Special Relativity Essentials

An *event* is a location and a time: (x, y, z, c t). Events in different inertial reference frames are related by the Lorentz Transformation:

$$x' = \gamma(x - \beta ct), \quad ct' = \gamma(ct - \beta x), \quad \text{where } \beta = v/c, \text{ and } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

(primed frame moving in +x direction with speed v)

Different observers do not agree on the time duration or spatial distance between events, but they all agree on the *spacetime interval* s between events. The interval s is Lorentz-invariant (the same in every frame).

$$s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - (c\Delta t')^2$$

If the interval is less than zero, $s^2 < 0$, the events have a *time-like* separation and can be causally related. If $s^2 > 0$, the separation is *space-like*, and the events cannot be causally related.

Time dilation, "Moving clocks run slower" : $\Delta\tau = \frac{\Delta t}{\gamma}$ where τ is the *proper time*, the time measured in the rest frame of the particle, which is the time read on the face of the moving clock. Note that proper time is Lorentz-invariant; everyone agrees on where the hands are on the face of the moving clock. You can derive $\tau = t/\gamma$ from the Lorentz-transformation.

Length contraction, "Moving sticks are shorter": $\Delta L_0 = \gamma \Delta L$ where L_0 is the proper length, the length measured in the rest frame of the stick.

The momentum **p** and energy E of a particle with rest mass m and velocity **v** are given by

$$\vec{p} = \gamma m \vec{v}$$
$$E = \gamma mc^2$$

Total energy and total momentum are conserved in any process for an isolated system.

The energy and momentum of a particle are related by:

$$E^2 = (pc)^2 + (mc^2)^2$$ (Energy is part kinetic and part rest mass energy)

Nothing can go faster than c. If you could make a particle go faster than c, you could affect the past, and hence, violate causality.

Relativistic velocity addition formula: $v_{tot} = \dfrac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$

Notice that when $v_1 = v_2 = c$, then $v_{tot} = c$.

Proof of $E^2 = (pc)^2 + (mc^2)^2$

Any 4-component object that transforms in different frames according to the Lorentz Transformation is a *4-vector*. If we multiply a 4-vector by a Lorentz-invariant constant, such as the proper time or the rest mass m, we get another 4-vector.

Notation: $x^\mu = (\, x, y, z, ct \,)$ where $\mu = 1, 2, 3, 4$, $x_\mu = (\, x, y, z, -ct \,)$

$x^\mu x_\mu = x^2 + y^2 + z^2 - (ct)^2 = s^2$ (repeated indices are always summed over)

$$\Delta x^\mu = (\, \Delta x, \Delta y, \Delta z, c\Delta t \,) = (\, \Delta\vec{r},\ c\Delta t)$$

$$\frac{\Delta x^\mu}{\Delta \tau} = \gamma \frac{\Delta x^\mu}{\Delta t} = (\, \gamma v_x, \gamma v_y, \gamma v_z, \gamma c \,) = (\gamma \vec{v}, \gamma c) = \text{four-velocity}$$

$$m\frac{\Delta x^\mu}{\Delta \tau} = m\gamma \frac{\Delta x^\mu}{\Delta t} = (\gamma m \vec{v},\ \gamma m c) = p^\mu = \text{four-momentum}$$

For any four-vector, x^μ, the dot product $x^\mu x_\mu = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2$ is invariant. Let's apply this to the 4-momentum. In the rest frame of the particle, $v = 0$, $\gamma = 1$, and so $p^\mu = (0, mc)$. In non-rest frame, $p^\mu = (\gamma m \vec{v},\ \gamma m c)$. Since $p^\mu p_\mu$ is invariant, we have $(\gamma m v)^2 - (\gamma m c)^2 = -(mc)^2$ We can rewrite this as

$(p)^2 - (E/c)^2 = -(mc)^2$ Now multiply through by c^2 and rearrange to get

$$E^2 = (pc)^2 + (mc^2)^2$$