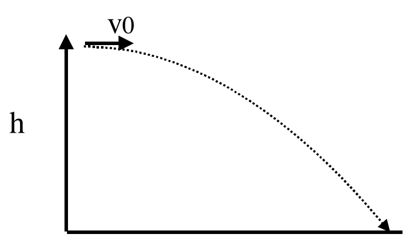
Gravity

Gravity is remarkable. An apple falls out of a tree- what made it accelerate? Nothing *touches* it. Saying the motion is due to "gravity" doesn't *explain* it. Einstein, in his general theory of relativity (1917) went very far in explaining *why* there is gravity. But let's follow Newton's path, and merely try to describe it...

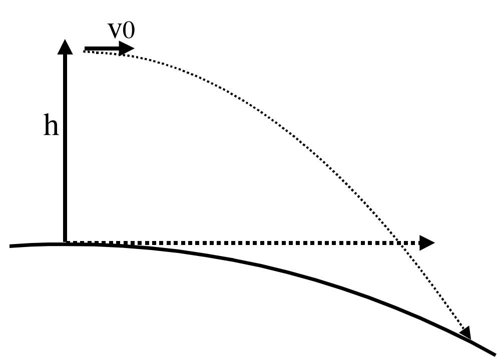
Gravity acts at a distance. Since the acceleration due to gravity is constant, gravity apparently acts at the top of trees as well as the bottom. How high does it reach? In Australia, gravity still acts towards the center of the earth. It appears that it is the earth itself which is doing the attracting.



Let's think about projectiles again. Suppose v0 is very big. So big, that in the few seconds it takes to fall "h", it has traveled a very long way horizontally.

Suppose it travels so far, that you finally notice the earth is *not* flat:

This projectile goes further than we thought in Chapter 4. It hits the ground later. What if v0 is bigger still? It hits further and farther away. Could it ever be going so fast that it *never* hits, that it keeps missing the ground? The answer is yes - the moon does this!



This was Newton's epiphany: the moon's motion is of the exact same nature - due to gravity - as falling apples. Gravity reaches up to the top of a tree, or mountains. Why not up to the moon? Why not further? Newton realized that there is *one* universal law of gravity, affecting all masses. One law that could explain and unify falling objects, projectiles, orbiting objects, and indeed all astronomy.

By 1650, the distance to the moon was well known: about 60 R_e (60 earth radii).

The moon's period T was also well known: about 27 days.

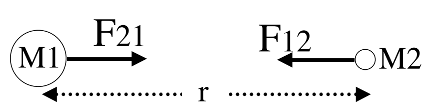
Newton could thus easily compute a(moon) = v(moon)^2 / R (using v = 2 Pi R/T)

Put in the numbers, you'll find a(moon) = 3E-3 m/s^2 = (1/3600)*g.

Conclusions: the moon is about 60 times further from the Earth's center than us, and a(moon) = (1/60^2)*g (whereas a(for any body near the earth) = g)

F(grav) on any object here, mg, is proportional to the mass. Newton III says the force should be equal (and opposite) on *both* objects that gravitationally attract: the formula for gravity must be symmetric in the masses. Newton put this all together:

If you have any *two objects, with masses M1 and M2 respectively, separated by a distance r (measured from center to center), there will be an attractive force of gravity between them*, given by



Newton's Universal law of Gravity: $\left|\mathbf{F}_{12}\right| = \left|\mathbf{F}_{21}\right| = F_{grav} = \frac{G M_1 M_2}{r^2}$

G is a constant of nature, $G = 6.67 \cdot 10^{-11}$ $N\ m^2 / kg^2$. (Not the same as g!)

G was not measured for > 100 years, by the way, until Cavendish (1798)

If an object is spherical (not a "point") it exerts an F(grav) on other bodies that is the exact *same* as if all the sphere's mass were concentrated at its center. (Newton worked for 20 years to prove this mathematically).

