that \( F = \Delta p/\Delta t \))

\[
F_{xi} = \frac{|\Delta p_{xi}|}{\Delta t_i} = \frac{|2mv_{xi}|}{\Delta t_i} = \frac{mv_{xi}^2}{L}
\]
The total pressure \( p \) on the wall due to all the molecules in the container is obtained by summing over all \( N \) molecules and dividing by the area \( A \) of the wall.

\[
p = \frac{\sum_i F_{xi}}{A} = \frac{m \sum_i v_{xi}^2}{AL} = \frac{Nm \sum_i v_{xi}^2}{V} \tag{3.20}
\]

In the last step we have replaced \( AL \) with the volume \( V \) of the box, and we have multiplied and divided by \( N \). The last expression \( \sum v_{xi}^2/N = \langle v_{xi}^2 \rangle \) is the average value of \( v_{xi}^2 \). (The brackets \( \langle \ldots \rangle \) represent an average over all molecules). Equation (3.20) now becomes

\[
p = \frac{Nm\langle v_{xi}^2 \rangle}{V} \tag{3.21}
\]

Finally, we relate \( \langle v_{xi}^2 \rangle \) to the average squared speed of a molecule \( \langle v^2 \rangle \). Because there is no preferred direction for the motion of the molecules, we must have

\[
\langle v_{xi}^2 \rangle = \langle v_x^2 \rangle = \langle v_i^2 \rangle \tag{3.22}
\]

The square of the speed of a molecule is \( v^2 = v_x^2 + v_y^2 + v_z^2 \). Therefore,

\[
\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle = 3\langle v_i^2 \rangle \tag{3.23}
\]

and equation (3.21) becomes

\[
pV = Nm\frac{1}{3}\langle v^2 \rangle \tag{3.24}
\]

a result obtained by the Swiss mathematician Daniel Bernoulli at the amazingly early date of 1760.

Comparing this expression with the empirical relation

\[
pV = NkT
\]

we see that our derived expression matches the ideal gas law if we assume that the average kinetic energy of a molecule \( \langle KE \rangle = \frac{1}{2}m\langle v^2 \rangle \) is related to the absolute temperature \( T \) by

\[
\frac{1}{2}m\langle v^2 \rangle = \frac{3}{2}kT \tag{3.25}
\]

Our derivation has not only predicted the form of the ideal gas law but has also shown the physical meaning of temperature. Equation (3.25) says that temperature is proportional to the average kinetic energy per molecule. For a gas, “hotter” means “more kinetic energy per molecule, faster moving molecules.” In Chapter 15 we will give a more general definition of temperature, one that applies to solids as well as gases, but the basic idea remains the same: *Temperature is a measure of the energy per atom.*
3.8 The Mean Free Path and Diffusion

In this section we examine some further features of the kinetic theory of gases. In particular, we look at the consequences of the fact that gases are made of molecules that have a nonzero size. In the derivation of the ideal gas law, we made the assumption that molecules collide only with the walls of the container and not with each other. In fact, because of their nonzero size, molecules do collide with each other, but these intermolecule collisions do not alter either the average speed of the molecules nor the validity of equation (3.21), so the ideal gas law remains valid in the presence of collisions. [An alternate derivation of (3.21), in the presence of collisions, is explored in Problem 3.36.]

The average distance that a molecule travels between collisions with other molecules is called the mean free path. As one might expect, the mean free path depends on both the size and density of the gas molecules; collisions are more frequent if the molecules are larger and if there are more of them. If we approximate the molecules as spheres of radius \( R \), two molecules will overlap and hence collide if their centers come within \( 2R \) of each other (see Fig. 3.2). The collision cross section is defined as the collision area that a molecule presents as it moves through space. We now show that the mean free path \( \lambda \), the collision cross section \( \sigma \), and the number density \( n \) of molecules (\( n = \) number per volume\(^1\)) are related by the simple equation

\[
\lambda \approx \frac{1}{n\sigma}
\]  

(3.26)

Consider a molecule that undergoes a large number \( N \) of collisions and in so doing moves along a zigzag path of total length \( L \) (see Fig. 3.3). Since \( \lambda \) is defined as the mean distance traveled between collisions, we have \( L = N\lambda \). As the molecule travels the distance \( L \), it sweeps through a volume of length \( L \) and cross-sectional area \( \sigma \), colliding with the \( N \) molecules it encounters whose centers are within this volume \( V = L\sigma = N\lambda\sigma \). Assuming that the number density \( N/V \) in this zigzag tube-shaped volume is the same as the number density \( n \) in the sample as a whole, we have

\[
n = \frac{N}{V} = \frac{N}{N\lambda\sigma} = \frac{1}{\lambda\sigma}
\]

which gives the desired result (3.26). This derivation is not rigorously correct because we have not taken proper account of the fact that the other molecules are moving. A more careful calculation yields \( \lambda = 1/\sqrt{2n\sigma} \).

**Example 3.2**

Given that the mean radius of an air molecule (either \( \text{O}_2 \) or \( \text{N}_2 \)) is about \( R = 0.15 \) nm, what is the approximate mean free path \( \lambda \) of the molecules in air at atmospheric pressure and room temperature? (1 atm \( \approx 1.01 \times 10^5 \) Pa)

---

\*We are assuming so-called hard-sphere collisions, that is, a short-range repulsive interaction only, no long-range interaction.

\^Note well that, here, \( n \) is *not* the number of moles, as in (3.17). It is the number of molecules per unit volume.
To use the formula $\lambda \approx 1/(n\sigma)$, we must compute the number density $n$ and the cross section $\sigma$. From the ideal gas law (3.18), we have

$$n = \frac{N}{V} = \frac{p}{kT} = \frac{1.01 \times 10^5 \text{ Pa}}{(1.38 \times 10^{-23} \text{ J/K}) \times (300 \text{ K})} = 2.45 \times 10^{25} \text{ m}^{-3}$$

The collision cross section is

$$\sigma = \pi (2R)^2 = 4\pi (0.15 \times 10^{-9} \text{ m})^2 = 2.8 \times 10^{-19} \text{ m}^2$$

So we have

$$\lambda \approx \frac{1}{n\sigma} = \frac{1}{(2.45 \times 10^{25}) \times (2.8 \times 10^{-19})} = 1.4 \times 10^{-7} \text{ m} = 140 \text{ nm}$$

This is about 40 times greater than the average distance between nearest-neighbor molecules in the gas.

The convoluted zigzag motion of an individual molecule as it collides with its neighbors is an example of **diffusive motion** or **diffusion**. We will show next that with diffusion the average magnitude of the displacement $D$ attained by a molecule in a time $t$ is proportional to $t^{1/2}$,

$$D_{\text{dif}} \propto \sqrt{t}$$

This is to be contrasted with ordinary **ballistic motion**, which is the straight-line, constant-velocity motion of a projectile in the absence of friction or gravity. For ballistic motion, with speed $v$, the net displacement is $D = vt$; that is, displacement is proportional to time,

$$D_{\text{bal}} \propto t$$

Compared to a ballistic projectile, a diffuser with the same speed does not get very far.

Diffusion is also called **random-walk motion** because the motion is that of a bewildered walker who wanders about, making frequent, random changes in direction. The standard random-walk problem is usually formulated like this: A walker moving in 1, 2, or 3 dimensions, takes a series of steps labeled $i = 1, 2, \ldots N$. The $i^{th}$ step is described by a vector $d_i$, which has length $d$, and a random direction. After $N$ steps, the net displacement of the walker, relative to his starting point, is

$$\mathbf{D} = \sum_{i=1}^{N} \mathbf{d}_i \quad \text{(3.27)}$$
In the case of a gas, the walker is a molecule moving in 3 dimensions, taking steps of average length the mean free path (see Fig. 3.4). We seek the average distance of the walker from the origin after $N$ steps.

The word “average” must be carefully defined since there is more than one kind of average. We will use the root-mean-square average distance, which is defined as the square root of the average of the distance squared,

$$D_{\text{rms}} = \sqrt{\langle D^2 \rangle} \quad (3.28)$$

Here the brackets $\langle \ldots \rangle$ indicate an average over all possible $N$-step random walks, often called the ensemble average. The rms average is usually chosen over other possible averages (such as the average of the absolute value of the displacement, $\langle |D| \rangle$) because it is much easier to work with mathematically.

We now show that after $N$ steps, each of length $d$, the rms average distance of a random walker from the starting point is

$$D_{\text{rms}} = \sqrt{N}d \quad (3.29)$$

We begin by noting that the magnitude of the vector $D$ is $D = |D| = \sqrt{D \cdot D}$. The dot product of $D$ with itself is

$$D \cdot D = D^2 = \left( \sum_{i=1}^{N} d_i \right) \cdot \left( \sum_{j=1}^{N} d_j \right) = \sum_{i=1}^{N} d_i^2 + \sum_{i \neq j} d_i \cdot d_j \quad (3.30)$$

We now take the ensemble average, the average over all possible $N$-step walks, and obtain

$$\langle D^2 \rangle = \langle Nd^2 \rangle + \left( \sum_{i \neq j} \langle d_i \cdot d_j \rangle \right) = Nd^2 \quad (3.31)$$

Since the direction of $d_i$ is random, the quantity $d_i \cdot d_j (i \neq j)$ is positive as often as it is negative, and so $\langle \sum d_i \cdot d_j \rangle$ averages to zero. Taking the square root of (3.31), we have our desired result

$$\sqrt{\langle D^2 \rangle} = D_{\text{rms}} = \sqrt{N}d$$
Example 3.3

An aimless physics student performs a random walk by taking 1-meter long steps, at a rate of one step each second, with each step in a completely random direction. How long does it take, on average, for the student to wander to a point 1 kilometer from his starting point?

