Last time:
- Spacetime, addition of velocities, Lorentz transformations

Today:
- Relativistic momentum and energy
- Review EM Waves and SR

HW03 due, beginning of class; HW04 assigned

Next week:
- Intro to quantum
- Exam I (Thursday, 2/10)

“Logic will get you from A to B. Imagination will take you everywhere.”
- Albert Einstein

PH300 Modern Physics SP11

2/3 Day 7:
Questions?
Relativistic Momentum & Energy
Review

Next Week:
Intro to Quantum
Exam I (in class)

Lorentz Transformations

Two clocks (one at A and one at B) are synchronized. A third clock flies past A at a velocity \( \mathbf{v} \). The moment it passes A all three clocks show the same time \( t_0 = 0 \) (viewed by observers in A and B. See left image.)

What time does the third clock show (as seen by an observer at B) at the moment it passes the clock in B? The clock at B is showing \( t_1 = 1 \) s at that moment. Use Lorentz transformation!

\[
\begin{align*}
A) \quad & \gamma (t_1 - t_0) \\
B) \quad & \gamma^2 (t_1 - t_0)(1 - \mathbf{v}^2/c^2) \\
C) \quad & \gamma^2 (t_1 - t_0)(1 + \mathbf{v}^2/c^2) \\
D) \quad & (t_1 - t_0) / \gamma \\
E) \quad & \gamma (t_1 - t_0)(1 + \mathbf{x}'/c^2)
\end{align*}
\]

The moving clock shows the proper time interval!! \( \Delta t_{\text{proper}} = \Delta t / \gamma \)

Hint: Use the following frames:

Two clocks (one at A and one at B) are synchronized. A third clock flies past A at a velocity \( \mathbf{v} \). The moment it passes A all three clocks show the same time \( t_0 = 0 \) (viewed by observers in A and B. See left image.)

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\end{align*}
\]

The moving clock shows the proper time interval!! \( \Delta t_{\text{proper}} = \Delta t / \gamma \)

Hint: Use the following systems:

The clock travels from A to B with speed \( \mathbf{v} \). Assume A is at position \( x = 0 \), then B is at position \( x = \mathbf{v} \cdot t, t = (t_1 - t_0) \).

Use this to substitute \( x \) in the Lorentz transformation:

\[
t' = \gamma (t - \frac{\mathbf{v}^2 t}{c^2}) = \gamma t (1 - \frac{\mathbf{v}^2}{c^2}) = \frac{t}{\gamma}
\]

\( \Rightarrow \) We get exactly the expression of the time dilation!

Relativistic Mechanics
Momentum

The classical definition of the momentum $p$ of a particle with mass $m$ is: $p = mu$.

In absence of external forces the total momentum is conserved (Law of conservation of momentum):

$$\sum_{i=1}^{n} p_i = \text{const.}$$

Due to the velocity addition formula, the definition $p = mu$ is not suitable to obtain conservation of momentum in special relativity!!

$\rightarrow$ Need new definition for relativistic momentum!

Conservation of Momentum

If $u_1 = -u_2$ we find:

$$p_{tot, before} = 0$$
$$p_{tot, after} = 0$$

System $S'$ is moving to the right with the velocity $v = u_1$.

We will use relativistic velocity transformations here.

Classical Momentum

$$P_1, \text{before} = m(u_x, u_y)$$
$$P_2, \text{before} = m(-u_x, -u_y)$$
$$P_{tot, \text{before}} = m(0, 0)$$
$$P_1, \text{after} = m(u_x, -u_y)$$
$$P_2, \text{after} = m(-u_x, u_y)$$
$$P_{tot, \text{after}} = m(0, 0)$$

$\rightarrow P_{tot, \text{before}} = P_{tot, \text{after}}$

Galileo (classical):

$$P_1, \text{before} = m(0, u_y)$$
$$P_2, \text{before} = m(-2u_x, -u_y)$$
$$P_{tot, \text{before}} = m(-2u_x, 0)$$
$$P_1, \text{after} = m(0, -u_y)$$
$$P_2, \text{after} = m(-2u_x, u_y)$$
$$P_{tot, \text{after}} = m(-2u_x, 0)$$

$\rightarrow P_{tot, \text{before}} = P_{tot, \text{after}}$

Velocity Transformation (3D)

Classical:

$$u'_x = u_x - v$$
$$u'_y = u_y$$
$$u'_z = u_z$$

Relativistic:

$$u'_x = \frac{u_x - v}{1 - u_x v / c^2}$$
$$u'_y = \frac{u_y}{\gamma(1 - u_x v / c^2)}$$
$$u'_z = \frac{u_z}{\gamma(1 - u_x v / c^2)}$$

Lorentz Transformation

Use:

$$u'_x = \frac{u_x - v}{1 - u_x v / c^2}$$
$$u'_y = \frac{u_y}{\gamma(1 - u_x v / c^2)}$$
$$u'_z = \frac{u_z}{\gamma(1 - u_x v / c^2)}$$

