Linear Momentum

**Definition:** Linear momentum of a mass \( m \) moving with velocity \( \vec{v} \):

\[
\vec{p} \equiv m \vec{v}
\]

Momentum is a vector. Direction of \( \vec{p} = \) direction of velocity \( \vec{v} \).

units \([p] = \text{kg} \cdot \text{m/s}\) (no special name)

(No one seems to know why we use the symbol \( p \) for momentum, except that we couldn't use "m" because that was already used for mass.)

**Definition:** Total momentum of several masses: \( m_1 \) with velocity \( \vec{v}_1 \), \( m_2 \) with velocity \( \vec{v}_2 \), etc..

\[
\vec{p}_{\text{tot}} \equiv \vec{p}_1 + \vec{p}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2
\]

Momentum is an extremely **useful** concept because momentum is **conserved**. It is especially useful for analyzing collisions between particles.

**Conservation of Momentum:** You can never create or destroy momentum; all we can do is transfer momentum from one object to another. Therefore, the total momentum of a system of masses isolated from external forces (forces from outside the system) is constant in time. Similar to Conservation of Energy – always true, no exceptions. (Proof will be given later)

Two objects, labeled A and B, collide. \( v = \) velocity before collision, \( v' \) (v-prime) = velocity after collision.

Before

\[
\begin{align*}
m_A & \quad \vec{v}_A \\
m_B & \quad \vec{v}_B
\end{align*}
\]

After

\[
\begin{align*}
m_A & \quad \vec{v}_A' \\
m_B & \quad \vec{v}_B'
\end{align*}
\]

Conservation of momentum guarantees that

\[
\vec{p}_{\text{tot}} = m_A \vec{v}_A + m_B \vec{v}_B = m_A \vec{v}_A' + m_B \vec{v}_B'
\]

**Types of collisions**

- **elastic collision:** total KE is conserved (KE before = KE after)

superball on concrete: KE just before collision = KE just after (almost!)
**inelastic collision**: some KE is lost to thermal energy, sound, etc

**perfectly inelastic collision** (or totally inelastic collision): 2 objects collide and stick together

All collisions between macroscopic (large) objects are inelastic – you always dissipate some KE in a collision. However, you can have an elastic collision between atoms: air molecules are always colliding with each other, but do not lose their KE.

### 1D Collisions

In 1D, we represent direction of vectors \( p \) and \( v \) with a sign. (+) = right  (–) = left

\[ \begin{align*}
  v_A &= +2 \text{ m/s} \quad \Rightarrow \text{moving right} \\
  v_B &= -3 \text{ m/s} \quad \Rightarrow \text{moving left}
\end{align*} \]

Notation Danger!! Sometimes \( v = |\vec{v}| = \text{speed (always positive)} \). But in 1D collision problems, symbol "\( v \)" represents *velocity*: \( v \) can (+) or (–).

**1D collision example**: 2 objects, A and B, collide and stick together (a perfectly inelastic collision). Object A has initial velocity \( v \), object B is initially at rest. What is the final velocity \( v' \) of the stuck-together masses?

\[ \begin{align*}
  \text{Before} & \quad \text{After} \\
  m_A \quad v & \quad \bigcirc \quad m_B \text{ (at rest)} & \quad \bigcirc \quad m_A + m_B \\
  p_{\text{tot (before)}} &= p_{\text{tot (after)}} \\
  m_A v_A + m_B v_B &= (m_A + m_B) v' \\
  m_A v &= (m_A + m_B) v' \\
  v' &= \left( \frac{m_A}{m_A + m_B} \right) v
\end{align*} \]

Notice that \( v' < v \), since \( m_A / (m_A + m_B) < 1 \).
Another 1D collision example (recoil of a gun). A gun of mass $M$ fires a bullet of mass $m$ with velocity $v_b$. What is the recoil velocity $v_G$ of the gun?

\[ p_{\text{tot}} = 0 \text{ (both at rest)} \]

**Before**

\[ p_{\text{tot(before)}} = 0 = p_{\text{tot(after)}} \]

\[ 0 = M v_G + m v_b \]

\[ M v_G = -m v_b \]

\[ v_G = -\left(\frac{m}{M}\right)v_b \]

$v_b = 500 \text{ m/s}$, $m = 10 \text{ gram} = 0.01 \text{ kg}$, $M = 3 \text{ kg}$ \Rightarrow $v_G = -\frac{0.010}{3} \cdot 500 = -1.7 \text{ m/s}$

Quite a kick! This is how rockets work! Rocket fuel is thrown out the back of the rocket; rocket recoils forward. There is NO WAY to make a rocket go forward in space except by throwing mass out the back. Any other means of propulsion would violate Conservation of Momentum. (Sorry Star Trek fans, warp drive is impossible.)

Incidentally, why is the barrel of a rifle so long?

Ans: $v = a \cdot t$ \Rightarrow long barrel, more time to accelerate, bigger $v$

To prove that momentum is conserved in collisions, we need the concept of *impulse*, which relates force to changes in momentum.