Using equation (3.29), with step length \( d = 1 \) m and final displacement \( D_{\text{rms}} = 1000 \) m, we have for the number of steps \( N \),

\[
N = \left( \frac{D_{\text{rms}}}{d} \right)^2 = \left( \frac{1000}{1} \right)^2 = 10^6
\]

That is, on average, the student needs to take one million steps in order to get one kilometer away from his starting point. At 1 second per step, this walk will take nearly 12 days.

We can now derive a formula for how far a molecule has diffused in a gas after a time \( t \). This is a random-walk problem in which the step length is the mean free path \( \lambda \) of the molecule. The rms average speed of our molecular random walker is, according to (3.25),

\[
v_{\text{rms}} = \sqrt{\frac{3kT}{m}} \tag{3.32}
\]

We can define a collision time, \( \tau \), as the mean time between collisions (that is, between steps) by the equation

\[
v_{\text{rms}} = \frac{\lambda}{\tau} \tag{3.33}
\]

The time \( t \) required to make \( N \) steps is then \( t = N\tau \). We can now rewrite (3.29) in terms of the time \( t \) of the walk and other microscopic variables

\[
D_{\text{rms}} = \sqrt{N\lambda} = \sqrt{\frac{t}{\tau} \lambda} = \sqrt{\frac{v_{\text{rms}} t}{\tau} \lambda} = \sqrt{Av_{\text{rms}} \sqrt{t}}
\]

Finally, using (3.26) and (3.32), we have

\[
D_{\text{rms}} \approx \frac{1}{\sqrt{\pi n\tau}} \left( \frac{3kT}{m} \right)^{1/4} \sqrt{t} \tag{3.34}
\]

This is a very useful formula for a present-day experimental physicist. However, to a physicist working in the late 1800s, it was interesting, but not especially useful, since none of the quantities \( \sigma, m, \) or \( D_{\text{rms}} \) could be measured experimentally. (Recall that at that time there was not even any good way to measure Avogadro’s number.) However, in 1905 Einstein derived a similar formula for the case of larger particles, particles suspended in air or water and large enough to be seen and measured in a microscope.
Example 3.4

A bottle of ammonia (NH₃) is opened in a closed room where the air is perfectly still. Estimate how long it will take for the evaporating ammonia molecules to diffuse from one side of the room to the other, a distance of 5 m. The mean radius of an ammonia molecule is about $R = 0.15$ nm.

The size of the ammonia molecule is very nearly the same as that of an O₂ or N₂ molecule, so we can use the values of $\sigma$, the collision cross section, and $n$, the number density of molecules, from Example 3.2 above. The mass of an ammonia molecule is

$$m = \frac{\text{(mass of 1 mole)}}{\text{(Avogadro's number)}} = \frac{17\text{ g}}{6 \times 10^{23}} = 2.8 \times 10^{-26}\text{ kg}$$

Solving equation (3.34) for the time $t$, we have

$$t \approx (D_{\text{rms}})^2 n \sigma \left( \frac{m}{3kT} \right)^{1/2}$$

$$= 5^2 \times (2.5 \times 10^{25}) \times (2.8 \times 10^{-19}) \times \left( \frac{2.8 \times 10^{-26}}{3 \times 1.38 \times 10^{-23} \times 300} \right)^{1/2}$$

$$\approx 3 \times 10^5 \text{ s} \approx 3 \text{ days}$$

(3.35)

This long time seems to contradict everyday experience. If you open a bottle of ammonia, the smell will usually permeate the room in minutes or perhaps, at most, an hour. That is because, under normal circumstances, the mixing of air (indoors or out) is accomplished primarily by convection, not diffusion. The point here is that diffusion is very slow compared to convection; if you want to be sure that gases or liquids are mixed, you must stir them.

3.9 Brownian Motion

In 1828 a Scottish botanist, Robert Brown, discovered that tiny pollen grains, when suspended in water and viewed under a microscope, exhibited an irregular jiggling motion, which was later dubbed Brownian motion (see Fig. 3.5). Brown initially thought that this motion was due to some “life force”; however, he quickly established that tiny particles of any composition, whether organic or inorganic, suspended in any fluid, whether liquid or gas, also exhibited this erratic motion. In the decades that followed, Brownian motion was carefully studied and a variety of explanations were suggested. In the late 1880s, it was proposed that Brownian motion is caused by the random motion of the molecules in the surrounding fluid. A small suspended particle experiences a continuous, irregular pounding from all sides because of collisions with the molecules of the fluid. This explanation, which presumed the existence of atoms, was controversial; many thought Brownian motion was due to convection currents or vibrations transmitted through the fluid — explanations that did not require the existence of atoms.

In 1905 the twenty-six-year-old Einstein, having failed to procure an academic position, was supporting himself and his young family by working in a Swiss Patent Office as a “technical expert, third-class.” Though not very prestigious, this job brought him financial security and allowed him enough leisure
time to pursue his own studies of physics. In that year, working in his spare time, Einstein wrote six history-making papers — a creative outburst rivaled only by the work of the young Isaac Newton, two and a half centuries before. The six articles, all published in the German scientific journal *Annalen der Physik* (Annals of Physics) consisted of a paper on the photoelectric effect (Section 4.3), for which he would receive the Nobel Prize in 1922; two papers on special relativity, the second of which contained the formula $E = mc^2$; a paper on the viscosity of dilute solutions, which turned out to have enormous practical importance and which is the Einstein paper most cited in the scientific literature; and two papers on the theory of Brownian motion.

Einstein was able to derive an equation for the motion of a Brownian particle, an equation similar to (3.34), but one containing quantities that could be measured experimentally. Einstein showed that the average displacement $D_{\text{rms}}$ in a time $t$ of a spherical particle* of radius $a$, suspended in a fluid with viscosity $\eta$, is given by

$$D_{\text{rms}} = \sqrt{\frac{kT}{3\pi \eta a}} \sqrt{t} \quad (3.36)$$

(The viscosity of a fluid is a measure of its resistance to flow: thicker, more syrupy fluids have higher viscosity.)

In a series of painstaking measurements, the French experimentalist Jean-Baptiste Perrin verified Eq. (3.36) and used it to determine experimentally Boltzmann’s constant $k_B$, and hence, Avogadro’s number $N_A = R/k_B$. Perrin also determined $N_A$ experimentally using two other methods: One method relied on a theory of the rotation of Brownian particles, a theory also due to Einstein; the other involved determining the distribution of suspended

*The Brownian particle is assumed to be much larger than an atom, but small enough to remain suspended in the fluid. The Brownian motion of larger-than-atom-sized particles is similar to, but not the same as the random-walk motion of individual atoms; the Brownian motion equation (3.36) cannot be derived from the atom motion equation (3.34).
Jean Baptiste Perrin
(1870–1942, French)

Perrin was professor of chemistry at the University of Paris, but is best remembered for two important contributions to modern physics. His investigations of cathode rays, which he showed conclusively to carry negative charge, preceded by a couple of years Thomson’s proof that these “rays” were actually the negative particles that we now call electrons. In 1913 Perrin published the results of his extensive studies of Brownian motion, which yielded a precise value for Avogadro’s number, provided convincing direct proof of the atomic hypothesis, and earned him the Nobel Prize for physics in 1926.

3.10 Thomson’s Discovery of the Electron*

*In the last four sections of this chapter, we describe three of the pivotal experiments of modern physics: the discovery of the electron, the measurement of its charge, and the discovery of the atomic nucleus. We believe that every well-educated scientist should know some of the details of these experiments. Nevertheless, we will not be using this material again, so if you are pressed for time, you could omit these sections without loss of continuity.

The main purpose of this chapter has been to introduce the principal characters in the story of atomic physics — atoms and molecules, electrons, protons, and neutrons — and to describe one of the first successes of atomic theory, the kinetic theory of gases. This early history, culminating in the acceptance of the atomic hypothesis by the scientific community, is not regarded as part of “modern physics,” and in this book we will content ourselves with the brief description already given. On the other hand, the more recent history of atomic physics is very much a part of modern physics, and from time to time we will give more detailed descriptions of some of its experimental highlights. In particular, we conclude this chapter by describing three key experiments: the experiments of J. J. Thomson and Millikan, which together identified the electron and its principal properties, and the so-called Rutherford scattering experiment, which established the existence of the atomic nucleus.

The discovery of the electron is generally attributed to J. J. Thomson (1897) for a series of experiments in which he showed that “cathode rays” were in fact a stream of the negative particles that we now call electrons. Cathode rays, which had been discovered some 30 years earlier, were the “rays” emitted from the cathode, or negative electrode, of a cathode ray tube — a sealed glass tube containing two electrodes and low-pressure gas (Fig. 3.6). When a large potential is applied between the cathode and anode, some of the gas atoms ionize and an electric discharge occurs. Positive ions hitting the cathode eject electrons, which are then accelerated toward the anode. With the arrangement shown in Fig. 3.6, some of the electrons pass through the hole in the anode and coast on to the far end of the tube. At certain pressures the rays can be seen by the glow that they produce in the gas, while at lower pressures they produce a fluorescent patch where they strike the end of the tube (as in a standard television tube).

