PH300 Modern Physics SP11

2/1 Day 6:
Questions?
Spacetime
Addition of Velocities
Lorentz Transformations

Thursday:
Relativistic Momentum & Energy

Last time:
• Time dilation and length contraction

Today:
• Spacetime
• Addition of velocities
• Lorentz transformations

Thursday:
• Relativistic momentum and energy

HW03 due, beginning of class; HW04 assigned

Next week:
Intro to quantum
Exam I (in class)

“The only reason for time is so that everything doesn’t happen at once.”
- Albert Einstein

Spacetime Diagrams (1D in space)

Recall: Lucy plays with a fire cracker in the train.
Ricky watches the scene from the track.
Example: Lucy in the train

In Lucy’s frame: Walls are at rest

Lucy concludes: Light reaches both sides at the same time.

Example: Ricky on the tracks

In Ricky’s frame: Walls are in motion

Ricky concludes: Light reaches left side first.

Frame S’ as viewed from S

Frame S’ is moving to the right at \( v = 0.5c \). The origins of S and S’ coincide at \( t=t'=0 \). Which shows the world line of the origin of S’ as viewed in S?

These angles are equal

This is the time axis of the frame S’

This is the space axis of the frame S’

Frame S’ as viewed from S

In S: \((x,ct) = (3,3)\)

In S’: \((x',ct') = (1.8,2)\)

Both frames are adequate for describing events – but will give different spacetime coordinates for these events, in general.

Spacetime Interval
Distance in Galilean Relativity

The distance between the blue and the red ball is:

$\sqrt{(3m)^2 + (4m)^2} = \sqrt{25m^2} = 5m$

If the two balls are not moving relative to each other, we find that the distance between them is "invariant" under Galileo transformations.

Remember Lucy?

Event 1 – firecracker explodes
Event 2 – light reaches detector
Distance between events is $h$

Remember Ricky?

Event 1 – firecracker explodes
Event 2 – light reaches detector
Distance between events is $c\Delta t$

But distance between $x$-coordinates is $\Delta x'$

We can write

$h^2 = (c\Delta t')^2 - (\Delta x')^2$

And Lucy got $h = c\Delta t$

since $\Delta x = 0$

Spacetime interval

Say we have two events: $(x_1, y_1, z_1, t_1)$ and $(x_2, y_2, z_2, t_2)$. Define the spacetime interval (sort of the "distance") between two events as:

$\Delta s^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$

With:

$\Delta x = x_2 - x_1$
$\Delta y = y_2 - y_1$
$\Delta z = z_2 - z_1$
$\Delta t = t_2 - t_1$

The spacetime interval has the same value in all reference frames! I.e. $\Delta s^2$ is "invariant" under Lorentz transformations.

Spacetime interval

$\Delta s^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$

$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$

The spacetime interval has the same value in all reference frames! I.e. $\Delta s^2$ is "invariant" under Lorentz transformations.
Here is an event in spacetime.
The blue area is the future on this event.
The pink is its past.

Here is an event in spacetime.
The yellow area is the "elsewhere" of the event. No physical signal can travel from the event to its elsewhere!

Now we have two events A and B as shown on the left.
The space-time interval \((\Delta s)^2\) of these two events is:
A) Positive  B) Negative  C) Zero

If \((\Delta s)^2\) is negative in one frame of reference it is also negative in any other inertial frame! \((\Delta s)^2\) is invariant under Lorentz transformation). \(\rightarrow\) Causality is fulfilled in SR.

\((\Delta s)^2 > 0:\) Time-like events (A \(\rightarrow\) D)
\((\Delta s)^2 < 0:\) Space-like events (A \(\rightarrow\) B)
\((\Delta s)^2 = 0:\) Light-like events (A \(\rightarrow\) C)

\((\Delta s)^2\) is invariant under Lorentz transformations.

Example: Wavefront of a flash

Wavefront = Surface of a sphere with radius \(ct\):
\((ct)^2 - x^2 - y^2 - z^2 = 0\)

Spacetime interval for light-like event: \((\Delta s)^2 = 0\)

Einstein: "c" is the same in all inertial systems. Therefore: \((ct)^2 - x^2 - y^2 - z^2 = 0\) in all inertial systems!
(Here we assumed that the origins of S and S' overlapped at \(t=0\).)
An object moves from event $A=(x_1,t_1)$ to event $B=(x_2,t_2)$. As seen from $S$, its speed is
\[ u = \frac{\Delta x}{\Delta t}, \] where $\Delta x = x_2 - x_1$, $\Delta t = t_2 - t_1$.

As seen from $S'$, its speed is
\[ u' = \frac{\Delta x'}{\Delta t'}, \] where $\Delta x' = x_2' - x_1'$.

**Velocity transformation (1D)**

**Velocity transformation in 3D**

In a more general case we want to transform a velocity $\vec{v}$ (measured in frame $S$) to $\vec{v}'$ in frame $S'$. Note that $\vec{v}$ can point in any arbitrary direction, but $\vec{v}$ still points along the $x$-axes.