Algebra

$\rightarrow P_{tot, \text{before}} \neq P_{tot, \text{after}}$
Conservation of momentum is extremely useful in classical physics. For the new definition of relativistic momentum we want:

1. At low velocities the new definition of \( p \) should match the classical definition of momentum.

2. The total momentum (\( \Sigma p \)) of an isolated system of bodies is conserved in all inertial frames.

Relativistic Momentum

The time dilation formula implies that \( dt = \gamma d\tau \). We can therefore rewrite the definition of the relativistic momentum as follows:

\[
p = \gamma m \frac{d\mathbf{r}}{dt} = \gamma m \mathbf{u}
\]

An important consequence of the Lorentz-factor \( \gamma \) is that no object can be accelerated past the speed of light. 😈

Relativistic Force

We can define the classical force using Newton’s law:

\[
F = ma
\]

This is equivalent to:

\[
F = \frac{dp}{dt}
\]

Using the definition of the relativistic momentum we obtain a suitable definition for a relativistic force:

\[
F = \frac{d}{dt} (\gamma m \mathbf{u})
\]

with \( \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \)

Classical vs. Relativistic Momentum

An electron has a mass \( m = 9 \cdot 10^{-31} \text{kg} \). The table below shows the classical and relativistic momentum of the electron at various speeds (units are \( 10^{-22} \text{kg} \cdot \text{m/s} \)):

<table>
<thead>
<tr>
<th>( u )</th>
<th>( p = m u ) classical</th>
<th>( p = m u ) relativistic</th>
<th>difference [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1c</td>
<td>0.273</td>
<td>0.276</td>
<td>1.1</td>
</tr>
<tr>
<td>0.5c</td>
<td>1.36</td>
<td>1.57</td>
<td>15.4</td>
</tr>
<tr>
<td>0.9c</td>
<td>2.46</td>
<td>5.63</td>
<td>128.9</td>
</tr>
<tr>
<td>0.99c</td>
<td>2.7</td>
<td>19.2</td>
<td>611.1</td>
</tr>
</tbody>
</table>

Particle A has half the mass but twice the speed of particle B. If the particles’ momenta are \( p_A \) and \( p_B \), then:

\[\text{a) } p_A > p_B \quad \text{b) } p_A = p_B \quad \text{c) } p_A < p_B \]

\( \gamma \) is bigger for the faster particle.
Relativistic Force
A particle with mass $m$ is at rest at $x = 0, t = 0$, and experiences a constant force, $F$.

Find the velocity $u$ of the particle as a function of time $t$

Force acting on the particle: $F$
Relativistic force: $F \equiv \frac{d}{dt} (\gamma mu)$. 
Therefore: $F \cdot dt = d(\gamma mu)$
Integrating both sides: $F \cdot t = \gamma mu = p$
(Remember, $F$ is a constant!)

Example: Relativistic force (cont.)
Now: Solve $\gamma \cdot m \cdot u = F \cdot t$ for the velocity $u$.
Dividing by $\gamma$ yields: $m \cdot u = \frac{F \cdot t}{\gamma} = F \cdot t \cdot \left(1 - \frac{(u/c)^2}{1}ight)$
Square both sides: $m^2 \cdot u^2 = F^2 \cdot t^2 \cdot \left(1 - \frac{(u/c)^2}{1}ight)$
Bring $u$ to the left: $u^2 \left(m^2c^2 + F^2t^2\right) = F^2 t^2 c^2$
Divide by term in bracket and take the square root:
$$u = \frac{Fc}{\sqrt{(Ft^2) + (mc)^2}}$$

Energy
Similar to the definition of the relativistic momentum we want to find a definition for the energy $E$ of an object that fulfills the following:

1. At low velocity, the value $E$ of the new definition should match the classical definition.
2. The total energy ($\Sigma E$) of an isolated system of bodies should be conserved in all inertial frames.

Relativistic Kinetic Energy
The relativistic kinetic energy $K$ of a particle with a rest mass $m$ is:
$$K = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$$
Note: This is very different from the classical $K = \frac{1}{2}mv^2$.

For slow velocities the relativistic energy equation gives the same value as the classical equation! Remember the binomial approximation for $\gamma$:
$$\gamma \approx 1 + \frac{1}{2}v^2/c^2$$
$$K = \gamma mc^2 - mc^2 \approx mc^2 + \frac{1}{2}mc^2v^2/c^2 - mc^2 = \frac{1}{2}mv^2$$

Total Energy
We rewrite the equation for the relativistic kinetic energy and define the total energy of a particle as:
$$E = \gamma mc^2 = K + mc^2$$

This definition of the relativistic mass-energy $E$ fulfills the condition of conservation of total energy.
(Not proven here, but we shall see several examples where this proves to be correct.)
**Rest Energy**

\[ E = \gamma mc^2 = K + mc^2 \]

In the particle's rest frame, its energy is its rest energy, \( E_0 \). What is the value of \( E_0 \)?

