Method of Images:

Remember, if \( \nabla^2 V = \rho / \varepsilon_0 \) in a region, and if we can guess a \( V \) which obeys \( \nabla^2 V \), and which is correct at boundary of our region, we're done, we have \( V(r) \)!

Consider charge \( q \), distance \( d \) from \( \sigma \) sheet of conductor

What is \( V(r^2) \) throughout all space? (Not simple, 'cause you induce a complicated \( \sigma \) on sheet!)

We know \( V = 0 \) on conductor. (Grounded) equipotential!

We know \( V = 0 \) off at \( \infty \). so we know \( V \) on giant boundary (wall on left, \( \infty \) everywhere else)

and we want \( V \) in rest of space. (The "right half" of universe)

We will use a trick. I can create an imaginary scenario where \( V = 0 \) everywhere on \( x=0 \) plane and \( \nabla^2 V = \rho / \varepsilon_0 \) in "right half", and \( V(r) \) is easy to compute.

If I do this, we're done. By uniqueness, that is the unique sol'n.

It satisfies all our boundary conditions and Poisson's eq'n.

(And, we won't need to know the complicated \( \sigma \) on the sheet, which nature generates to cancel \( \hat{E} \) out to 0 inside conductor.)
The trick: Consider this problem:

\[ V(r) \text{ here is just} \quad \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{r - d} + \frac{q}{r + d} \right) \text{ by inspection} \]

1) \( V(\infty) = 0 \). Good!

2) \( V(\infty) = 0 \). Distance is same \( r = \pm g \)!

\( \nabla^2 V(r) = \rho / \varepsilon_0 \), for right half of universe, (this is 0 everywhere except \( q \delta^{(3)}(r - d) \), it's perfectly fine)

So for right half of universe our \( V \) has correct B.C. solves Poisson. So we're done.

That's it!

\[ V(x, y, z) \bigg|_{x>0} = \frac{q}{4\pi\varepsilon_0} \left( \frac{1}{\sqrt{(x-d)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+d)^2 + y^2 + z^2}} \right) \]
Important comments.

1. This \( V \) is _dead wrong_ for \( x < 0 \) (where \( V = 0 \), right?!)

2. There are only a _limited_ \# of situations where this method works (e.g. conducting sphere + conducting sheet + conductor cylinder).

But important for people who think about antennas!

3. The _method_ is: make up a _new_ situation, where you add _image charge(s)_ in _special spots_ (with special \( q \)'s)

   _Such that_:
   - Images are not located in the region of space you want to know \( V(r) \) for (?!)
   - \( V \) at boundaries of region you're interested in is _same_ as your (real) situation

And done! \( V = \sum_{i \in \text{images}} \frac{q_i}{4\pi\epsilon_0 \left| r - r_i \right|} \) easy to write down!

\( \sqrt{ } \)

_only valid in the region you were considering, not all space._

• Some \( q_i \)'s are real
• Some are _fictitious_.

• So, this method works if have bunch of \( q \)'s outside sheet (just add bunch of matching image charges!)
By the way, since we know \( V(x>0, y, z) \), we can easily figure out \( \sigma \) on the sheet!

Remember, \( \vec{E} = \frac{\sigma(y, z)}{\varepsilon_0} \times \) just outside the conductor.

But \( \vec{E} = -\nabla V \), so

\[
\frac{\sigma}{\varepsilon_0} = -\frac{\partial V}{\partial x} \bigg|_{x=0}
\]

This derivative is not so hard:

\[
\frac{\sigma(y, z)}{\varepsilon_0} = \frac{\partial}{\partial x} \left( \frac{(x-d)}{(x-d)^2 + y^2 + z^2)^{3/2}} + \frac{(x+d)}{(x-d)^2 + y^2 + z^2)^{3/2}} \right) \bigg|_{x=0}
\]

\[
= \frac{\sigma(y, z)}{4 \pi \varepsilon_0} \left( \frac{2}{(2^2 + y^2 + z^2)^{3/2}} \right)
\]

- It peaks at \( y = z = 0 \) and then "fades out" \( \checkmark \)
- It's negative everywhere \( \checkmark \) (it would've been hard to guess!)
- Griffiths shows \( \int \sigma(y, z) \, dy \, dz = -q \) \( \checkmark \)

So our fictitious image charge, \(-q\), is physically manifested by \(-q\) "smeared" appropriately on surface of conductor.

(There really is not any "-q" at \( x = -d \)!!)

(Oh, one more thing. If you try \( \text{E}_{y} = -\frac{\partial V}{\partial y} \), you'll get 0 at \( x = 0 \), check for yourself.)
Work + Energy:

\[ W_{\text{2 charges in "fictitious" situation}} = \frac{1}{4\pi\varepsilon_0} \frac{(+q)(-q)}{(2d)^2} \]

\[ W_{\text{real situation, +q outside grounded plane}} = \frac{1}{2} \varepsilon_0 \int \int \int E^2 \, dx \, dy \, dz = \frac{1}{2} W_{\text{fictitious world, only right half of universe contributes!}} \]

Why not same? See Griffiths... but here's my explanation:

(i) In "fictitious world", you bring \( -q \) to \( x = -d \) (no work!), then bring \( +q \) to \( x = +d \), doing \( -\) work all the way.

(ii) In "real world", you bring \( +q \) to \( x = d \), and it "sees" an image charge which is always farther away than what it would be in scenario i, with image fixed at \( -d \).

(Because image is always at "-x")

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*Note that, when \( q \) is a", \( x = +d \), the force on it (at that spot) is just \( \frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2d)^2} \) \(-x\). Because \( \vec{E}(x) \) is same whether in real world or "image charge" world, since \( \vec{E} = -\vec{\nabla} V \).