Static Equilibrium

An object is in static equilibrium (it is not moving) IF
1) it is not translating (not moving up, down, left, or right) AND
2) it is not rotating (not spinning CW or CCW)

(We are talking about motion in a 2D plane here.)

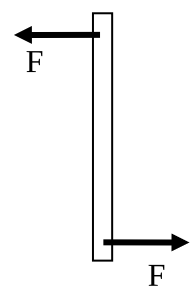
If a stationary mass is acted on by several forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, ..$, then in order to NOT translate, the net force must be zero.

$$\vec{F}_{net} = \vec{F}_{total} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots = 0$$

$$\Rightarrow \; F_{1x} + F_{2x} + F_{3x} + \ldots = 0 \,, \quad F_{1y} + F_{2y} + F_{3y} + \ldots = 0$$

$$\Rightarrow \; \boxed{\sum F_x = 0 \,, \quad \sum F_y = 0}$$ Equilibrium possible, but not guaranteed.

Even though the net force is zero, the object might not be in static equilibrium. Here is a case (two forces acting on a bar) where the net force is zero, but the forces cause the object to *rotate*:



In order to guarantee static equilibrium, we must have
1) net force = 0 AND
2) net torque = 0

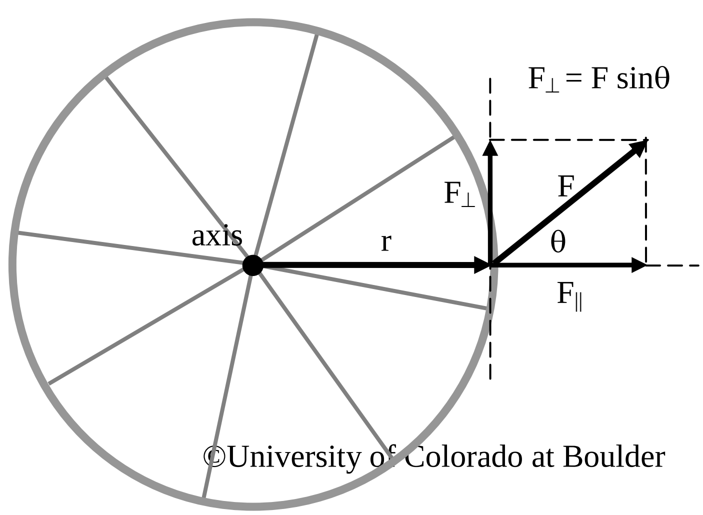
Remember torque:

torque (pronounced "tork") is a kind of "rotational force": $\vec{\tau} \equiv \vec{r} \times \vec{F}$

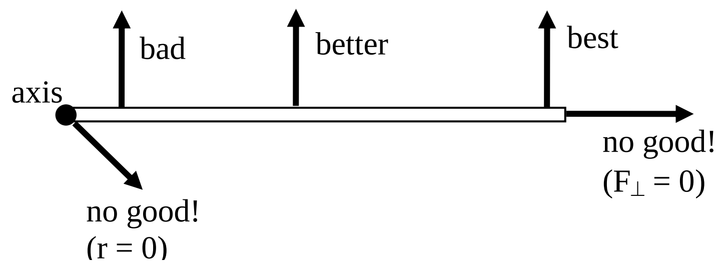
magnitude of torque: $\boxed{|\tau| \equiv r \cdot F_{\perp}}$ Unit of torque = $[\tau] = [r][F] = \text{m N}$

r = "lever arm" = distance from axis of rotation to point of application of force
$F_{\perp}$ = component of force perpendicular to lever arm vector

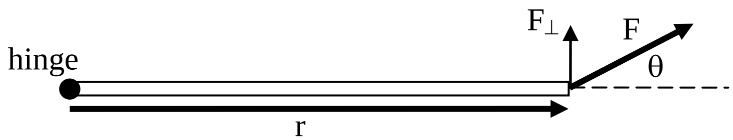
Example: Wheel on a fixed axis: Notice that only the perpendicular component of the force will rotate the wheel. The component of the force parallel to the lever arm ($F_{\|}$) has no effect on the rotation of the wheel.



If you want to easily rotate an object about an axis, you want a large lever arm r and a large perpendicular force $F_\perp$:

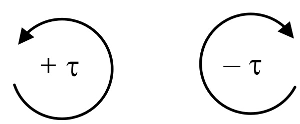


Example: Pull on a door handle a distance r = 0.8 m from the hinge with a force of magnitude F = 20 N at an angle $\theta = 30^o$ from the plane of the door, like so:



$\tau = r\,F_\perp = r\,F\sin\theta = (0.8\text{ m})(20\text{ N})(\sin 30^o) = 8.0\text{ m}\cdot\text{N}$

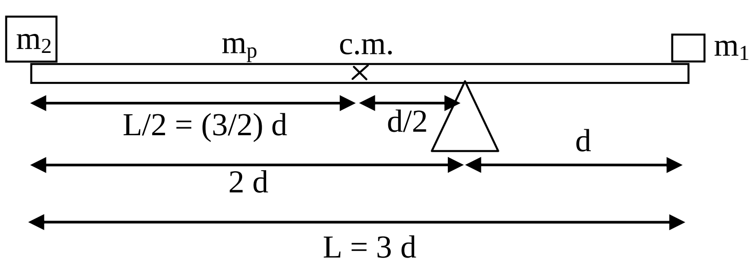
Torque has a sign (+ or –) :
Positive torque causes counter-clockwise (CCW) rotation.
Negative torque causes clockwise (CW) rotation.