It had been shown by William Crookes (1879) and others that cathode rays normally travel in straight lines and that they carry momentum. (When directed at the mica vanes of a tiny “windmill,” they caused the vanes to rotate.) Crookes had also shown that the rays can be bent by a magnetic field, the direction of deflection being what would be expected for negative charges. All of this suggested that the rays were actually material particles carrying particles in a gravitational field. All three methods produced the same value of \( N_A \) within a few percent. It was this concordance of different measurements of Avogadro’s number that convinced almost all* the remaining skeptics of the atomic hypothesis.

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*Not quite all. For instance, Ernst Mach (of Mach’s number) died in 1916, still refusing to accept the reality of atoms.
negative charge. Unluckily, attempts to deflect the rays by a transverse electric field had failed, and this failure had led to the suggestion that cathode rays were not particles at all, but were instead some totally new phenomenon — possibly some kind of disturbance of the ether.

J. J. Thomson carried out a series of experiments that settled these questions beyond any reasonable doubt. He showed that if the rays were deflected into an insulated metal cup, the cup became negatively charged; and that as soon as the rays were deflected away from the cup, the charging stopped. He showed, by putting opposite charges on the two deflecting plates in Fig. 3.6, that cathode rays were deflected by a transverse electric field. (He was also able to explain the earlier failures to observe this effect. Previous experimenters had been unable to achieve as good a vacuum as Thomson’s. The remaining gas in the tube was ionized by the cathode rays, and the ions were attracted to the deflecting plates, neutralizing the charge on the plates and canceling the electric field.) After further experiments with magnetic deflection of cathode rays, Thomson concluded, “I can see no escape from the conclusion that they are charges of negative electricity carried by particles of matter.” Accepting this hypothesis, he next measured some of the particles’ properties. By making each measurement in several ways, he was able to demonstrate the consistency of his measurements and to add weight to his identification of cathode rays as negatively charged particles.

Thomson measured the speed of the cathode rays by applying electric and magnetic fields. The force on each electron in fields $E$ and $B$ is the well known Lorentz force

$$\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(3.37)

(where $\mathbf{v}$ is the electron’s velocity). Using an $E$ field alone, he measured the deflection of the electrons. After removing $E$ and switching on a transverse magnetic field $B$, he adjusted $B$ until the magnetic deflection was equal to the previous electric deflection. Under these conditions $\mathbf{E} = \mathbf{v} \times \mathbf{B}$, or since $\mathbf{v}$ and $\mathbf{B}$ were perpendicular,

$$\mathbf{v} = \frac{\mathbf{E}}{B}$$

(3.38)

(In Thomson’s experiments $v$ was of order $0.1c$. Given his experimental uncertainties, this means that his experiments can be analyzed nonrelativistically.) By measuring the heating of a solid body onto which he directed the electrons, he got a second estimate of $v$, which agreed with the value given by (3.38) considering his fairly large uncertainties. (See Problem 3.43.)
Knowing the electrons’ speed he could next find their “mass-to-charge” ratio, m/e. For example, as we saw in Chapter 2 — Eq. (2.48) — a B field causes electrons to move in a circular path of radius

\[ R = \frac{mv}{eB} \]  

(3.39)

(In discussing relativity, we used \( v \) for the relative speed of two frames and \( u \) for the speed of a particle. Here we have reverted to the more standard \( v \) for the electron’s speed.) Thus, measurement of \( R \) (plus knowledge of \( v \) and \( B \)), gave the ratio \( m/e \). Similarly, you can show (Problem 3.40) that an E field alone deflects the electrons through an angle

\[ \theta = \frac{eEl}{mv^2} \]  

(3.40)

where \( l \) denotes the length of the plates that produce \( E \). Thus, by measuring the deflection \( \theta \) produced by a known \( E \) field, he could again determine the ratio \( m/e \).

Even though Thomson used several different gases in his tube and different metals for his cathode, he always found the same value for \( m/e \) (within his experimental uncertainties). From this observation, he argued correctly that there was apparently just one kind of electron, which must be contained in all atoms. He could also compare his value of \( m/e \) for electrons with the known values of \( m/e \) for ionized atoms. (The mass to charge ratio for ionized atoms had been known for some time from experiments in electrolysis — the conduction of currents through liquids by transport of ionized atoms.) Thomson found that even for the lightest atom (hydrogen) the value of \( m/e \) was about 2000 times greater than its value for the electron. The smallness of \( m/e \) for electrons had to be due to the smallness of their mass or the largeness of their charge, or some combination of both. Thomson argued (again correctly, as we now know) that it was probably due to the smallness of the electrons’ mass.

### 3.11 Millikan’s Oil Drop Experiment

A frustrating shortcoming of Thomson’s measurements was that they allowed him to calculate the ratio \( m/e \), but not \( m \) or \( e \) separately. The reason for this is easy to see: Newton’s second law for an electron in \( E \) and \( B \) fields states that

\[ ma = -e(E + v \times B) \]

Evidently, any calculation of the electron’s motion is bound to involve just the ratio \( m/e \) and not \( m \) or \( e \) separately. One way around this difficulty was to study the motion of some larger body, such as a droplet of water, whose mass \( M \) could be measured and which had been charged by the gain or loss of an electron. In this way one could measure the ratio \( M/e \). Knowing the mass \( M \) of the droplet, one could find \( e \), and thence the mass \( m \) of the electron from the known value of \( m/e \).

The need to measure \( e \) (or \( m \)) separately was recognized by Thomson and his associates, who quickly set about measuring \( e \) using drops of water. However, the method proved difficult (for example, the water drops evaporated rapidly), and Thomson could not reduce his uncertainties below about 50%.
In an effort to avoid the problem of evaporation, the American physicist Robert Millikan tried using oil drops and quickly developed an extremely effective technique. His apparatus is sketched in Fig. 3.7. Droplets of oil from a fine spray were allowed to drift into the space between two horizontal plates. The plates were connected to an adjustable voltage, which produced a vertical electric field $E$ between the plates. With the field off, the droplets all drifted downward and quickly acquired their terminal speed, with their weight balanced by the viscous drag of the air. When the $E$ field was switched on, Millikan found that some of the drops moved down more rapidly, while others started moving upward. This showed that the drops had already acquired electric charges of both signs, presumably as a result of friction in the sprayer. The method by which Millikan measured these charges was ingenious and intricate, but the essential point can be understood from the following simplified account. (For some more details, see Problem 3.45.)

By adjusting the electric field $E$ between the plates, Millikan could hold any chosen droplet stationary. When this was the case, the upward electric force must exactly have balanced the downward force of gravity:

$$qE = Mg$$  \hspace{1cm} (3.41)

where $q$ was the charge on the droplet. Since $E$ and $g$ were known, it remained only to find $M$ in order to determine the charge $q$.

To measure $M$, Millikan switched off the $E$ field and observed the terminal speed $v$ with which the droplet fell. (This speed was very small and so easily measured.) From classical fluid mechanics, it was known that the terminal speed of a small sphere acted on by a constant force $F$ is

$$v = \frac{F}{6\pi \eta r}$$  \hspace{1cm} (3.42)

where $r$ is the radius of the sphere and $\eta$ the viscosity of the gas in which it moves. In the present case $F$ is just the weight of the droplet,

$$F = Mg = \frac{4}{3} \pi r^3 \rho g$$  \hspace{1cm} (3.43)

where $\rho$ is the density of the oil. Substitution of (3.43) in (3.42) gives the speed of the falling droplet as

$$v = \frac{2r^2 \rho g}{9\eta}$$  \hspace{1cm} (3.44)

Since $\rho$, $g$, and $\eta$ were known, measurement of $v$ allowed Millikan to calculate the droplet’s radius $r$ and thence, from (3.43), its mass $M$. Using (3.41), he could then find the charge $q$.

From time to time it was found that a balanced droplet would suddenly move up or down, indicating that it had picked up an extra ion from the air.
Millikan quickly learned to change the charge of the droplet at will by ionizing the air (by passing X-rays through the apparatus, for example). In this way he could measure not only the initial charge on the droplet but also the change in the charge as it acquired extra positive and negative ions.

In the course of several years, starting in 1906, Millikan observed thousands of droplets, sometimes watching a single droplet for several hours as it changed its charge a score or more times. He used several different liquids for his droplets and various procedures for changing the droplets’ charges. In every case the original charge \( q \) and all subsequent changes in \( q \) were found to be integer multiples (positive and negative) of a single basic charge,

\[
e = 1.6 \times 10^{-19} \text{ C}
\]

That the basic unit of charge was the same as the electron’s charge was checked by pumping most of the air out of the apparatus. With few air molecules, there could be few ions for the droplet to pick up. Nevertheless, Millikan found that X-rays could still change the charge on the droplets, but only in the direction of increasing positive charge. He interpreted this, correctly, to mean that X-rays were knocking electrons out of the droplet. Since the changes in charge were still multiples of \( e \) (including sometimes \( e \) itself), it was clear that the electron’s charge was itself equal to the unit of charge \( e \) (or rather \( -e \), to be precise).