**Velocity transformation (3D)**

The velocity $\vec{v}=(u_x, u_y, u_z)$ measured in $S$ is given by:
\[ u_x = \frac{u_x - vt}{1 - uv/c^2}, \quad u_y = \frac{u_y}{\gamma(-u_xv/c^2)}, \quad u_z = \frac{u_z}{\gamma(-u_xv/c^2)}, \] where $\Delta x = x_2 - x_1$ ...

To find the corresponding velocity components $u'_x$, $u'_y$, $u'_z$ in the frame $S'$, which is moving along the $x$-axes in $S$ with the velocity $v$, we use again the Lorentz transformation:
\[ x'_1 = \gamma(x_1 - vt), \quad y'_1 = \gamma(x_1 - vx_1/c^2), \] and so on...

**Velocity transformation (3D) (aka. “Velocity-Addition formula”)**

\[ u'_x = \frac{u_x - v}{1 - u_xv/c^2}, \quad u'_y = \frac{u_y}{\gamma(-u_xv/c^2)}, \quad u'_z = \frac{u_z}{\gamma(-u_xv/c^2)} \]

**Some applications**
Suppose a spacecraft travels at speed \( v = 0.5c \) relative to the Earth. It launches a missile at speed \( 0.5c \) relative to the spacecraft in its direction of motion. How fast is the missile moving relative to Earth? (Hint: Remember which coordinates are the primed ones. And: Does your answer make sense?)

\[ a) \ 0.8 \ c \quad b) \ 0.5 \ c \quad c) \ \boxed{c} \quad d) \ 0.25 \ c \quad e) \ 0 \]

\[
\begin{align*}
x' &= \gamma(x - vt) \\
\gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \\
u' &= \frac{u - \gamma v}{\sqrt{1 - \gamma^2 v^2}}
\end{align*}
\]

The "object" could be light, too!
Suppose a spacecraft travels at speed \( v = 0.5c \) relative to the Earth. It shoots a beam of light out in its direction of motion. How fast is the light moving relative to the Earth? (Get your answer using the formula).

\[ a) \ 1.5c \quad b) \ 1.25c \quad c) \ \boxed{c} \quad d) \ 0.75c \quad e) \ 0.5c \]

\[ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \]

A note of caution:

The way the Lorentz and Galileo transformations are presented here assumes the following:
An observer in \( S \) would like to express an event \((x,y,z,t)\) (in his frame \( S \)) with the coordinates of the frame \( S' \), i.e. he wants to find the corresponding event \((x',y',z',t')\) in \( S' \). The frame \( S' \) is moving along the x-axes of the frame \( S \) with the velocity \( v \) (measured relative to \( S \)) and we assume that the origins of both frames overlap at the time \( t=0 \).

\[
\begin{align*}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma(t - \frac{v}{c^2} x)
\end{align*}
\]

Application: Lorentz transformation
Two clocks (one at \( A \) and one at \( B \)) are synchronized. A third clock flies past \( A \) at a velocity \( v \). The moment it passes \( A \) all three clocks show the same time \( t_0 = 0 \) (viewed by observers in \( A \) and \( B \). See left image.)

What time does the third clock show (as seen by an observer at \( B \)) at the moment it passes the clock in \( B \)? The clock at \( B \) is showing \( t_1 = 1s \) at that moment. Use Lorentz transformation!

A) \( \gamma (t - v/c) \)  
B) \( \gamma (t - \gamma v/c) \)  
C) \( \gamma (t - v/c^2) \)  
D) \( (t_1 - t_0)/\gamma \)  
E) \( \gamma (t_1 - t_0)(1 + v/c^2) \)  

\[ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \]
Two clocks (one at A and one at B) are synchronized. A third clock flies past A at a velocity $v$. The moment it passes A all three clocks show the same time $t_0 = 0$ (viewed by observers in A and B. See left image.)

What time does the third clock show (as seen by an observer at B) at the moment it passes the clock in B? The clock at B is showing $t_1 = 1s$ at that moment. Use Lorentz transformation!

A) $\gamma (t_1 - t_0)$  
B) $\gamma^2 (t_1 - t_0) (1 - v/c^2)$  
C) $\gamma^2 (t_1 - t_0) (1 + v/c^2)$  
D) $(t_1 - t_0)/\gamma$  
E) $\gamma (t_1 - t_0) (1 + vx'/c^2)$

The moving clock shows the proper time interval! $\Delta t_{\text{prop}} = \Delta t / \gamma$

The clock travels from A to B with speed $v$. Assume A is at position $x = 0$, then B is at position $x = vt$. $t = (t_1 - t_0)$

Use this to substitute $x$ in the Lorentz transformation:

$t' = \gamma \left( t - \frac{v^2 t}{c^2} \right) = \gamma t \left( 1 - \frac{v^2}{c^2} \right) = \frac{t}{\gamma}$

We get exactly the expression of the time dilation!