Energy: Recall, we invented \( V(r) \) = "Voltage"

\[
V(r) = - \int_{\text{origin where } r=0}^{r} \mathbf{E} \cdot d\mathbf{l} \tag{1-2)(-36)}
\]

\[\leftarrow \text{So, given } \mathbf{E}, \text{ can always compute } V.\]

Then we showed, mathematically \( \mathbf{E} \) with Maxwell's help

\[
V(r) = \frac{1}{4\pi\varepsilon_0} \int_{\text{vol}} \rho(r') d\mathbf{r'} \tag{1-2)(-36)}
\]

\[\leftarrow \text{So, given } \rho, \text{ can always compute } V.\]

\[
\mathbf{E} = - \nabla V \leftarrow \text{So, once have } V, \text{ you know } \mathbf{E}. \tag{1-2)(-36)}
\]

\[
\nabla^2 V = - \frac{\rho}{\varepsilon_0} \leftarrow \text{So, } \ "\ " " " \ " " \ " \ \rho. \tag{1-2)(-36)}
\]

But what is it? What does it mean?

Think of moving a tiny rest charge \( q \) through \( \mathbf{E} \) fields from \( a \) to \( b \). "Electric force \( = q \mathbf{E} \), so \( \mathbf{F}_{\text{electric}} = -q \mathbf{E} \) to "fight the field."

To go from \( a \) to \( b \), you do \( \mathbb{W}_{\text{ext}} = \int_{a}^{b} \mathbf{F}_{\text{you}} \cdot d\mathbf{l} \)

\[
= -q \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} \tag{1-2)(-36)}
\]

Look up \( \Delta \rho \),

\[
\mathbb{W}_{\text{ext}} = q \Delta V_{ab} = q \left( V(b) - V(a) \right) \tag{1-2)(-36)}
\]

\[\text{So voltage carries info about work/energy!} \]
In III0, if you do work, can talk about stored potential energy. Here, \[ PE = q V \] (from prev. page)

(Note, always ambiguity, can define \( PE = 0 \) anywhere)

so \( V(\vec{r}) = \frac{PE}{q} = \text{the potential energy per unit charge} \)

\[ \text{could call this } p.e. = U(\vec{r}) = q V(\vec{r}) \]

Griffiths calls it \( W \), it's work needed by you to get \( q \) to this point.

That's "\( PE \) of a charge \( q \) in presence of others". But what about work to get others together?! Start from scratch, build up any given distribution of \( q \)'s, how much work?

That will tell us "stored electrostatic energy of system"

So let's bring in, one at a time, \( q_1, q_2, q_3 \ldots \) and figure out total work.
Bring in $g_1$. No other $g$'s $\implies$ no work.

Bring in $g_2$. $g_1$ is there, producing field.

So $W_2 = g_2 \cdot V_{\text{caused by } g_1} = g_2 \cdot \left( \frac{1}{\mu \varepsilon_0} \cdot \frac{g_1}{R_{12}} \right)

Now bring in $g_3$.

$W_3 = g_3 \cdot V_{\text{caused by } g_1, g_2} = g_3 \cdot \frac{1}{\mu \varepsilon_0} \left( \frac{g_1}{R_{13}} + \frac{g_2}{R_{23}} \right)

Total so far is $W_1 + W_2 + W_3$

$W_{\text{system}} = \frac{1}{\mu \varepsilon_0} \left( \frac{g_1 g_2}{R_{12}} + \frac{g_1 g_3}{R_{13}} + \frac{g_2 g_3}{R_{23}} \right)$

See pattern? Extends to any #.

Add all pairs $\frac{g_i g_j}{R_{ij}}$ but don't compute "self energy" $i=j$

Add all pairs $\frac{g_i g_j}{R_{ij}}$ and don't double count.

or, do double count and then divide by 2!!

So $W_{\text{system}} = \frac{1}{2 \cdot \mu \varepsilon_0} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{g_i g_j}{R_{ij}}$

Note: Could be negative!
Can reorganize this

\[ W_{\text{sys}} = \frac{1}{2} \sum_{i=1}^{n} q_i \left( \sum_{j \neq i}^{\infty} \frac{q_j}{4\pi \epsilon_0 r_{ij}} \right) \]

What's in parens? It looks like \( \tilde{V}(\vec{r}_i) \)

The voltage you get at point "i" due to all the other charges at all points \( j \neq i \), (Being careful not to include "self energies")

So \( W_{\text{sys}} = \frac{1}{2} \sum_{i=1}^{n} q_i \tilde{V}(\vec{r}_i) \)

\[ W_{\text{sys}} = \frac{1}{2} \int dq \cdot \tilde{V}(\vec{r}) = \frac{1}{2} \int \tilde{V}(\vec{r}) \rho(r') \, dr' \]

Here \( \tilde{V}(\vec{r}) \) = the potential at point \( \vec{r} \) due to all \( \rho \) except right at \( \vec{r} \), but this is irrelevant issue for \( \rho(1) \)

There is no charge in an infinitesimal volume...
\[ W_{\text{sys}} = \frac{1}{2} \int \rho(r) V(r) \, d^3r \equiv \text{Total Energy of a system.} \]

where is it located? In the \( E \) fields! Let's see how...

\[ \rho = \epsilon_0 \nabla \cdot \vec{E} \], so \( W_{\text{sys}} = \frac{\epsilon_0}{2} \int_{\text{vol}} (\nabla \cdot \vec{E}) \cdot \vec{V} \, d^3r \]

But \( \int u \, dv = uv - \int v \, du \), or in 3-D see eq'n 1.59

\[ W_{\text{sys}} = \frac{\epsilon_0}{2} \left[ \left( \int_{\text{vol}} \vec{V} \cdot \nabla \cdot \vec{E} \, d^3r \right) - \int_{\text{vol}} \vec{E} \cdot \nabla \vec{V} \, d^3r \right] \]

If volume is "all space", \( \vec{V} \to 0 \) far away!

So, as long as all charges are localized (no good for "infinite sheet", e.g.)

\[ W_{\text{sys}} = \frac{\epsilon_0}{2} \int_{\text{vol}} \vec{E} \cdot \nabla \vec{V} \, d^3r \]

But \( \nabla \vec{V} = -\vec{E} \), so

\[ W_{\text{sys}} = \frac{1}{2} \epsilon_0 \int_{\text{vol}} E^2 \, d^3r \]

Cool. It's \( E \) that stores the energy,

\[ \frac{1}{2} \epsilon_0 E^2 \] tells you "stored energy/m^3".

[But, should really only use this with continuous \( \rho \)'s.]

[If have discrete point charges, go back to the sum formula.]