<u>Example:</u> What is the gravitational attraction between 2 humans, 1 meter apart?

$$F_{grav} = \frac{GM_1M_2}{r^2} = 6.67 \cdot 10^{-11} \frac{N\,m^2}{kg^2} \frac{(65\,kg)(65\,kg)}{(1m)^2} = 3 \cdot 10^{-7}\,N$$

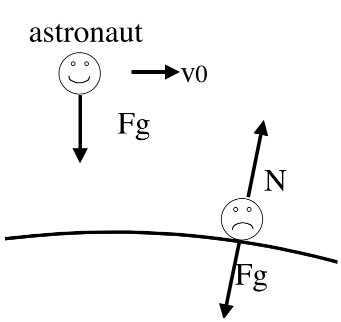
That's a *small* force. The weight of a mosquito is about 1000 times more than that.

<u>Example:</u> What is the force that the earth applies to an object with mass *m*?

$$F_{grav} = \frac{GM_1M_2}{r^2} = \frac{GM_{earth}m}{r_{earth}^2} = 6.67 \cdot 10^{-11} \frac{N\,m^2}{kg^2} \frac{(6 \cdot 10^{24}\,kg)(m)}{(6.4 \cdot 10^6\,m)^2} = 9.8 \frac{m}{s^2} m$$

Oh of course! The answer is *m*g, just as we've been using all semester.

If you climb 5 m to the top of a tree, r =6.4E6 m + 5 m => *no* noticeable numerical difference. Even at the top of Mt. Everest, taking r= 6.4E6 m + 8E3 m, the answer is still unchanged to two sig figs. At the orbital height of the Shuttle, h=250 km *above the surface*, the appropriate r for the formula (to find F(grav) on the shuttle, or the astronauts) is r(earth)+h = 6.4E6 m+2.5E5 m = 6.65E6 m.

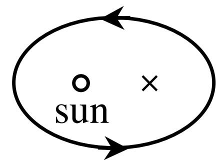


Still a fairly small numerical change. When you square it, the answer is slightly different (down to about 93% what it was at the surface.) We talked about this in Ch. 5. The astronauts aren't weightless, they're just in freefall (the *only* force acting on them is gravity).

Kepler's Laws, and Planetary Motion:

Kepler was 70 years before Newton. He discovered some patterns in the data of Tycho Brahe (this was before telescopes.) He summarized the data with the following laws, but he didn't understand *why* they were true.

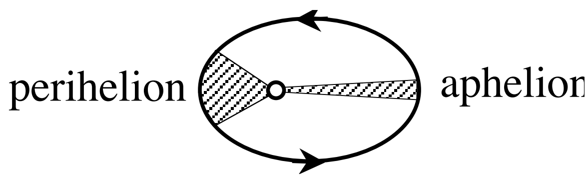
1) The orbit of all planets is elliptical, with the sun at a focus.



2) Planets "sweep out" equal areas in equal times.

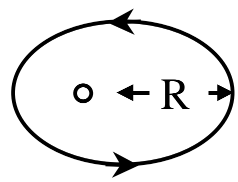
(Closest to sun = *perihelion* = fastest.

Farthest from sun = *aphelion* = slowest)



3) The period, T, of a planet and its distance R from the sun (technically, the "semimajor axis R of the elliptical orbit") satisfy the curious relation R^3 / T^2 =constant

(and it's the *same* constant for *all* the planets around the sun.)



Newton was able to show that these laws arise directly, mathematically, from his universal law of gravitation. (Compelling proof that his law of gravity was correct.)

Example: Let's prove Kepler's third law for the special case of circular orbits.

Circular orbit means a = v^2/R. Remember, N-II says F_net = *m*a = *m*v^2/R,

where *m* = mass of the planet, and R = radius of the planet's orbit around the sun.

What supplies this F_net? What's the physical force on the planet?

There's *only* gravity acting, so F_net = $F_{grav} = \frac{G M_{sun} m}{R^2}$.