An object is in rotational equilibrium only when the net torque ***about any axis*** is zero (when the negative torques cancel the positive torques). $\Rightarrow$ $\boxed{\sum_i \tau_i = 0}$

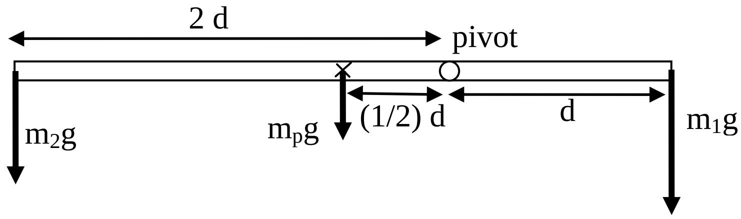
(Recall, from Chapter 10, that $\tau_{net} = I\alpha$. If the object is not rotating, we must have $\alpha = 0$, so $\tau_{net} = 0$.)

Example: Use of torque in static equilibrium. A uniform plank of mass m_p and length L is balanced on a pivot as shown with two masses m_1 and m_2 at the ends.
Knowns: m_p, m_2, d, L Unknown: m_1 = ?

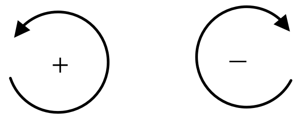


c.m. = "center of mass" or "center of gravity" of the plank is located at the middle of the plank (since it is a uniform plank). Key Idea about torque and c.m : As far as torques are concerned, the plank acts as if all its mass were concentrated at the center of mass.

Torque diagram showing forces and lever arms:



Plank will balance only if $\sum \tau = 0$, where $\tau = \pm\, r\ F_{\perp}$



$$+2d\,(m_2 g) \quad + \quad \left(\frac{d}{2}\right)(m_p g) \quad - \quad d\,(m_1 g) \quad = \quad 0$$

g's and d's cancel in this equation:

$$+2m_2 \quad + \quad \frac{m_p}{2} \quad - \quad m_1 \quad = \quad 0 \quad \Rightarrow \quad m_1 \quad = \quad 2m_2 + \frac{m_p}{2}$$

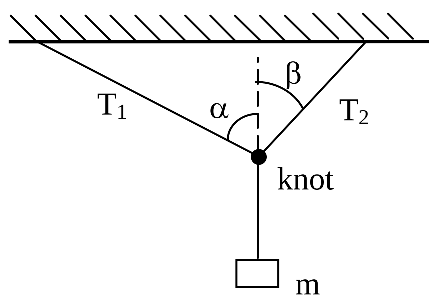
Notice that since the g's cancel, this plank will balance on the moon too.

Example: Static equilibrium problem with $\mathbf{F}_{net} = 0$, but no torque.

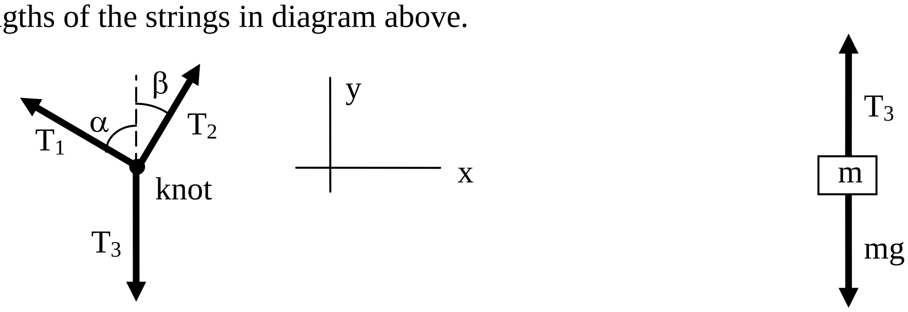
A mass m hanging from two strings like so:

Knowns: m, α, β

Unknowns: string tensions T_1 = ? T_2 = ?



Forces on knot: Notice that lengths of force arrows in diagram below have nothing to do with the lengths of the strings in diagram above.



Because the forces on the mass m must cancel, the tension in the bottom string = T_3 = mg.

