INSTRUCTORS MANUAL: TUTORIAL 6
Separation of Variables & Multipole Expansion

Goals:
1. Review older method of calculating potential, and connect it to newer material (course scale learning goals 5c, 10)
2. Expand solution into Taylor series (course scale learning goal 5a)
3. Understand where (and physically why) solutions to Laplace’s equation are valid (course scale learning goal 1)
4. See the interconnected nature of “separate” topics in E&M (course scale learning goal 3)
5. Introduce multipole expansion (by connecting it to older material)

This tutorial is based on:
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Tutorial Summary:
Students solve the potential by direct integration, then expand their solution into a Taylor series. They must explain where (in space) the separable solutions to Laplace’s equation are valid. Then they compare the terms of their Taylor series to the terms that result from separation of variables. Finally, they see how these terms are related to monopole and dipole moments.
Reflections on this tutorial

Part 1  
Students spend a considerable amount of time on this section, and many forgot the notation of primed versus un-primed coordinates (which was covered in the very first tutorial). There was also confusion about what to substitute for script-r. We observed productive arguments about signs (e.g. which "way" to integrate, and whether script r should be written as z+z' or z-z'). We encouraged students to hash these issues out for themselves.

In question (ii) students had a surprising amount of difficulty with the expansion. In earlier versions of this Tutorial we had given them the formula for Taylor expansion (see below this paragraph), but students had trouble figuring out what x_0 and ε needed to be, there was e.g. discussion about whether they were expanding about 0, 1, or some ill-defined "large z0". A few students admitted that they couldn't remember what a Taylor expansion even was. As a result, this problem took a lot longer than anticipated. Some students got a little discouraged here since they already had the exact answer in front of them and did not yet see the point of making this approximation. The following equivalent equations may be of use to students, but were removed from the tutorial because we found that including them caused further confusion of x, ε, x_0, and z. We leave it to the instructor’s discretion how long to let students work on the Taylor expansion section of this tutorial. (If you use our pretests, we have included a question about Taylor expanding ln(1+epsilon) to prime students for this.)

\[
f(x_0 + \epsilon) = \frac{f(x_0)}{0!} \epsilon^0 + \frac{f'(x_0)}{1!} \epsilon^1 + \frac{f''(x_0)}{2!} \epsilon^2 + ... \\
f(x) = \frac{f(x_o)}{0!} + \frac{f'(x)|_{x_o}}{1!}(x - x_o) + \frac{f''(x)|_{x_o}}{2!}(x - x_o)^2 + ...
\]

Part 2  
For question (i) there was significant discussion. Most (if not all) the students didn’t know, or had forgotten, where this form for the potential comes from. They knew that it was from separation of variables, but instructors may need to help them remember that it results from solving Laplace’s equation. Once Laplace’s equation was written, many students are quick to say that it’s where there is no charge. Among faculty there has been some discussion about regions of validity. The given equation is certainly valid for r>d (and then the boundary condition at infinity eliminates all non-negative powers of r) But for r<d, since there is charge present, Laplace's equation is not valid throughout this region, and it is not guaranteed that this form of the solution is appropriate. (You might consider a simpler, related problem of a single point charge at point z=-d . You will find that outside r>d, you want only inverse powers of r, but inside r<d, you want only positive powers. However, the details of this justification are perhaps beyond the level of this course.)

For question (ii), students know that the A’s have to be zero, though some don’t recognize right away that the P’s were 1 on the z-axis. Students didn’t think right away to match these terms with their Taylor series terms, but eventually saw how it made sense. One thing to watch for is that in their final answer, students leave out the Legendre polynomials. They either forget about them since they were 1 on the z-axis, or they think that they vanish. To prompt students to compare these terms to the terms from their Taylor expansion, the words “boundary
condition” could be used. By now, students are very familiar with applying boundary conditions. They could be told that one boundary condition is that on the z-axis, their solution must match what they solved previously and Taylor expanded. This is the section that most of the students were working on when the hour was over.

**Part 3** Very few students actually made it this far in the hour – unfortunate since this is the “punch line” of the tutorial. For those that did make it this far, question (i) seemed very clarifying when their monopole moment was Q, and dipole made sense physically. Question (ii) was a real puzzler for the students. They all agreed that the monopole wouldn’t change, since shifting it doesn’t change the total amount of charge. Some students just couldn’t make sense of why the dipole term vanishes. One student kept saying, “The choice of the origin shouldn’t change what’s happening physically.” They could see that the dipole term vanishes mathematically, but still couldn’t justify it physically.

