Currents make B-fields

We have seen that charges make E-fields: \[ \text{dE} = \frac{1}{4\pi \varepsilon_0} \frac{\text{dQ}}{r^2} \hat{r} \, . \]

Currents make B-fields according to the **Biot-Savart Law**:

\[ \text{dB} = \frac{\mu_0}{4\pi} \frac{\text{d}\ell \times \hat{r}}{r^2} \]

where \( \mu_0 = \text{constant} = 4\pi \times 10^{-7} \) (SI units).

\( \text{dB} \) is the element of B-field due to the element of current \( \text{d}\ell \). \( \text{d}\ell \) is an infinitesimal length of the wire, with direction given by the current. The total B-field due to the entire current is \( \vec{B} = \int \text{dB} = \frac{\mu_0 I}{4\pi} \int \frac{\text{d}\ell \times \hat{r}}{r^2} \).

This can be a very messy integral!

This law was discovered experimentally by two French scientists (Biot and Savart) in 1820, but it can be derived from Maxwell's equations.

**Example of Biot-Savart:** B-field at the center of a circular loop of current I, with radius R. Here the integral turns out to be easy.

\[ |\text{d}\ell \times \hat{r}| = \text{d}\ell |\hat{r}| \sin 90^\circ = 1 \implies \text{dB} = \frac{\mu_0 I}{4\pi} \frac{\text{d}\ell}{R^2} \]

We can replace the vector integral

\( \vec{B} = \int \text{dB} \) with the scalar integral \( B = \int \text{dB} \)

because all of the dB's point in the same direction.

\[ B = \int \text{dB} = \frac{\mu_0 I}{4\pi R^2} \int \text{d}\ell = \frac{\mu_0 I}{2R} \, . \]

The full field at all positions near a current loop requires very messy integrations, which are usually done numerically, on a computer. The full field looks like this:
Another, more difficult example of the Biot-Savart Law: B-field due to a long straight wire with current $I$. The result of a messy integration is

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

This formula can be derived from a fundamental law called Ampere's Law, which we describe below. The B-field lines form circular loops around the wire.

To get directions right for both these examples (B due to wire loop, B due to straight wire), use "Right-hand-rule II": With right hand, curl fingers along the curly thing, your thumb points in direction of the straight thing.

**Force between two current-carrying wires:**

Current-carrying wires exert magnetic forces on each other. Wire2 creates a B-field at position of wire 1. Wire1 feels a force due to the B-field from wire 2: $F_{on1} = I_1 \vec{L} \times \vec{B}_2$

$$\Rightarrow F_{on1} = I_1 L \vec{B}_2 = I_1 L \frac{\mu_0 I_2}{2\pi r}.$$  

force per length between wires $= \frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi r}$

- Parallel currents attract
- Anti-parallel currents repel.

"Going my way? Let's go together.
Going the other way? Forget you!"
Gauss's Law for B-fields

B-field lines are fundamentally different from E-field lines in this way: E-field lines begin and end on charges (or go to $\infty$). But B-field lines always form closed loops with no beginning or end. A hypothetical particle which creates B-field lines in the way a electric charge creates E-field lines is called a magnetic monopole. As far as we can tell, magnetic monopoles, magnetic charges, do not exist. There is a fundamental law of physics which states that magnetic monopoles do not exist.

Recall the electric flux through a surface $S$ is defined as $\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a}$. In the same way, we define the magnetic flux through a surface as $\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{a}$. Gauss's law stated that for any closed surface, the electric flux is proportional to the enclosed electric charge:

$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enc}}}{\varepsilon_0}$. But there is no such thing as "magnetic charge", so the corresponding equation for magnetic fields is $\oint \mathbf{B} \cdot d\mathbf{a} = 0$

This equation, which has no standard name, is one of the four Maxwell Equations. It is sometimes called "Gauss's Law for B-fields".

Ampere's Law

Ampere's Law gives the relation between current and B-fields:

For any closed loop $\mathcal{L}$, $\oint_{\mathcal{L}} \mathbf{B} \cdot d\mathbf{\ell} = \mu_0 I_{\text{enclosed}}$, where $I_{\text{enclosed}}$ is the current through the loop $\mathcal{L}$.
(It will turn out the Ampere's Law is only true for constant current $I$. If the current $I$ is changing in time, Ampere's Law requires modification.) Ampere's Law for steady currents, like Gauss's Law, is a fundamental law of physics. It can be shown to be equivalent to Biot-Savart Law.

We can use Ampere's Law to derive the B-field of a long straight wire with current $I$.

**B-field of a long straight wire:**

![Diagram of a long straight wire with a circular loop labeled $\mathcal{L}$ and a radius $r$. The current $I$ flows along the wire, and the B-field is tangential to the loop.]

$L = \text{imaginary circular loop of radius } r$

We know that $\mathbf{B}$ must be tangential to this loop; $\mathbf{B}$ is purely azimuthal; $\mathbf{B}$ can have no radial component toward or away from the wire. How do we know this? A radial component of $\mathbf{B}$ is forbidden by Gauss's Law for B-fields. Alternatively, we know from Biot-Savart that $\mathbf{B}$ is azimuthal. So, in this case, $\mathbf{B} \cdot d\ell = B d\ell$. Also, by symmetry, the magnitude of $\mathbf{B}$ can only depend on $r$ (distance from the wire): $B = B(r)$.

$$\oint_{\mathcal{L}} \mathbf{B} \cdot d\ell = \oint B \, d\ell = B \oint d\ell = B \cdot 2\pi r = \mu_0 I$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

Like Gauss's Law, Ampere's Law is always true, but it is only useful for computing $\mathbf{B}$ if the situation has very high symmetry.

**B-field due to a solenoid.** solenoid = cylindrical coil of wire

It is possible to make a uniform, constant B-field with a solenoid. In the limit that the solenoid is very long, the B-field inside is uniform and the B-field outside is virtually zero.

Consider a solenoid with $N$ turns, length $L$, and $n = N/L = \# \text{ turns/meter}$
Permanent Magnets

Currents make B-fields. So where's the current in a permanent magnet (like a compass needle)? An atom consists of an electron orbiting the nucleus. The electron is a moving charge, forming a tiny current loop—an "atomic current". In most metals, the atomic currents of different atoms have random orientations, so there is no net current, no B-field.

In ferromagnetic materials (Fe, Ni, Cr, some alloys containing these), the atomic currents can all line up to produce a large net current.
In a magnetized iron bar, all the atomic currents are aligned, resulting in a large net current around the rim of the bar. The current in the iron bar then acts like a solenoid, producing a uniform B-field inside:

Why do permanent magnets sometimes attract and sometimes repel? Because parallel currents attract and anti-parallel current repel.