Combine these, and notice that *m* cancels out. Thus, no matter what the mass of the *planet* is, we will have $\frac{v^2}{R} = \frac{GM_{sun}}{R^2}$, or if you prefer, $\boxed{v = \sqrt{\frac{GM_{sun}}{R}}}$.

If you have an orbit around something else, replace M(sun) with M(central object).

Remember that v = 2 Pi R/T, plug this in the 1st Eqn: $\frac{(4\pi^2 R^2 / T^2)}{R} = \frac{GM_{sun}}{R^2}$.

Now rearrange: $\boxed{\frac{R^3}{T^2} = \frac{GM_{sun}}{4\pi^2}}$. This is Kepler's third law. The right side is a constant for *all* planets. Newton has proven, and *generalized*, Kepler's third law.

Earth satellites (the moon, communications satellites, whatever) must also satisfy R^3/T^2=constant, but with a new constant (use M_earth on the right side now.) Conclusion: any satellite in orbit at some radius has a certain, definite, predictable period (and thus, velocity), independent of mass. We're doing rocket science here!

Example: Consider low earth orbit (e.g. the shuttle) so R=R(earth) + a little bit.

$v = \sqrt{\frac{GM_{earth}}{R}} \approx 8\,km/sec \approx 5\,mi/sec$. Just under 20,000 mi/hr.

Any slower and the shuttle will not be a circular orbit, it will fall. Any faster, and it will be go to a *higher orbit*. The period (time for the shuttle to go around the earth once) is about T = 2 Pi R/v = 90 minutes.

If you could throw a baseball (and have neither friction nor mountains get in the way) horizontally at 20,000 mi/hr, it would ALSO be in orbit! (R is pretty similar for the baseball and the shuttle, because R is measured from the earth's *center*.)

T is often easy to measure. (The period of the earth is one year, the period of the moon is 27 days, etc). So you can use Kepler's third law to deduce R. For the earth, we *know* R(from the sun) and T: we can figure out the Sun's mass. Cool!
When Cavendish measured G in 1798, he used the distance and period of the moon to deduce the mass of the earth. Cavendish weighed the earth! (Knowing the weight and size of the earth, we could then deduce our average density, and learn e.g. that most of the earth is not dirt, but iron...)

<u>Example:</u> Consider a satellite whose orbital period is exactly 24 hours. Think about this - the earth rotates once in 24 hours, and in that time the satellite has also run around the earth once. From *our* perspective on the ground, that satellite is at rest above us! This is convenient for TV or satellite communication - the satellite is always above the same spot. You aim your dish at it and don't have to keep tracking the satellite. We call this ***geosynchronous orbit***.. How high up is such a satellite?
$R^3 = \frac{GM_{earth}}{4\pi^2} T^2$. Plugging in all the constants, with T=24 hrs, and taking the cube root (figure out how to do that on your calculator) I get R=42,000 km.
This is a *very* high orbit. Since R_earth = 6,000 km, it is 36,000 km above the ground. (Compare to the shuttle, which is only about 250 km up)

Geosynchronous satellites have $v = \sqrt{GM_{earth} / R_{sat}}$ = 3 km/sec, *slower* than the shuttle. The farther out you are in orbit, the slower you go. (Bigger distance, but still bigger T.) Puzzler: Do you think the moon's speed is larger or smaller than 3 km/sec?

Energy considerations.

Last chapter, we said U(grav)=mgy, but that assumed *constant* gravity.

