Capacitors

A capacitor is simply two pieces of metal near each other, separated by insulator or air. A capacitor is used to store charge and energy.

A parallel-plate capacitor consists of two parallel plates separated by a distance \( d \), each plate with area \( A \). If \( A \) large and \( d \) small, the plates are effectively infinite planes, and the E-field is uniform and entirely in-between the plates.

Charges are always on the inside surfaces, because (+) attracts (–). The outside surfaces remain uncharged.

"Charge Q on a capacitor" always means +Q on one plate, –Q on the other plate. Capacitors are charged by transferring (–) charge from one plate to the other. Taking (–) charge off a plate leaves behind an equal-sized (+) charge.

The charges make an E-field, which means a voltage difference between the plates. The "voltage V on a capacitor" always means the voltage difference \( \Delta V \) between the plates.

\[
|\Delta V| = E \cdot d = V \Rightarrow V \propto E \propto Q \Rightarrow \text{ratio} \frac{Q}{V} = \text{constant}
\]

Definition: capacitance \( C \) of a capacitor:

\[
C = \frac{Q}{V}
\]

If we double the charge \( Q \), the voltage \( V \) doubles, but the ratio \( Q/V \) remains constant.

(Remember! \( Q \) means +Q and –Q, \( V \) means \( \Delta V \).)

units [C] = coulomb / volt = farad (F)

Big capacitance (1F) ⇒ can store a big Q with a small V
Small capacitance (nF = 10\(^{-9}\) F) ⇒ small Q stored with a big V

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For a parallel-plate capacitor, with air or vacuum between the plates, it turns out that

\[ C = \frac{\varepsilon_0 A}{d} \]  

(air or vacuum separating plates)

\(\varepsilon_0\) ("epsilon-naught") is a constant related to the constant \(k\) in Coulomb's law.

\[ k = \frac{1}{4\pi \varepsilon_0}, \quad \varepsilon_0 = \frac{1}{4\pi k} = 8.85 \times 10^{-12} \text{ (SI units)} \]

Note that this formula means \(C\) increases as \(d\) decreases. Why? If \(Q\) is kept fixed, we have the same magnitude E-field (because the \(Q\) creates the \(E\)). Smaller \(d\) and same-sized \(E\) means smaller voltage \(V = E d\). Same \(Q\) and smaller \(V\) means bigger \(C = Q/V\).

### Stored Energy in Capacitors

It takes work to charge a capacitor, because it is difficult to transfer more electrons from the (+) plate to the (–) plate. The work required to transfer a charge \(q\) across a voltage difference "\(V\)" = \(\Delta V\) is \(\Delta PE = q \Delta V\).

When we charge up a capacitor from \(q_{\text{initial}} = 0\) to \(q_{\text{final}} = Q\), we transfer electrons one at a time. The first electron is easy to transfer since \(V = \Delta V = 0\) initially, but the later electrons take more and more work to transfer as \(Q\) (and \(\Delta V\)) builds up.

Total work to charge capacitor = electrostatic potential energy stored in capacitor =

\[ U = \frac{1}{2} Q V \]
(We use $U$ for energy to avoid confusion with $E$ for electric field.)
Why the $(1/2)$? Why not $\Delta PE = W_{\text{ext}} = QV$? While transferring the total charge $Q$, the voltage difference increased from 0 to $V$. The average value was $(1/2)V$.

Can rewrite this in various ways using $C = Q/V$, $Q = CV$, $V = Q/C$:

$$U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2} \frac{Q^2}{C}$$

Where is this energy? The E-field contains the energy. It takes work to create an E-field. It turns out that the energy per volume (the energy density) of the E-field is given by

$$u = \frac{U}{\text{Vol.}} = \frac{1}{2} \varepsilon_0 E^2$$

The energy $U = (1/2)QV$ of a charged capacitor is in the E-field between the plates. If we pull the plates apart, keeping the charge $Q$ fixed, we increase the volume which contains E-field and the total energy increases. It was hard to pull the plates apart, because opposite charges attract.

The work we did went into creating more E-field (same size E-field over larger volume).

The capacitance $C$ of a capacitor can be increased by placing an insulator ("dielectric") between plates. The dielectric is polarized by the charges on the plates.

For fixed $Q$ on the plates, the E-field between the plates is reduced when a dielectric is inserted because the polarization charge on the dielectric partially cancels the charge $Q$ on the plates. smaller $E \Rightarrow$ smaller $V = |\Delta V| = E \Delta d$, smaller $V$ and same $Q$ on plates $\Rightarrow$ larger $C = Q/V$.

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