It is worth emphasizing the double importance of Millikan’s beautiful experiments. First, he had measured the charge of the electron, \( q = -e \). Combined with the measurement of \( m/e \) by Thomson and others, this also determined the mass of the electron (as about \( 1/2000 \) of the mass of the hydrogen atom, as Thomson had guessed). Second, and possibly even more important, he had established that all charges, positive and negative, come in multiples of the same basic unit \( e \).

### 3.12 Rutherford and the Nuclear Atom

As soon as the electron had been identified by Thomson, physicists began considering its role in the structure of atoms. In particular, Kelvin and Thomson developed a model of the atom (usually called the Thomson model), in which the electrons were supposed to be embedded in a uniform sphere of positive charge, somewhat like the berries in a blueberry muffin. The nature of the positive charge was unknown, but the small mass of the electron suggested that whatever carried the positive charge must account for most of the atom’s mass.

To investigate these ideas, it was necessary to find an experimental probe of the atom, and the energetic charged particles ejected by radioactive substances proved suitable for this purpose. Especially suitable was the alpha particle, which is emitted by many radioactive substances and which Rutherford had identified (1909) as a positively ionized helium atom. (In fact, it is a helium atom that has lost both of its electrons; that is, it is a helium nucleus.) It was found that if alpha particles were directed at a thin layer of matter, such as a metal foil, the great majority passed almost straight through, suffering only small deflections. It seemed clear that the alpha particles must be passing through the atoms in their path and that the deflections must be caused by the electric fields inside the atoms. All of this was consistent with the Thomson model, which necessarily predicted only small deflections, since the electrons
are too light and the field of the uniform positive charge too small to produce large changes in the velocity of a massive alpha particle.

However, Ernest Rutherford and his two assistants, Hans Geiger and Ernest Marsden, found (around 1910) that even with the thinnest of metal foils (of order 1 μm) a few alpha particles were deflected through very large angles — 90° and even more. Since a single encounter with Thomson’s atom could not possibly cause such a deflection, it was necessary to assume (if one wished to retain the Thomson model) that these large deflections were the result of many encounters, each causing a small deflection. But Rutherford was able to show that the probability of such multiple encounters was far too small to explain the observations.

Rutherford argued that the atom must contain electric fields far greater than predicted by the Thomson model, and large enough to produce the observed large deflections in a single encounter. To account for these enormous electric fields he proposed his famous nuclear atom, with the positive charge concentrated in a tiny massive nucleus. (Initially, he did not exclude the possibility of a negative nucleus with positive charges outside, but it was quickly established that the nucleus was positive.)

According to Rutherford’s model, the majority of alpha particles going through a thin foil would not pass close enough to any nuclei to be appreciably deflected. (Collisions with atomic electrons would not cause significant deflections because the electron is so much lighter than the alpha particles.) On the other hand, a few would come close to a nucleus, and these would be the ones scattered through a large angle. A simplifying feature of the model is that the large deflections occur close to the nucleus, well inside the electron orbits, and are unaffected by the electrons. That is, one can analyze the large deflections in terms of the Coulomb force of the nucleus, ignoring all the atomic electrons.

Rutherford was able to calculate the trajectory of an alpha particle in the Coulomb field of a nucleus and hence to predict the number of deflections through different angles; he also predicted how this number should vary with the particle’s energy, the foil’s thickness, and other variables. These predictions were published in 1911, and all were subsequently verified by Geiger and Marsden (1913), whose results provided the most convincing support for the nuclear atom.

The experiment of Geiger and Marsden is shown schematically in Fig. 3.8. A narrow stream of alpha particles from a radioactive source was directed toward a thin metal foil. The number of particles deflected through an angle θ was observed with the help of a zinc sulfide screen, which gave off a visible scintillation when hit by an alpha particle; these scintillations could be observed through a microscope and counted.

To calculate the number of deflections expected on the basis of Rutherford’s model of the atom, let us consider a single alpha particle of mass m, charge q = 2e, and energy E. Since we are interested in large deflections

![Diagram of the experiment](image-url)
FIGURE 3.9
Trajectory of the alpha particle in the Coulomb field of a nucleus. The initial and final momenta are labeled \( \mathbf{p}_i \) and \( \mathbf{p}_f \); the direction labeled \( x \) is used as \( x \) axis in Section 3.13.

(\( \theta \) more than a couple of degrees, say) we suppose that the particle passes reasonably close to the nucleus of an atom and well inside the atomic electrons, whose presence we can therefore ignore. We denote the nuclear charge by \( Q = Ze \) and, for simplicity, suppose the nucleus to be fixed. (This is a good approximation since most nuclei are much heavier than the alpha particle.) The only force acting on the alpha particle is the Coulomb repulsion of the nucleus,

\[
F = k \frac{Qq}{r^2} = \frac{2Ze^2}{r^2} \tag{3.45}
\]

where \( k \) is the Coulomb force constant* \( k = 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \). Under the influence of this inverse-square force the alpha particle follows a hyperbolic path as shown in Fig. 3.9.

We can characterize the path followed by any particular alpha particle by its impact parameter \( b \), defined as the perpendicular distance from the nucleus to the alpha particle’s original line of approach (Fig. 3.9). Our first task is to relate the angle of deflection \( \theta \) to the impact parameter \( b \). We defer this somewhat tedious exercise in mechanics to Section 3.13, where we will prove that

\[
b = \frac{Zke^2}{E \tan(\theta/2)} \tag{3.46}
\]

where \( E \) denotes the energy of the incident alpha particles. Notice that a large scattering angle \( \theta \) corresponds to a small impact parameter \( b \), and vice versa, just as one would expect.

Let us now focus on a particular value of the impact parameter \( b \) and the corresponding angle \( \theta \). All particles whose impact parameter is less than \( b \) will be deflected by more than \( \theta \) [Fig. 3.10(a)]. Thus all those particles that impinge on a circle of radius \( b \) (and area \( \pi b^2 \)) are scattered by \( \theta \) or more. If the original beam of particles has cross-sectional area \( A \) [Fig. 3.10(b)], the proportion of particles scattered by \( \theta \) or more is \( \pi b^2/A \). If the total number of particles is \( N \), the number scattered through \( \theta \) or more by any one atom in the foil is

\[
\text{(number scattered through } \theta \text{ or more by one atom)} = N \frac{\pi b^2}{A} \tag{3.47}
\]

*The Coulomb force constant \( k \) is often written in the form \( k = 1/(4\pi \varepsilon_0) \), where \( \varepsilon_0 \) is called the permittivity of the vacuum.
Section 3.12 • Rutherford and the Nuclear Atom

This must now be multiplied by the number of target atoms that the beam of alpha particles encounters. If the target foil has thickness \( t \) and contains \( n \) atoms in unit volume, the number of atoms that the beam meets is [Fig. 3.10(b)]

\[
\text{(number of target atoms encountered)} = nAt \quad (3.48)
\]

Combining (3.47) and (3.48), we find that the total number of alpha particles scattered through \( \theta \) or more is

\[
N_{sc}(\theta \text{ or more}) = \frac{N\pi b^2}{A} \times nAt = \pi Nntb^2 \quad (3.49)
\]

By differentiating (3.49) with respect to \( \theta \), we can find the number of particles emerging between \( \theta \) and \( \theta + d\theta \); and finally, by elementary geometry we can find the number that hit unit area on the zinc sulfide screen at angle \( \theta \) and distance \( s \) from the foil. We again defer the details of the calculation to Section 3.13. The result is that

\[
n_{sc}(\theta) = \text{number of particles per unit area at } \theta = \frac{Nnt}{4s^2} \left( \frac{Zke^2}{E} \right)^2 \frac{1}{\sin^4(\theta/2)} \quad (3.50)
\]

This important result is called the Rutherford formula.

Some features of the Rutherford formula would be expected of almost any reasonable atomic model. For example, it is almost inevitable that \( n_{sc}(\theta) \) should be proportional to \( N \), the original number of alpha particles, and inversely proportional to \( s^2 \), the square of the distance to the detector. On the other hand, several features are specific to Rutherford’s assumption that each large angle deflection results from an encounter with the tiny but massive charged nucleus of a single target atom. Among the features specific to Rutherford’s model are

1. \( n_{sc}(\theta) \) is proportional to the thickness \( t \) of the target foil.
2. \( n_{sc}(\theta) \) is proportional to \( Z^2 \), the nuclear charge squared.
3. \( n_{sc}(\theta) \) is inversely proportional to \( E^2 \), the incident energy squared.
4. \( n_{sc}(\theta) \) is inversely proportional to the fourth power of \( \sin(\theta/2) \).
Geiger and Marsden were able to check each of these predictions separately and found excellent agreement in all cases. Their observations of item (4), the variation the scattering with the angle $\theta$, are plotted in Fig. 3.11, which shows their measurements taken at 14 different angles* and fitted to a curve of the form

$$K / \sin^4(\theta/2).$$

These and their other equally beautiful results established Rutherford’s nuclear atom beyond reasonable doubt and paved the way for the modern quantum theory of the atom.