The knot is a point object; there are no lever arms here, so no possibility of rotation, so we don't have to worry about torques. Apply equations of static equilibrium to the knot:

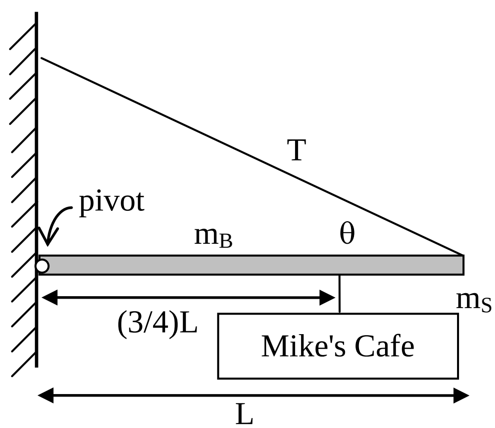
$$\sum F_x = 0 \quad \Rightarrow \quad -T_1 \sin\alpha + T_2 \sin\beta = 0 \quad \text{OR} \quad T_1 \sin\alpha = T_2 \sin\beta$$
$$[\ |F_{left}| = |F_{right}|\ ]$$

$$\sum F_y = 0 \quad \Rightarrow \quad +T_1 \cos\alpha + T_2 \cos\beta - mg = 0 \quad \text{OR} \quad +T_1 \cos\alpha + T_2 \cos\beta = mg$$
$$[\ |F_{up}| = |F_{down}|\ ]$$

Now have 2 equations in 2 unknowns (T_1 and T_2), so we can solve. (I'll let you do that!)

Harder Example: Static equilibrium problem with $\mathbf{F}_{net} = 0$ **and** $\sum \tau = 0$.

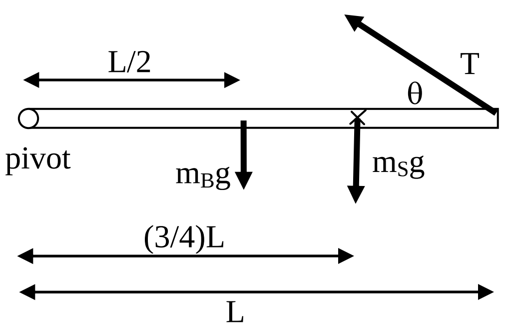
A store sign with mass m_s is hung from a uniform bar of mass m_b and length L. The sign is suspended from a point ¾ of the way from the wall. The bar is held up with a cable at an angle θ as shown. What is the tension T in the cable?



Knowns: m_B , m_S , L, θ

Unknown: tension T = ?

Torque diagram, showing forces and lever arms about the pivot:



Choose pivot as the axis:
$\sum \tau = 0 \quad \Rightarrow$

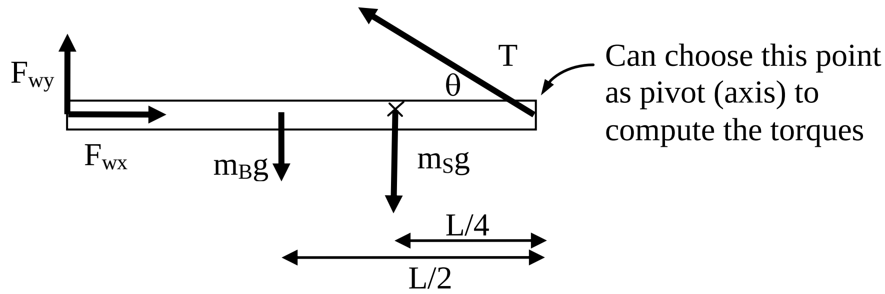
$$+L\,(T\sin\theta) - (3/4)L(m_S g) - (L/2)(m_B g) = 0$$

L's cancel, so

$$T\sin\theta = \tfrac{3}{4} m_S g + \tfrac{1}{2} m_B g \quad \Rightarrow \quad T = \frac{\left(\tfrac{3}{4} m_S + \tfrac{1}{2} m_B\right) g}{\sin\theta}$$

Another question: The wall is exerting some force $\vec{F}_W$ on the left end of the bar. What are the components F_{wx} and F_{wy} of this force?

All forces on bar:



Method I: Assume we know tension T (from previous problem). Then can use

$$\sum F_x = 0\,, \quad \sum F_y = 0 \quad \Rightarrow \quad F_{wx} = T\cos\theta\,, \quad F_{wy} = m_B g + m_S g - T\sin\theta$$

Method II: Assume that we do not know tension T.

Torque is always computed with respect to some axis or pivot point. If the object is not moving at all, we can pick <u>any</u> point as the axis. We can always pretend that the object is about to rotate about that point. Let us choose the right end of the bar as our pivot point. Then the tension force does not produce any torque (since the lever arm is zero), and the (unknown) variable T does not appear in our torque equation.

$$\sum \tau = 0 \quad \Rightarrow$$

$$\frac{L}{4} m_S g \quad + \quad \frac{L}{2} m_B g \quad - \quad L F_{wy} \quad = \quad 0 \qquad \text{(L's cancel)}$$

$$F_{wy} \quad = \quad \frac{m_S g}{4} \quad + \quad \frac{m_B g}{2}$$

(Still have to get F_{wx} using method I above.)

What is the magnitude of the total force on the bar from the wall?

$$F_w = \left|\vec{F}_w\right| = \sqrt{F_{wx}{}^2 + F_{wy}{}^2}$$