Newton never wrote $F_{\text{net}} = m \cdot a$. He wrote an equivalent relation using momentum:

\[ \vec{F}_{\text{net}} = \frac{\Delta \vec{p}}{\Delta t} \]

Net force is the rate of change of momentum.

Is this the same as $F_{\text{net}} = m \cdot a$?

Check: $\vec{p} = m \vec{v}$, $\Delta \vec{p} = m \Delta \vec{v}$ (assuming $m = \text{constant}$) \Rightarrow

\[ \vec{F}_{\text{net}} = \frac{\Delta \vec{p}}{\Delta t} = \frac{m \Delta \vec{v}}{\Delta t} = m \vec{a} \]
"impulse" = change in momentum = force × time

To change the momentum of an object, you must apply a net force for some time interval.

The term "impulse" is usually reserved for situations in which a BIG force acts for a short time to cause a rapid change in momentum. Like a bat hitting a baseball:

\[ \mathbf{I} = \Delta \mathbf{p} = \mathbf{F}_{\text{net}} \cdot \Delta t \]

Example:

\[
m_{\text{baseball}} = 0.30 \text{ kg} , \quad v_i = -42 \text{ m/s} , \quad v_f = +80 \text{ m/s} , \quad \text{duration of bat/ball collision} = \Delta t = 0.010 \text{ s}
\]

What is the impulse? And what is the size of the average force exerted by the bat on the ball?

\[
I = m(v_f - v_i) = (0.30 \text{ kg})(80 \text{ m/s} - (-42 \text{ m/s})) = 0.30(122) = +37 \text{ kg m/s} \quad \text{(Impulse is to the right.)}
\]

\[
F = \frac{\Delta p}{\Delta t} = \frac{37 \text{ kg m/s}}{0.010 \text{ s}} = 3700 \text{ N} \approx 800 \text{ lbs} \quad \text{Bat exerts a BIG force for a short time.}
\]

**Proof that momentum is conserved**

Now finally, we are ready for the proof that momentum is conserved in collisions. We are going to show that Newton's 3rd Law guarantees that

\[
(\text{total momentum before collision}) = (\text{total momentum after collision})
\]
We will show that when two objects (A and B) collide, the total momentum \( \vec{p}_{\text{tot}} = \vec{p}_A + \vec{p}_B \) remains constant because \( \Delta \vec{p}_A = -\Delta \vec{p}_B \); that is, the change in momentum of object A is exactly the opposite the change in momentum of object B. Since the change of one is the opposite of the change of the other, the total change is zero: \( \Delta \vec{p}_{\text{ext}} = \Delta \vec{p}_A + \Delta \vec{p}_B = \Delta \vec{p}_A - \Delta \vec{p}_A = 0 \).

Here's the proof: When two objects collide, each exerts a force on the other. NIII says that each feels the same-sized force \( F \), but in opposite directions. Each object experiences the same-sized force for the same duration \( \Delta t \). So each object receives the same-sized impulse \( I = \Delta p = F \cdot \Delta t \), but with opposite directions. Done.

1D collision:

Before: \( \begin{align*}
m_A & \quad v_A \\
m_B & \quad v_B
\end{align*} \)

During: \( \begin{align*}
F & \quad (\text{Each feels same magnitude } F, \\
& \quad \text{for same duration } \Delta t.)
\end{align*} \)

\( \Delta p_A = -F \Delta t < 0 \quad \Delta p_B = +F \Delta t > 0 \)

\[ \Rightarrow \Delta p_A + \Delta p_B = 0 \quad \Rightarrow \Delta (p_A + p_B) = 0 \quad \Rightarrow p_A + p_B = \text{constant} \]

The total momentum is constant, if all forces acting are internal to system, so the system is isolated from outside forces. If there are forces from outside the system, then the system's total momentum can change. But any momentum change of the system must be due to transfer of momentum between the system and its surroundings.

Example of Conservation of Energy and Momentum: The Ballistic Pendulum. The ballistic pendulum is a simple device which can accurately measure the speed of a bullet. It consists of a block of wood hanging from some strings. When a bullet is fired into the block, the kick from the bullet cause the block to swing upward. From the height of the swing, the speed of the bullet can be determined.

Bullet of mass \( m \), with unknown initial velocity \( v_1 \), is fired into a large wooden block of mass \( M \), hanging at rest from strings.

\[ p_{\text{tot}} = m v_1 \]
Immediately after collision, bullet is buried in block, but block has not yet had time to move. Impulse from bullet gives block+bullet a velocity $v_2$.

Momentum conservation $\Rightarrow m \cdot v_1 = (M + m) \cdot v_2 \quad (1)$

Momentum is conserved, but KE is not. Most of the bullet's initial KE has been converted to thermal energy: bullet and block get hot. Some KE is left over: $KE = \frac{1}{2}(m + M) v_2^2$

Block+bullet rise to max height $h$, which is measured.