One final consequence of Rutherford’s analysis deserves mention. The Rutherford formula (3.50) was derived by assuming that the force of the nucleus on the alpha particles is the Coulomb force

$$F = \frac{kqQ}{r^2}. \quad (3.51)$$

This is true provided that the alpha particles remain outside the nucleus at all times. If the alpha particles penetrated the nucleus, the force would not be given by (3.51) and the Rutherford formula would presumably not hold. Thus the fact that Geiger and Marsden found the Rutherford formula to be correct in all cases indicated that all alpha particles were deflected before they could penetrate the target nuclei. Since it was easy to calculate the minimum distance of the alpha particle from the center of any nucleus, this gave an upper limit on the nuclear radius, as we see in the following example.

Example 3.5

When 7.7-MeV alpha particles are fired at a gold foil ($Z = 79$), the Rutherford formula agrees with the observations at all angles. Use this fact to obtain an upper limit on the radius $R$ of the gold nucleus.

Since the Rutherford formula holds at all angles, none of the alpha particles penetrate inside the nucleus; that is, for all trajectories the distance $r$ from the alpha particle to the nuclear center is always greater than $R$

$$r > R. \quad (3.52)$$

(See Fig. 3.12.) Now, the minimum value of $r$ occurs for the case of a head-on collision, in which the alpha particle comes instantaneously to rest. At this

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point its kinetic energy is zero \((K = 0)\) and its potential energy \(U = 2Zke^2/r_{\min}\), is equal to its total energy; that is, \(U = E = 7.7 \text{ MeV}\). Thus

\[
\frac{2Zke^2}{r_{\min}} = E
\]  

whence

\[
r_{\min} = \frac{2Zke^2}{E}
\]

Since \(R < r\) for all orbits, it follows that \(R < r_{\min}\) and hence

\[
R \leq \frac{2Zke^2}{E}
\]

We have used the sign \(\leq\) to emphasize that this is only an approximate result. In the first place, the nuclear radius is itself only an approximate notion. Second, we are ignoring the size of the alpha particle. [We could improve on (3.55) by replacing \(R\) by \(R + R_a\).] Finally, it is possible that non-Coulombic, nuclear forces may have an appreciable effect even a little before the alpha particle penetrates the nucleus.

Before we substitute numbers into the inequality (3.55), it is convenient to note that the Coulomb constant, \(k = 9 \times 10^9 \text{N} \cdot \text{m}^2/\text{C}^2\), almost always appears in atomic and nuclear calculations in the combination \(ke^2\). Since \(ke^2/r\) is an energy, \(ke^2\) has the dimensions of energy \(\times\) length and is conveniently expressed in the units \(\text{eV} \cdot \text{nm}\) or, what is the same thing, \(\text{MeV} \cdot \text{fm}\). To this end, we detach one factor of \(e\) and write

\[
ke^2 = (8.99 \times 10^9 \text{N} \cdot \text{m}^2/\text{C}^2) \times (1.60 \times 10^{-19} \text{C}) \times e
\]

\[
= 1.44 \times 10^{-9} \text{e} \cdot \text{N} \cdot \text{m}^2/\text{C}
\]

Now the unit \(\text{N} \cdot \text{m}/\text{C} = \text{J}/\text{C}\) is the same as a volt. Multiplied by \(e\), this gives an electron volt; therefore, \(ke^2 = 1.44 \times 10^{-9} \text{eV}\) or

\[
ke^2 = 1.44 \text{ eV} \cdot \text{nm} = 1.44 \text{ MeV} \cdot \text{fm}
\]

Returning to the inequality (3.55), we find that

\[
R \leq \frac{2Zke^2}{E} = \frac{2 \times 79 \times (1.44 \text{ MeV} \cdot \text{fm})}{7.7 \text{ MeV}}
\]
or
\[ R \leq 30 \text{ fm} \tag{3.57} \]

That is, the fact that Geiger and Marsden found the Rutherford formula valid for 7.7 MeV alpha particles on gold implied that the gold nucleus has radius less than 30 fm. Since the radius of the gold nucleus is in fact about 8 fm, this was perfectly correct.

If the incident energy \( E \) were steadily increased, some of the alpha particles would eventually penetrate the target nucleus, and the Rutherford formula would cease to hold, first at \( \theta = 180^\circ \) (that is, for head-on collisions), then as the energy increased, at smaller angles as well. From (3.55) we see that the energy at which the Rutherford formula first breaks down is
\[ E \approx \frac{2Zke^2}{R} \tag{3.58} \]

For gold this energy is about 30 MeV, an energy that was not available to Rutherford until much later. However, for aluminum, with \( Z \) only 13, it is about 6 MeV (Problem 3.49), and Rutherford was able to detect some departure from his formula at large angles.

### 3.13 Derivation of Rutherford’s Formula*

In deriving the Rutherford formula (3.50), we omitted two slightly tedious calculations. First, the relation (3.46) between impact parameter \( b \) and scattering angle \( \theta \): If we call the initial and final momenta of the alpha particle \( \mathbf{p}_i \) and \( \mathbf{p}_f \), the angle between \( \mathbf{p}_i \) and \( \mathbf{p}_f \) is the scattering angle \( \theta \), as shown in Fig. 3.13. Since the alpha particle’s energy is unchanged, \( \mathbf{p}_i \) and \( \mathbf{p}_f \) are equal in magnitude and the triangle on the left of Fig. 3.13 is isosceles. If we denote the change in momentum by \( \Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i \), it is clear from the construction shown that
\[ \Delta \mathbf{p} = 2p_i \sin \frac{\theta}{2} \tag{3.59} \]

Now, we know from Newton’s second law \( (d\mathbf{p}/dt = \mathbf{F}) \) that
\[ \Delta \mathbf{p} = \int_{-\infty}^{\infty} \mathbf{F} \, dt \]
Section 3.13 • Derivation of Rutherford’s Formula

To evaluate this integral, we choose our $x$ axis in the direction of $\Delta \mathbf{p}$. This direction was indicated in Fig. 3.9, where we labeled the polar angle of the alpha particle as $\phi$. With these notations

$$\Delta \mathbf{p} = \int_{-\infty}^{\infty} F_x \, dt = \int_{-\infty}^{\infty} \frac{2Zke^2}{r^2} \cos \phi \, dt \quad (3.60)$$

This integral can be easily evaluated by a trick that exploits conservation of angular momentum. (Since the force on the alpha particle is radially outward from the nucleus, the angular momentum $L$ about the nucleus is constant.) Long before the collision $L = bp_i$, whereas we know that at all times

$$L = mr^2 \omega = mr^2 \frac{db}{dt}$$

Equating these two expressions for $L$, we find that

$$\frac{db}{dt} = \frac{bp_i}{mr^2}$$

Thus we can rewrite (3.60) as

$$\Delta \mathbf{p} = 2Zke^2 \int \cos \phi \, \frac{dt}{r^2} \frac{db}{d\phi}$$

$$= 2Zke^2 \int \cos \phi \, \frac{mr^2}{bp_i} \, d\phi$$

$$= \frac{2Zke^2}{bp_i} \int_{\phi_i}^{\phi_f} \cos \phi \, d\phi$$

$$= \frac{2Zke^2}{bp_i} (\sin \phi_f - \sin \phi_i) \quad (3.61)$$

Comparing Figs. 3.9 and 3.13, we see that $\phi_f = -\phi_i$ and that $\theta + 2\phi_f = 180^\circ$. Therefore, $\phi_f = 90^\circ - \theta/2$ and $\sin \phi_f = \cos(\theta/2)$. Thus (3.61) becomes

$$\Delta \mathbf{p} = \frac{4Zke^2m}{bp_i} \cos \frac{\theta}{2} \quad (3.62)$$

Equating the two expressions (3.59) and (3.61) for $\Delta \mathbf{p}$, we see that

$$2bp_i \sin \frac{\theta}{2} = \frac{4Zke^2m}{bp_i} \cos \frac{\theta}{2}$$

which we can solve for $b$:

$$b = \frac{Zke^2}{E \tan(\theta/2)} \quad (3.63)$$

since $E = p_i^2/2m$. This completes the proof of the relation (3.46).
The other step that was omitted in Section 3.12 was the derivation of the final result, the Rutherford formula (3.50), from Equation (3.49),

\[ N_{\text{sc}}(\theta \text{ or more}) = \pi Nntb^2 \]

Substituting (3.63), we can rewrite the latter as

\[ N_{\text{sc}}(\theta \text{ or more}) = \pi Nnt \left( \frac{Zke^2}{E \tan(\theta/2)} \right)^2 \]

(3.64)

This gives the number of alpha particles deflected through \( \theta \) or more. The number that emerge between \( \theta \) and \( \theta + d\theta \) is found by differentiating (3.64) to give (see Problem 3.51)

\[ N_{\text{sc}}(\theta \text{ to } \theta + d\theta) = \pi Nnt \left( \frac{Zke^2}{E} \right)^2 \cos(\theta/2) \sin^3(\theta/2) d\theta. \]

(3.65)

Now, at a distance \( s \) from the target (where the detector is placed) the particles emerging between \( \theta \) and \( \theta + d\theta \) are distributed uniformly over the ring-shaped surface shown in Fig. 3.14. This ring has area

\[ \text{area of ring} = (2\pi s \sin \theta) \times (s d\theta) \]

(3.66)