We must really go back to our Ch. 8 formula, $\Delta U = -\int_{\mathbf{r0}}^{\mathbf{rf}} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, and use

F=G M1 M2/r^2, not mg. The dot product has a "cos θ" in it, as usual. It's subtle:

if we're integrating UP in radius, gravity is DOWN, the angle between **F** and **dr** is

180 degrees, cos θ= -1, so this *cancels* the - sign in the formula.

$$\Delta U = +\int_{r0}^{rf} \frac{G M_1 M_2}{r^2}\, dr = G M_1 M_2 \left(-\frac{1}{r}\right)\Bigg|_{r0}^{rf}$$

$$U(r_f) - U(r_0) = G M_1 M_2 \left(\frac{1}{r_0} - \frac{1}{r_f}\right)$$

From this, we can argue that $\boxed{U_{grav}(r) = -G M_1 M_2 / r}$.   (NOTE: 1/r, *not* 1/r^2)

Think about that - sign, it's there to give back the formula we just *derived* for Δ U.

This definition of U(r) says the potential energy is *zero* when r gets very large.

A reasonable choice: far away objects feel nothing, no force, no potential energy....

(You can make other choices, the zero of U(r) is arbitrary, but this one is common.)

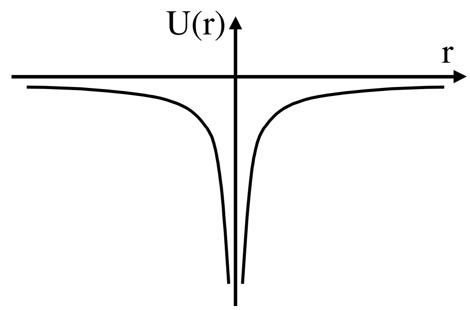
According to this formula, any object (closer than infinity) has a *negative* potential energy, meaning you would have to ADD energy to it to move it away to infinity, where it's defined to have zero potential energy. (Like a ball in the basement, which we said has negative potential energy, if we define U=0 at ground level.)

Here's a sketch of the potential energy

U(r)=-GM1M2/r.

This graph is called a "potential well".

The potential energy is HIGHER (or in this case, less negative, same thing) as you move out to larger radius r, which makes sense.



(As you move *away* from the earth, your potential energy is *higher*.)

I drew it symmetrical to remind you that the variable r is radius, from the center. You can head away from the earth in any direction you want, U(r) rises as you go off any which way. U(r) depends *only* on radius: the potential energy 20 feet off the ground is the same no matter where around the sphere you are.

<u>Escape velocity:</u> If you throw a rock straight up it goes up, stops, then comes down. Think of energy conservation - total energy is conserved: it starts off all kinetic, turns into gravitational potential, then turns back into kinetic on the way down. What if you throw it *very* hard? Can it ever escape, and never return?

The answer is yes. You need to give it a critical amount of initial speed, called the *escape velocity*, v_esc.

If v0<v_esc, it falls back down. If v0>v_esc, it runs away forever.

If v0 =v_esc, then the rock will JUST barely make it off to r=   : it will have zero energy out there (K=0, U=0, E_tot=0) But energy is always conserved: in this critical case, the initial total energy must have *also* been zero. Let's write this out:

For any object near the surface, $E_{tot,0} = K_0 + U_0 = \frac{1}{2} m v_0^2 - \frac{GM_{earth}\, m}{R_{earth}}$.

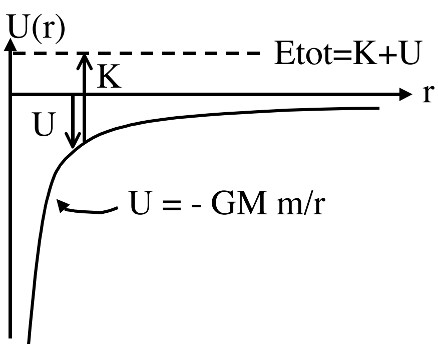
If we equate this to E(final)=0 (to find the critical initial, or *escape*, velocity)

$$v_{esc} = \sqrt{\frac{2GM_{earth}}{R_{earth}}}.$$

m has cancelled out. This is Sqrt[2] times bigger than the low-earth orbital velocity we found on p. 5. Only 40% more initial speed will let the shuttle escape earth!