Conservation of energy $\Rightarrow$ $KE_i + PE_j = KE_f + PE_f$

$$\frac{1}{2} (M + m) v_2^2 = (M + m) g h \quad (2)$$

Now have 2 equations [(1) and (2)] in two unknowns ($v_1$ and $v_2$). So you can solve for the velocity of the bullet $v_1$ terms of the knowns ($m$, $M$, $g$, and $h$).

**Elastic Collisions**

Before:

$$\begin{align*}
& \text{m}_A \quad v_A \\
& \text{m}_B \\
\end{align*}$$

After:

$$\begin{align*}
& \text{m}_A \quad \text{v}’_A \\
& \text{m}_B \quad \text{v}’_B
\end{align*}$$

In a collision between two masses, momentum is ALWAYS conserved (when there are no outside forces). So, for an isolated system, we can always write:

$$\bar{p}_{tot} = m_A \bar{v}_A + m_B \bar{v}_B = m_A \bar{v}’_A + m_B \bar{v}’_B$$

If the collision is elastic, then KE is also conserved, so we can also write:

$$KE_{\text{Before}} = KE_{\text{After}} \quad , \quad \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A v'_A^2 + \frac{1}{2} m_B v'_B^2$$

If the initial conditions (masses and initial velocities) are known, and we seek the final velocities, then we have two equations (Conserv of p, Conserv of KE) in two unknowns ($v_A'$ and $v_B'$), and it is possible to solve. But the algebra gets very messy, because of the squared terms in the KE equation.
It turns out that when the collision is elastic, the relative velocity of the two objects (velocity of one relative to the other) is reversed, according to the equation:

\[
(\vec{v}_B - \vec{v}_A) = -(\vec{v}'_B - \vec{v}'_A) \quad \text{(elastic collision)}
\]

Because this equation has no squared terms, it is much easier to use than the KE conservation equation. This equation says that the relative velocity of approach before the collision is the negative of the relative velocity after the collision. The proof of this equation is in the Appendix.

**Example of elastic collision in 1D:** A mass \(m_A = 10m\) with initial velocity \(v_A\) collides head-on with a mass \(m_B = m\) that is at rest. What are the final velocities, \(v'_A\) and \(v'_B\), of the two masses?

Before:

\[
\begin{array}{c}
10m \\
\vec{v}_A \\
\text{(rest)}
\end{array}
\]

After:

\[
\begin{array}{c}
10m \\
\vec{v}'_A \\
\text{} \\
\vec{v}'_B
\end{array}
\]

Here \(v_B\) (initial velocity of object B) is zero, so Conservation of Momentum gives:

\[
10m v_A = 10v'_A + m v'_B \quad \text{(m's cancel)} \Rightarrow 10v_A = 10v'_A + v'_B \quad (*)
\]

Because the collision is elastic (meaning KE is conserved), we can write

\[
(\vec{v}_B - \vec{v}_A) = -(\vec{v}'_B - \vec{v}'_A), \quad (v_B = 0) \Rightarrow v_A = v'_B - v'_A, \quad v'_B = v_A + v'_A
\]

Substitution into (*) gives ..

\[
10v_A = 10v'_A + v_A + v'_A, \quad 9v_A = 11v'_A, \quad v'_A = \frac{9}{11}v_A
\]

\[
v'_B = v_A + v'_A = \frac{11}{11}v_A + \frac{9}{11}v_A = \frac{20}{11}v_A
\]

Notice that the big mass is slowed by the collision (makes sense) and the little mass is shot forward with a velocity that is larger than the initial velocity of the big mass.

**Appendix.** Proof of \((\vec{v}_B - \vec{v}_A) = -(\vec{v}'_B - \vec{v}'_A)\) for elastic collisions.

Working in 1D, so we can drop the "vector arrow" notation. Conservation of momentum gives
m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \quad (1)

Conservation of KE gives

m_A v_A^2 + m_B v_B^2 = m_A v'_A^2 + m_B v'_B^2 \quad (2) \quad \text{[We've cancelled out all the (1/2) factors.]}

We can rearrange these equations to put all the m_A terms on one side and all the m_B terms on the other:

(1) \Rightarrow m_A (v_A - v'_A) = m_B (v'_B - v_B) \quad (3)

m_A (v_A^2 - v'_A^2) = m_B (v'_B^2 - v_B^2)

(2) \Rightarrow m_A (v_A + v'_A)(v_A - v'_A) = m_B (v'_B + v_B)(v'_B - v_B) \quad (4)

[We have used the identity \((x^2 - y^2) = (x + y)(x - y)\)]

If we divide equation (4) by equation (3), we get:

\[
\frac{m_A (v_A + v'_A)(v_A - v'_A)}{m_A (v_A - v'_A)} = \frac{m_B (v'_B + v_B)(v'_B - v_B)}{m_B (v'_B - v_B)}
\]

Notice that almost everything cancels out in this equation, leaving only

\((v_A + v'_A) = (v'_B + v_B)\), which is the same as \((v_B - v_A) = -(v'_B - v'_A)\)