To find the number of particles per unit area at distance \( s \), we must divide the number (3.65) by this area, to give

\[ n_{\text{sc}}(\theta) = \frac{N_{\text{sc}}(\theta \text{ to } \theta + d\theta)}{2\pi s^2 \sin \theta d\theta} = \frac{Nnt}{4s^2} \left( \frac{Zke^2}{E} \right)^2 \frac{1}{\sin^3(\theta/2)} \]

where we have used the identity \( \sin(\theta) = 2 \sin(\theta/2) \cos(\theta/2) \). This completes our derivation of the Rutherford formula.
### Problems for Chapter 3

#### SECTION 3.2 (Elements, Atoms, and Molecules)

**3.1** One of the triumphs of nineteenth-century science was the discovery by Mendeleev of the periodic table of the elements, which brought order to the confusing multitude of known elements, as described in Chapter 10. A crucial step in this discovery was to list the elements in the correct order — that is, as we now know, in order of increasing atomic number (which we define in Section 3.4, as the number of electrons in the neutral atom). At the time, atomic number was an unknown concept, and chemists listed the elements in order of atomic mass. Fortunately, these two orderings are very nearly the same. Use the periodic table inside the back cover of the book (where both parameters are listed) to find out how many times the two orderings are different. The exception involving argon and potassium was a stumbling block in the development of the periodic table since it was found necessary to reverse the order of these two (when ordered by mass) to get the periodic table to make sense. We will see in Chapter 16 why it is that atomic number and atomic mass increase almost perfectly in step.

<table>
<thead>
<tr>
<th>Nucleon</th>
<th>Name for a proton or neutron</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit of charge $e$</td>
<td>$e = -(\text{electron charge}) = (\text{proton charge})$</td>
</tr>
<tr>
<td>Atomic radii</td>
<td>$0.1\ \text{nm} = 10^{-10}\ \text{m}$ (3.1)</td>
</tr>
<tr>
<td>Nuclear radii</td>
<td>$\approx \text{a few fm} = \text{a few } 10^{-13}\ \text{m}$ (3.3)</td>
</tr>
<tr>
<td>Atomic number of an atom $Z$</td>
<td>$Z = (\text{number of electrons in neutral atom}) = (\text{number of protons in nucleus})$ (3.4)</td>
</tr>
<tr>
<td>Ions</td>
<td>Atoms that have lost or gained one or more electrons</td>
</tr>
<tr>
<td>Relative masses of $e, p, \text{ and } n$</td>
<td>$m_p \approx m_n \approx 2000\ m_e$ (3.5)</td>
</tr>
<tr>
<td>Mass number $A$</td>
<td>$A = \text{number of nucleons (protons + neutrons) in nucleus}$ (3.7)</td>
</tr>
<tr>
<td>Isotopes</td>
<td>Two or more atoms with the same atomic number $Z$ but different neutron number. Same chemical properties, but different masses</td>
</tr>
<tr>
<td>Atomic mass unit, $u$</td>
<td>$1\ u = \frac{1}{12}(\text{mass of neutral atom of } ^{12}\text{C}) \approx (\text{mass of } ^1\text{H atom})$ (3.12)</td>
</tr>
<tr>
<td>Avogadro’s constant, $N_A$</td>
<td>$N_A = 6.022 \times 10^{23}\ \text{objects/mole}$ (3.15)</td>
</tr>
<tr>
<td>Mole (mol)</td>
<td>$1\ \text{mole} = N_A\ \text{objects}$</td>
</tr>
<tr>
<td>Ideal gas law</td>
<td>$pV = nRT = Nk_B T$ (3.17) and (3.18)</td>
</tr>
<tr>
<td>Universal gas constant, $R$</td>
<td>$R = 8.31\ \text{J/K} \cdot \text{mole}$</td>
</tr>
<tr>
<td>Boltzmann’s constant, $k_B$</td>
<td>$k_B = R/N = 1.38 \times 10^{-23}\ \text{J/K} = 8.62 \times 10^{-5}\ \text{eV/K}$ (3.19)</td>
</tr>
<tr>
<td>Mean free path, $\lambda$</td>
<td>Mean distance between collisions (Sec. 3.8)</td>
</tr>
<tr>
<td>Diffusive motion</td>
<td>Random-walk motion. Displacement $D_{\text{diff}} \propto \sqrt{t}$ (Sec. 3.8)</td>
</tr>
<tr>
<td>Brownian motion</td>
<td>Jiggling of small particles due to impacts of atoms (Sec. 3.9)</td>
</tr>
<tr>
<td>Thomson’s discovery of the electron*</td>
<td>Sec. 3.10</td>
</tr>
<tr>
<td>Millikan’s oil-drop experiment*</td>
<td>Measurement of electron charge (Sec. 3.11)</td>
</tr>
<tr>
<td>Rutherford’s scattering experiment*</td>
<td>Established nuclear model of atom (Secs. 3.12 and 3.13)</td>
</tr>
</tbody>
</table>
3.2 • The elements are generally listed in order of their “atomic number” $Z$ (which is just the number of electrons in a neutral atom of the element). Hydrogen is first with $Z = 1$, helium next with $Z = 2$, and so on. There are 90 elements that occur naturally on earth, but they are not just numbers 1 through 90. What are they? What are the atomic numbers of the known artificial elements? (You can find this information in the periodic table inside the back cover.)

3.3 • With what mass of oxygen would 2 g of hydrogen combine to form water $(H_2O)$?

[HINT: All you need to know is the ratio of the masses of the atoms concerned; you can find these masses, listed in atomic mass units, inside the back cover.]

3.4 • With what mass of carbon would 1 g of hydrogen combine to form acetylene, $C_2H_2$? (Read the hint to Problem 3.3.)

3.5 • With what mass of nitrogen would 1 g of hydrogen combine to form ammonia $(NH_3)$? (Read the hint to Problem 3.3.)

3.6 • (a) With what mass of oxygen would 1 g of carbon combine to form carbon monoxide $CO$? (b) With what mass of oxygen would 1 g of carbon combine to form carbon dioxide $CO_2$? (In practice, when oxygen and carbon combine — as in a car engine — both CO and $CO_2$ are formed in proportions that depend on conditions such as the temperature.)

SECTION 3.4 (Some Atomic Parameters)

3.7 • Make a table showing the numbers of electrons, protons, neutrons, and nucleons in the most common form of the following atoms: He, Li, Na, Ca. (The necessary information is in Appendix D.)

3.8 • Tabulate the numbers of electrons, protons, neutrons, and nucleons in the most common form of the following neutral atoms: hydrogen, carbon, nitrogen, oxygen, aluminum, iron, lead. (The necessary information can be found in Appendix D.) On the basis of this information, can you suggest why lead is much denser than aluminum?

3.9 • Make a table similar to Table 3.5 showing all the stable isotopes for hydrogen, helium, oxygen, and aluminum, include their percentage abundances. (Use the information in Appendix D.)

3.10 • Use Appendix D to find four elements that have only one type of stable atom (that is, an atom with no stable isotopes).

3.11 • Two atoms that have the same mass number $A$ but different atomic numbers $Z$ are called isobars. (Isobar means “of equal mass.”) For example, $^4$He and $^3$H are isobars. Use the data in Appendix D to find three examples of pairs of isobars. Find an example of a triplet of isobars (that is, three different atoms with the same mass number).

3.12 • It is found that the radius $R$ of any nucleus is given approximately by

$$ R = R_0 A^{1/3} $$

where $A$ is the mass number of the nucleus and $R_0$ is a constant whose value depends a little on how $R$ is defined, but is about 1.1 fm. (a) What are the radii of the nuclei of helium, carbon, iron, lead, and lawrencium? (b) How does the volume of a nucleus (assumed spherical) depend on $A$? What does your answer tell you about the average density of nuclei?

SECTION 3.5 (The Atomic Mass Unit)

3.13 • The mass of a carbon 12 atom is $1.992648 \times 10^{-26}$ kg (with an uncertainty of 1 in the final digit). Calculate the value of the atomic mass unit to six significant figures in kg and in MeV/c².

3.14 • Find the mass of a $^{12}$C nucleus in u correct to five significant figures. (The mass of an electron is 0.000549 u. Ignore the binding energy of the electrons.)

3.15 • Using the natural abundances listed in Table 3.5 (or Appendix D) calculate the average atomic masses of naturally occurring carbon, chlorine, and iron. (Your answers should be in u and correct to three significant figures.)

3.16 • What is the mass (correct to the nearest u) of the following molecules: (a) water, $H_2O$; (b) laughing gas, $N_2O$; (c) ozone, $O_3$; (d) glucose, $C_6H_{12}O_6$; (e) ammonia, $NH_3$; (f) limestone, $CaCO_3$.

3.17 • For the great majority of atoms, it is found that the atomic mass (in atomic mass units) is close to twice the atomic number (an observation we will explain in Chapter 16). (a) Using the data in the periodic table inside the back cover, verify this observation by computing the ratio

$$ \frac{\text{atomic mass in u}}{\text{atomic number}} $$

for the elements with $Z = 10, 20, \ldots, 90$. Comment on your results. (b) In terms of the mass number $A$, your results should show that $A$ is close to (or in some cases a bit more than) $2Z$. What does this mean for the neutron number $N$ in relation to the proton number $Z$?