A: 0
B: \( c^2 \)
C: \( mc^2 \)
D: \( (\gamma - 1)mc^2 \)
E: \( \frac{1}{2} mc^2 \)

**Equivalence of Mass and Energy**

\[ E_1 = \gamma mc^2 = K + mc^2 \]
\[ E_2 = \gamma mc^2 = K + mc^2 \]

Total energy:

\[ E_{\text{tot}} = E_1 + E_2 = 2K + 2mc^2 \]

**Example:**

**Rest energy of an object with 1kg**

\[ E_0 = mc^2 = (1 \text{ kg}) \cdot (3 \cdot 10^8 \text{ m/s})^2 = 9 \cdot 10^{16} \text{ J} \]

\[ 9 \cdot 10^{16} \text{ J} = 2.5 \cdot 10^{10} \text{ kWh} = 2.9 \text{ GW} \cdot 1 \text{ year} \]

This is a very large amount of energy! (Equivalent to the yearly output of ~3 very large nuclear reactors.)

Enough to power all the homes in Colorado for a year!

**How does nuclear power work?**

Atomic cores are built from neutrons and protons. There are very strong attractive forces between them. The potential energy associated with the force keeping them together in the core is called the binding energy \( E_B \).

We now know that the total rest energy of the particle equals the sum of the rest energy of all constituents minus the total binding energy \( E_B \):

\[ Mc^2 = \Sigma (m_i c^2) - E_B \]
**Definitions:**

We redefined several physical quantities to maintain the conservation laws for momentum and energy in special relativity.

- Relativistic momentum: \( p = m \frac{dr}{dt_{\text{proper}}} = \gamma m \frac{dr}{dt} = \gamma m u \)

- Relativistic force: \( F = \frac{dp}{dt} = \frac{d}{dt} (\gamma m u) \)

- Relativistic Energy: \( E = \gamma mc^2 = K + mc^2 \)  
  \( (K: \text{kinetic energy}) \)

**Important Relation**

(This results from the previous definitions)

- Total energy of an object: \( E = \gamma mc^2 \)

- Relativistic momentum of an object: \( p = \gamma m u \)

- Energy – momentum relation: \( E^2 = (pc)^2 + (mc^2)^2 \)

  - Momentum of a massless particle: \( p = E/c \)
  - Velocity of a massless particle: \( u = c \)

**We started somewhere here:**

How do you generate light (electromagnetic radiation)?

- **Stationary charges** → constant E-field, no magnetic (B)-field
- **Charges moving at a constant velocity** → Constant current through wire creates a B-field, but B-field is constant
- **Accelerated charges** → changing E-field and changing B-field  
  EM radiation → both E and B are oscillating

**Electromagnetic Waves**
1-Dimensional Wave Equation

\[ \frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} \]

Solutions are sines and cosines:

\[ E_y = A \sin(k_x x - \omega_t t) + B \cos(k_x x - \omega_t t) \]

...with the requirement that:

\[ k_x^2 = \frac{\omega_t^2}{c^2} \quad \text{or} \quad c = \frac{\lambda}{T} \]

A specific solution is found by applying boundary conditions.

Light is a wave: Two slit interference

1. A plane wave is incident on the double slit.
2. Waves spread out behind each slit.
3. The waves interfere in the region where they overlap.
4. Bright fringes occur where the antinodal lines intersect the viewing screen.

Double-slit experiment

**Determining the space between peaks (H)**

\[ \Delta r = r_2 - r_1 \]

For constructive

\[ \Delta r = D \sin(\Theta) = m \lambda \]

If screen far away

\[ \Theta_1 = \Theta_2 = \Theta \]

\[ \Delta r = D \sin(\Theta) = m \lambda \]

Are they in phase?

What's the difference in path?

\[ H = m \lambda \]

\[ \Theta = m \lambda / D \]

Electromagnetic waves carry energy

\[ E_{\text{max}} = \text{peak amplitude} \]

\[ E(x,t) = E_{\text{max}} \sin(ax - bt) \]

Intensity = \[ \text{Power} = \frac{\text{energy/time}}{\text{area}} \sim (E_{\text{avg}})^2 \]

\[ \sim (\text{amplitude of wave})^2 \sim E_{\text{max}}^2 \]

Light shines on three black barrels filled w/ water:

Which barrel will heat up the fastest?

A) \( 2 > 1 > 3 \)
B) \( 1 > 2 > 3 \)
C) \( 1 = 2 > 3 \)
D) \( 1 = 3 > 2 \)
E) \( 2 > 1 = 3 \)

(Use \( E_{1\text{max}} = E_{2\text{max}} > E_{3\text{max}} \))

Does not depend on frequency/color!
Classical waves: Intensity $\sim E_{\text{max}}^2$

Classically:
Time average of the E-field squared:
same… independent of frequency.

Intensity only depends on the E-field amplitude but not on the color (frequency) of the light!