**Ideas to shorten this tutorial**
The Taylor expansion in Part 1, Question (ii), took a lot of time exercising mathematical skills and confusing the focus of the tutorial. Consider revising or using as pre-test.

**Relevant Homework Problems**

**Separation of Variables: Disk**
A disk of radius R has a uniform surface charge density $\sigma_0$. Way back on Set #2 you found the E-field along the axis of the disk (and on the midterm, you again solved a very similar (but harder) version of this where $\sigma$ was not uniform). You can check for yourself by direct integration, (but don't have to): I claim that along the z axis, (i.e. $\theta=0$),

$$V(r, \theta=0) = \frac{\sigma_0}{2\varepsilon_0} \left( \frac{r^2 + R^2}{r} \right)$$

i) Find the potential away from the axis (i.e nonzero $\theta$), for distances $r > R$, by using the result above and fiddling with the Legendre formula, Griffiths’ 3.72 on page 140. You will in principle need an infinite sum of terms here - but for this problem, just work out explicitly what the first two *non-zero* terms are.

ii) Griffiths Chapter 3.4 talks about the "multipole expansion". Look at your answer to part i, and compare it to what Griffiths says it should look like (generically) on page 148. Discuss - does your answer make some physical sense? Note that there is a "missing term" - why is that?
TUTORIAL 6:  
★ZEN & THE ART OF MULTIPOLE EXPANSION★

Separation of variables and multipole expansion are two sides of the same cosmic coin

Part 1 – Potential from a Line of Charge

A uniform line charge density \( \lambda \) extends from the origin to the point (0,0,-d).

i. Using the script-\( r \) technique from earlier in the course, find an expression for the potential for points on the positive \( z \)-axis, \( V(z) \).

Recall:

\[
V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')d\tau'}{|r|}
\]

\[\lambda\]
\[-z = -d\]
Your answer to part i. should have been: \( V(z) = \frac{\lambda}{4\pi\varepsilon_0} \ln(1 + \frac{d}{z}) \). (If you didn’t get this, find out where you went wrong.) In the next part, we will be interested in \( V(z) \) for large \( z \) (i.e. small \( \frac{d}{z} \)). Defining a small parameter \( \varepsilon = \frac{d}{z} \), we get \( V(z) = \frac{\lambda}{4\pi\varepsilon_0} \ln(1 + \varepsilon) \), which can be Taylor expanded in powers of \( \varepsilon \).

tii. Find the first two non-zero terms of this Taylor expansion.
Part 2 – Separation of Variables

i. Consider the charge distribution from Part 1. This problem does not have spherical symmetry. Could the potential have the form:  
   \[ V(r, \theta) = \sum_l \left( A_l \cdot r^l + \frac{B_l}{r^{l+1}} \right) \cdot P_l(\cos \theta) \]  
   If so, in what regions could the solution look like this?

ii. Assuming the potential can have the form  
   \[ V(r, \theta) = \sum_l \left( A_l \cdot r^l + \frac{B_l}{r^{l+1}} \right) \cdot P_l(\cos \theta) \]  
   in the region you specified above, find the two leading non-zero A’s and/or B’s. Do any terms vanish? Keep in mind that this potential is the same potential you solved by integrating in part 1, so when \( \theta = 0 \) (the z-axis), the answers must match.

   \[
   \begin{align*}
   P_0(x) &= 1 \\
   P_1(x) &= x \\
   P_2(x) &= \frac{3}{2} x^2 - \frac{1}{2} \\
   P_3(x) &= \frac{5}{2} x^3 - \frac{3}{2} x
   \end{align*}
   \]
Part 3 – Multipole expansion

A potential can be expanded into the form:

\[ V = \frac{1}{4\pi\varepsilon_0} \left( \frac{\text{"monopole"}}{r} + \frac{\text{"dipole"}}{r^2} + \frac{\text{"quadrupole"}}{r^3} + \ldots \right) \]

i. For this problem, what are the monopole and dipole moments? Do these answers make sense physically?

ii. Which terms would change if the charge distribution were shifted up by \( d/2 \), so that it was centered on the origin?

iii. Does your answer to part ii make sense physically? What is the physical significance of the dipole term when there is a net charge?