3.18 • (a) Calculate the mass of the $^4$He nucleus in atomic mass units by subtracting the mass of two electrons from that of a neutral 4 He atom. (See Appendix D, and give four decimal places.) (b) The procedure suggested for part (a) is theoretically incorrect because we have neglected the binding energy of the electrons. Given that the energy needed to pull both electrons far away from the nucleus is 80 eV, is the correct answer larger or smaller than implied by part (a), and by how much (in u)? In stating the mass of the nucleus, about how many significant figures can you give before this effect would show up?
SECTION 3.6 (Avogadro’s Number and the Mole)

3.19 • How many moles are in (a) 1 g of carbon? (b) 1 g of hydrogen molecules? (c) 10 g of water? (d) 1 ounce of gold?

3.20 • How many moles are in (a) 10 g of NaCl? (b) 2 kg of NH₃? (c) 10 cm³ of Hg (density 13.6 g/cm³)? (d) 1 pound of sugar (C₁₂H₂₂O₁₁)?

3.21 • A mole of O₂ molecules has mass 32 g (that is, the molecular “weight” of O₂ is 32 g/mole). What is the mass of a single O₂ molecule (in u and in grams)?

3.22 • What is the total charge of 1 mole of Cl⁻ ions? (The Cl⁻ ion is a chlorine atom that has acquired one extra electron.) This important quantity is called the faraday; it is easily measured since it is the charge that must pass to release 1 mole of chlorine in electrolysis of NaCl solution, (or, more generally, 1 mole of any monovalent ion in electrolysis of an appropriate solution). Once the electron charge was known, knowledge of the faraday let one calculate Avogadro’s number.

3.23 • (a) How many atomic mass units are there in one pound mass? (b) How many in one gram? Comment on your answer to part (b).

3.24 • How many C atoms are there in (a) 1 g of CO₂? (b) 1 mole CH₄? (c) 1 kmole of C₂H₆? In each case what fraction of the atoms are C atoms, and what fraction of the mass is C?

SECTION 3.7 Kinetic Theory

3.25 • Show that the number density (number per volume) of molecules in an ideal gas is \( p/kT \).

3.26 • What is the edge length of a cube that contains 1 billion \( (10^9) \) molecules of a gas at standard temperature and pressure (STP: \( T = 273 \text{ K}, \ p = 1 \text{ atm} = 1.01 \times 10^5 \text{ N/m}^2 \))?

3.27 • At what pressure would the density of He gas at \( T = 300 \text{ K} \) equal the density of ordinary water? Give your answer in atmospheres. \( [1 \text{ atm} = 1.01 \times 10^5 \text{ N/m}^2] \).

3.28 • Consider an ideal gas at standard temperature and pressure (STP: \( T = 273 \text{ K}, \ p = 1 \text{ atm} = 1.01 \times 10^5 \text{ N/m}^2 \)). Imagine that the molecules of the gas are equally spaced so that every molecule is at the center of a small cube. (a) What is the edge length of this cube, which is the average distance between nearest-neighbor molecules in the gas? (b) Qualitatively, how does this mean molecular separation distances compare with the diameter of a small molecule such as O₂?

3.29 • (a) Consider an ideal gas at temperature \( T \) with molecules of mass \( m \) and number density (number per volume) \( n \). Show that the number of molecules that strike a surface of area \( A \) in a time \( \Delta t \) is given roughly by \( (nA \Delta t/2) \sqrt{kT/m} \). To simplify this calculation, consider a wall perpendicular to the \( x \)-axis and make the radical assumption that the \( x \)-components of the velocites of all molecules have the same magnitude; that is, assume that all molecules have the same \( |v_x| \), with half the molecules moving to the right and the other half moving to the left. (b) For oxygen at standard temperature and pressure (STP: \( T = 273 \text{ K}, \ p = 1 \text{ atm} = 1.01 \times 10^5 \text{ N/m}^2 \))

3.30 • Consider a large container of oxygen gas at STP (\( T = 273 \text{ K}, \ p = 1 \text{ atm} = 1.01 \times 10^5 \text{ N/m}^2 \)). Assume a molecular radius of 0.15 nm. (a) How long does it take for a molecule to diffuse to a point 1 m from its starting point? (b) How long would it take for the molecule to diffuse the same distance at a pressure of 0.01 atm? (c) Solve for the pressure at which a molecule will diffuse a displacement of magnitude \( d \) in a time \( t \).

3.31 • (a) Show that for a gas, the mean free path \( \lambda \) between collisions is related to the mean distance between nearest neighbors \( r \) by the approximate relation \( \lambda \approx r^2/\sigma \) where \( \sigma \) is the collision cross-section. (b) Given that the molecular radius of a gas molecule such as \( \text{O}_2, \text{N}_2, \) or \( \text{CO}_2 \) is about 0.15 nm, estimate the value of \( r \) and \( \lambda \) for air at STP (standard temperature and pressure, \( T = 273 \text{ K}, \ p = 1.00 \text{ atm} = 1.01 \times 10^5 \text{ N/m}^2 \)). (c) Why, qualitatively, is \( \lambda \gg r \), for dilute gases?

3.32 • The Knudsen regime is the pressure regime in which the mean free path is comparable to the dimensions of the container. (a) Given the container diameter \( d \), the molecule’s collision cross section \( \sigma \), and the temperature \( T \), derive an expression for the pressure \( p \) at which the Knudsen regime occurs. (b) What is this pressure (in atmospheres) for a gas at room temperature (about 293 K) in a container with diameter 20 cm? Assume a molecular radius of 0.15 nm. A good mechanical pump can achieve a pressure of \( 10^{-4} \) atm; can such a pump produce Knudsen pressures in a 20-cm diameter container?

3.33 • The best vacuum that can be readily achieved in the lab is called “ultra-high vacuum” or UHV and is \( 10^{-10} \text{ torr} \) (mm Hg) \( \approx 10^{-7} \text{ atm} \). At that pressure and at room temperature (\( T = 293 \text{ K} \)), (a) how many air molecules are present in 1 cm³, (b) what is the mean free path \( \lambda \) of the molecules? Assume a molecular radius of 0.15 nm.

3.34 • An aimless physics student, wandering around on a flat plane, takes a 1-m step in a random direction each second. (a) After one year of continuous random walking, what is the student’s expected distance from his starting point? (b) If the student wandered in 3D space, rather than in a plane, but still took 1-m steps each second in random directions, would his expected distance from the origin be greater, less, or the same as before. Explain.
3.35 A gambler has $100 and makes a series of $1 bets that he can correctly guess the outcome of a coin flip—heads or tails. (The coin and the game are honest, so for every coin flip, there is a 50% chance that the gambler will guess correctly.) The gambler will quit when he has either won an additional $100 or has lost all his money. About how many times will the gambler bet before quitting?

[HINT: This situation is like a random walk in one dimension.]

3.36 Derive the ideal gas law without the assumption of no intermolecule collisions. To simplify this calculation, make the radical assumption that the x-components of the velocities of all molecules have the same magnitude; that is, assume that all molecules have the same \( |v_x| \), with half the molecules moving to the right and the other half moving to the left; make similar assumptions for y- and z-directions. Consider a gas of molecules with number density \( n \) in a cubical container, and consider only those molecules that are within a short distance \( \Delta x \) of one wall of the container where \( \Delta x \ll \lambda \), the mean free path. At any instant, half of those molecules are heading toward the wall and will collide with the wall in an average time \( \Delta t \), where \( \Delta x/\Delta t = |v_x| \). Proceed to compute the average force on the the wall due to molecular collisions, and then deduce the ideal gas law.

3.37 A Brownian particle is observed to diffuse 100 \( \mu \)m in 60 s, on average. (Diffusion distance means magnitude of the net displacement.) A second particle with twice the diameter and ten times the mass of the first is observed. How far, on average, will the second particle diffuse in 60 s at the same temperature?

3.38 A Brownian particle is observed to diffuse a distance of 10 \( \mu \)m in 20 s, on average. How far, on average, will the particle diffuse in 200 s?

3.39 A good microscope can readily resolve objects that are 1 \( \mu \)m in diameter (1 \( \mu \)m is about twice the wavelength of visible light). The unaided human eye can resolve objects no smaller than roughly 0.1 mm = 100 \( \mu \)m. Suppose a 1-\( \mu \)m diameter Brownian particle is observed (with a microscope) to diffuse an average distance of 5 times its diameter (5 \( \mu \)m) in 20 s. Under the same conditions of temperature and air visibility, another Brownian particle of 100 \( \mu \)m diameter is observed with the unaided eye.

How long will this particle take to diffuse an average distance of 5 times its diameter? Why was Brownian motion not discovered until after the invention of the microscope?

3.39 SECTION 3.9 Brownian Motion

3.40 In Thomson’s experiment electrons travel with velocity \( v \) in the x direction. They enter a uniform electric field \( E \), which points in the y-direction and has total width \( l \) (Fig. 3.15). Find the time for an electron to cross the field and the y-component of its velocity when it leaves the field. Hence show that its velocity is deflected through an angle \( \theta \approx eEl/(mv^2) \) (provided that \( \theta \) is small). Assume that the electrons are nonrelativistic, as was the case for Thomson.