If you throw the rock even harder, it has greater initial energy. It will escape, and *still* have some positive energy off at infinity, i.e. it will have K>0 far away.

Graphically, you can think of this like our old roller coaster pictures:



As always, K+U=E_tot = constant.

K>0, always. U<0 (for gravity)

The sum, E_tot, can be either + or -,

just depends how much energy you have to start with.

For the situation shown above, the sum is >0, you're "free", you will ultimately reach infinity with some + KE left over.

If the sum is negative, you're "bound", you will never reach infinity. (That's the case for all of us at the moment, and even the astronauts in the space shuttle.)

An object in circular orbit does *not* have Etot=0. It's in *orbit*, it's not running off to infinity. It has Etot<0, it is bound.   (Let's define M=M_earth, *m*=m_object)

For objects in circular orbit, E_tot =K+U= $\frac{1}{2}mv^2 - \frac{GMm}{R}$.

Using our earlier formula for v(orbital) (see notes, p 5), I find K= GMm/2R.

(Please try to check this result for yourself)

That means K=-U/2 (do you see this?) and so

E_tot = K+U = -U/2 + U = +U/2 = -GMm/2R = -K.

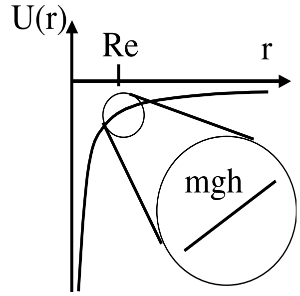
This is definitely all a little hard to understand at first, but it does make *some* sense.

- E_tot is NEGATIVE, any object in orbit is bound.
- K and U are intimately related for circular orbits (because v and R are related).
- The faster it goes (bigger K) the more negative E_tot is, that means it has *less* total energy. Yikes! Can that be? Faster means less energy? Yes, we just saw that objects in orbit go faster if they are *closer* to the earth. Closer objects are faster. They have LESS overall energy (they're lower down), but have *more* KE . Remember energy is the sum of K+U. In this case, as you go further out, K goes down, U goes up, but U *wins* (because of a curious factor of 2 in the equations)

All this weirdness is very real. When the space shuttle pilot wants to catch up to a satellite, she does NOT hit the gas (i.e. fire rockets directly aft). That would increase their energy => move them to a higher orbit. More energy, but *less* K, so they would watch the satellite go below them, and also start to move ahead of them! Very strange indeed. (Shuttle pilots need to study a lot of intro physics.)

Appendix: U(grav) near the earth.

Our new formula for gravitational potential energy, U(r)=-GM1M2/r, seems *way* different from the old formula U(h) = mgh. How can we reconcile them?



If you zoom in on that graph of U(r)=-GM1M2/r, near r=R(earth), it will *look* like a straight line. (Any curve looks straight if you zoom in on it). That's what we did before, when we said U(h) = mgh, we were zooming in on U(r) near the surface. But, how does it work out quantitatively? Just let $r = R_e + h$, and plug into our formula $U(r) = -\frac{GM_e m}{R}$, to get

$$U = -\frac{GM_e m}{R_e + h} = -\frac{GM_e}{R_e(1+\frac{h}{Re})}m.$$

It's a very useful mathematical fact that $\frac{1}{1+\varepsilon} \approx 1-\varepsilon$, if ε is small.

Here, h/Re is small (that's the assumption that we're near the surface: the height is small compared to the size of the earth), so the formula becomes

$$U \approx -\frac{GM_e}{R_e}(1-\frac{h}{Re})m = \text{constant} + \frac{GM_e}{R_e^2}h\,m$$

We showed (p.3) that G Me/Re^2 = g, so we have U= constant + mgh.

The value of the constant is irrelevant. Remember, you can always choose wherever you want to call U=0. (Adding a constant to U(r) doesn't change physics. All you care about is ΔU, and our constant cancels out when you subtract) So the formulas look different, but they are in complete agreement *if* you are near the earth's surface.