Figure 3.15

(Problem 3.40)

3.41 Suppose that the electrons in Thomson’s experiment enter a uniform magnetic field \( B \), which is in the \( z \)-direction (with axes defined as in Fig. 3.15) and has total width \( l \). Show that they are deflected through an angle \( \theta \approx eBl/(mv) \) (provided that \( \theta \) is small). Assume that the electrons are nonrelativistic.

3.42 The formulas for the deflection of an electron in an electric or magnetic field (Problems 3.40 and 3.41) are quite similar. Explain in words why both formulas have factors of \( v \) in the denominator and why the magnetic deflection has just one power of \( v \) where the electric has two.

3.43 Thomson directed his cathode rays at a metal body and measured the total charge \( Q \) that it acquired and its rise in temperature \( \Delta T \). From \( \Delta T \) and the body’s known thermal capacity, he could find the heat given to the body. Show that this heat should be

\[
\text{heat} = \frac{Qmv^2}{2e}
\]

where \( m \), \( v \), and \( e \) are the electron’s mass, speed (nonrelativistic), and charge. Combine this with the result of Problem 3.41 to give expressions for \( v \) and \( m/e \) in terms of measured quantities.

3.44 In one version of Thomson’s measurement of \( m/e \), nonrelativistic electrons are accelerated through a measured potential difference \( V_0 \). This means that their kinetic energy is \( K = eV_0 \). They are then passed into a known magnetic field \( B \) and the radius \( R = mv/(eB) \) of their circular orbit is measured. Eliminate the speed \( u \) from these two equations, and derive an expression for \( m/e \) in terms of the measured quantities \( V_0 \), \( B \), and \( R \).

3.45 In order to find the electron’s charge \( e \), Millikan needed to know the mass \( M \) of his oil drops, and this was actually the source of his greatest uncertainty in determining \( e \) (His value of \( e \) was about 0.5% low as a result.) However, to show that all charges are multiples of some basic unit charge, it was not necessary to
know $M$, which cancels out of the charge ratios. The following problem illustrates this point and gives some more details of Millikan’s experiment.

By switching the $E$ field off and on, Millikan could time an oil drop as it fell and rose through a measured distance $l$ (falling under the influence of gravity, rising under that of $E$ and gravity). The downward and upward speeds are given by (3.42) as

$$v_d = \frac{Mg}{6\pi \eta} \quad \text{and} \quad v_u = \frac{qE - Mg}{6\pi \eta} \quad (3.67)$$

Both speeds were measured in the form $v = \frac{l}{t}$, where $t$ is the time for the droplet to traverse $l$.

(a) By adding the two equations (3.67) and rearranging, show that

$$\frac{1}{t_d} + \frac{1}{t_u} = \frac{E}{6\pi \eta q} q = K q$$

where the quantity $K = \frac{E}{(6\pi \eta q)}$ was constant as long as Millikan watched a single droplet and did not vary $E$.

From (3.68) we see that the charge $q$ is proportional to the quantity $(1/t_d) + (1/t_u)$. Thus if it is true that $q$ is always an integer multiple of $e$, it must also be true that $(1/t_d) + (1/t_u)$ is always an integer multiple of some fixed quantity. Table 3.7 shows a series of timings for a single droplet (made with a commercial version of the Millikan experiment used in a teaching laboratory). That $t_u$ changes abruptly from time to time indicates that the charge on the droplet has changed as described in Section 3.11. (The times $t_d$ should theoretically all be the same, of course. The small variations give you a good indication of the uncertainties in all timings. For $t_d$ you should use the average of all measurements of $t_d$.)

Table 3.7

<table>
<thead>
<tr>
<th>$t_d$</th>
<th>15.2</th>
<th>15.0</th>
<th>15.1</th>
<th>15.0</th>
<th>14.9</th>
<th>15.1</th>
<th>15.1</th>
<th>15.0</th>
<th>15.2</th>
<th>15.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_u$</td>
<td>6.4</td>
<td>6.3</td>
<td>6.1</td>
<td>24.4</td>
<td>24.2</td>
<td>3.7</td>
<td>3.6</td>
<td>1.8</td>
<td>2.0</td>
<td>1.9</td>
</tr>
</tbody>
</table>

(b) Calculate $(1/t_d) + (1/t_u)$ and show that (within the uncertainties) this quantity is always an integer multiple of one fixed number, and hence that the charge is always a multiple of one fixed charge.

(c) Use Equation (3.44) for $v_d$ and the additional data in Table 3.8 to find the radius $r$ of the droplet.

(d) Finally, use (3.68) to find the four different charges $q$ on the droplet. What is the best estimate of $e$ based on this experiment?

### Table 3.8

<table>
<thead>
<tr>
<th>$l = 8.3 \times 10^{-4}$ m</th>
<th>(distance traveled by the droplet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 859$ kg/m$^3$</td>
<td>(density of oil)</td>
</tr>
<tr>
<td>$g = 9.80$ m/s$^2$</td>
<td>(acceleration of gravity)</td>
</tr>
<tr>
<td>$\eta = 1.60 \times 10^{-5}$ N·s/m$^2$</td>
<td>(viscosity of air, adjusted*)</td>
</tr>
<tr>
<td>$E = 1.21 \times 10^3$ N/C</td>
<td>(electric field)</td>
</tr>
</tbody>
</table>

* For very small droplets, there is a small correction to the formula (3.42) for the terminal velocity. For the experiment reported here, this correction amounts to a 12% reduction in the viscosity. We have included this correction in the value given.

### Section 3.12 (Rutherford and the Nuclear Atom)

3.46 • A student doing the Rutherford scattering experiment arranges matters so that she gets 80 counts/min at a scattering angle of $\theta = 10^\circ$. If she now moves her detector around to $\theta = 150^\circ$, keeping it at the same distance from the target, how many counts would she expect to observe in a minute? (This illustrates an awkward feature of the experiment, especially before the days of automatic counters. An arrangement that gives a reasonable counting rate at small $\theta$ gives far too few counts at large $\theta$, and one that gives a reasonable rate for large $\theta$ will overwhelm the counter at small $\theta$.)

3.47 • The Rutherford model of the atom could explain the large-angle scattering of alpha particles because it led to very large electric fields compared to the Thomson model. To see this, note that in the Thomson model the positive charge of an atom was uniformly distributed through a sphere of the same size as the atom itself. According to this model, what would be the maximum $E$ field (in volts/meter) produced by the positive charge in a gold atom ($Z = 79$, atomic radius $\approx 0.18$ nm)? What is the corresponding maximum field in Rutherford’s model of the gold atom, with the positive charge confined to a sphere of radius about 8 fm? (In the Thomson model the actual field would be even less because of the electrons. Note that since $ke^2 = 1.44$ eV·nm, it follows that $ke = 1.44$ eV·nm.)

3.48 • Consider several different alpha particles approaching a nucleus, with various different impact parameters, $b$, but all with the same energy, $E$. Prove that an alpha particle that approaches the nucleus head-on ($b = 0$) gets closer to the nucleus than any other.

3.49 • (a) If the Rutherford formula is found to be correct at all angles when 15-MeV alpha particles are fired at a silver foil ($Z = 47$), what can you say about the radius of the silver nucleus? (b) Aluminum has atomic number $Z = 13$ and a nuclear radius about $R_{Al} \approx 4$ fm. If one were to bombard an aluminum foil with alpha particles and slowly increase their energy, at about what energy would you expect the Rutherford formula to break down? You can make the estimate (3.58) a bit more realistic by taking $R$ to
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be \( R_{Al} + R_{He} \), where \( R_{He} \) is the alpha particle’s radius (about 2 fm).

3.50 ** In a student version of the Rutherford experiment \(^{210}\)Po is used as a source of 5.2-MeV alpha particles, which are directed at a gold foil (thickness 2 \( \mu \)m) at a rate of \( 10^6 \) particles per minute. The scattered particles are detected on a screen of area 1 cm\(^2\) at a distance of 12 cm. Use the Rutherford formula (3.50) to predict the number of alpha particles observed in 10 minutes at and 45°, given the following data:

\[
N = 10^6 \quad \text{(number of incident particles in 10 min)}
\]
\[
\rho = 19.3 \text{ g/cm}^3 \quad \text{(density of gold)}
\]
\[
m_{Au} = 197 \text{ u} \quad \text{(mass of gold atom)}
\]
\[
t = 2 \times 10^{-6} \text{ m} \quad \text{(thickness of foil)}
\]

\[
Z = 79 \quad \text{(atomic number of gold)}
\]
\[
E = 5.2 \text{ MeV} \quad \text{(energy of alpha particles)}
\]

[Note that the number density \( n \) of gold nuclei is \( \rho/m_{Au} \), and don’t forget that \( ke^2 = 1.44 \text{ MeV} \cdot \text{fm} \).]

SECTION 3.13 (Derivation of Rutherford’s Formula)

3.51 * (a) If \( N_{sc}(\theta \text{ or more}) \) denotes the number of alpha particles scattered by \( \theta \) or more, show that the number whose deflection is between \( \theta \) and \( \theta + d\theta \) is

\[
N_{sc}(\theta \text{ to } \theta + d\theta) = - \frac{dN_{sc}(\theta \text{ or more})}{d\theta} d\theta
\]

(b) Differentiate the expression (3.64) for \( N_{sc}(\theta \text{ or more}) \) and verify the result (3.65).