Magnetostatics: Steady current flow everywhere
\[ \nabla \cdot \vec{j} = 0 \]

No "individual charges", it's all steady flows, \( \left( \frac{\partial \vec{j}}{\partial t} \right) \) to fuss with.

Currents create magnetic fields!

It's an experimental fact. We have to do experiments to determine direction, magnitude. I cannot derive this!

We will see the formula later as one of Maxwell's equations, but historically, Biot & Savart deduced an equivalent result by careful experiments (following Oersted's original discovery, during a class lecture demo (!) that current \( \Rightarrow \vec{B} \))

Think of Coulomb\( \quad \)\( \frac{d\vec{E}}{dq} = \frac{1}{4\pi \epsilon_0} \frac{dq}{R^2} \quad \vec{E} \)

A small charge creates a small \( \vec{E} \) field \( dq \).

Biot - Savart is similar\( \quad \frac{d\vec{B}}{d\vec{l}} = \frac{\mu_0}{4\pi} \frac{d\vec{l} \times \vec{R}}{R^2} \)

due to

chunk of current \( \frac{d\vec{l}}{d\vec{l}} \) just like Coulomb! except for the cross product!
of course, in magnetostatics there are no isolated chunks of current like this, so really must sum over chunks:

\[ \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \cdot \hat{R}}{R^2} \, dl' \times \hat{R} \]

A constant. The magnetic partner of \( \varepsilon_0 \) = "permittivity"

\[ \frac{\mu_0}{4\pi} = 10^{-7} \frac{N}{A^2} \]

"permeability of free space"

Some authors don't like the vector on \( \frac{\vec{I}}{\varepsilon_0} \), shift it to \( d\vec{L} \), so \( \vec{I} \) is just the amount of current in \( d\vec{L} \) direction.

So \( \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \cdot \hat{R}}{R^2} \, dl' \times \hat{R} \)

Remember \( \hat{R} \):

\[ (\text{definition}) \rightarrow \]

\[ \hat{R} = \vec{r} - \vec{r}' \]

Units:

\[ \vec{F} = q \vec{V} \times \vec{B} \]

so \( [B] = \frac{N}{C \cdot m/s} = \frac{N}{A \cdot m} \)

1 Tesla = \( \frac{1}{A \cdot m} \)

\[ dB = \frac{\mu_0}{4\pi} \frac{\vec{I} \cdot \hat{R}}{R^2} \, dl' \]

\[ [\mu_0] = \text{Tesla} \cdot \frac{m}{A} = \frac{N}{A^2} \checkmark \]

\[ \hat{R} = \vec{r} - \vec{r}' \]

Point of interest:

\[ \hat{R} = \vec{r} - \vec{r}' \]
Example 1: Straight current segment \( I \) to \( \hat{z} \)

Let's figure out \( \vec{B} \) at \( P \) due to this chunk (shown hatched) that way, we can figure out \( \vec{B} \) from e.g. \( \vec{I} = \int \vec{B} \) by "summing chunk"

This finite chunk is itself a sum (integral) of infinitesimal chunks.

\[
\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I \, d\vec{r} \times \hat{r}}{R^2}
\]

Look at the figure and convince yourself that \( d\vec{r} \times \hat{r} \) points in \( \hat{y} \) direction by the right hand rule (RHR)

I will use \( S = \) distance to \( P \) from \( z \)-axis (cylindrical radial coordinate)

And I have defined an angle \( \alpha' \) in the picture.

Here \( d\vec{r} = d\vec{z}' \), and \( |d\vec{r} \times \hat{r}| = d\vec{r} \cdot 1 \cdot \sin \alpha' \)

Also, looking at picture, \( R^2 = S^2 + (z - z')^2 \), \( \sin \alpha' = \frac{S}{R} \)
So \( B \) at \( P = \frac{M_0 I}{4\pi S} \int_{S_1}^{S_2} \frac{dz'}{S^3 + (z' - \mathbf{2})^2} \)

\( S \) is a constant in this integral, \( + \) MMA gives me the result \(-\) (Griffiths works it out)

\[ B = \frac{M_0 I}{4\pi S} \left. \frac{\mathbf{z}' - \mathbf{2}}{\sqrt{S^3 + (\mathbf{z}' - \mathbf{2})^2}} \right|_{\mathbf{z}' = \mathbf{z}_2}^{\mathbf{z}' = \mathbf{z}_1} \]

I could also observe: \( \cos(\alpha_1) = \frac{\mathbf{z}_2 - \mathbf{z}_1}{\sqrt{S^3 + (\mathbf{z}_2 - \mathbf{z}_1)^2}} \)
(from this figure)

So \( \frac{1}{B} = \frac{M_0 I}{4\pi S} \left( \cos(\alpha_2 + \cos(\alpha_1) \right) \) into the page (RHR!)

If current is infinite \( \alpha_1 = 0, \alpha_2 = \frac{\pi}{2}, \) and we get \( -1 + 1 = 2 \) (in parenth.)

\( \overline{B} \) (long wire) = \( \frac{M_0 I}{2\pi S} \) (RHR sense)

If current is "half infinite, starting across from \( P \)

\( B = \frac{M_0 I}{4\pi S} \) \((\alpha_1 = \frac{\pi}{2})\)

e.t.c.
one more example: Ring of current, \( B \) on axis?

Here, \( R = \sqrt{a^2 + z^2} = \text{constant} \), so that's nice easy!

Unfortunately \( d\hat{z} \times \hat{R} \) points at a crazy angle, but if we sum over all \( dl' \)'s, only the vertical (z) component of those will survive! (Convince yourself!)

\[ d\hat{z} \times \hat{R} = dl', \text{ but the vertical component is} \]

\[ dl' \cos \alpha = dl' \frac{\theta a}{\sqrt{z^2 + a^2}} \]

\[ B_z = \frac{\mu_0 I}{4 \pi} \int \frac{dl' \times \hat{R}}{R^2} = \frac{\mu_0 I}{4 \pi} \frac{1}{a^2 + z^2} \frac{\theta a}{\sqrt{a^2 + z^2}} \int dl' 
\]

\[ B_z (0,0,z) = \frac{\mu_0 I \theta a^2}{2 \pi (a^2 + z^2)^{3/2}} \]
To wrap up:

- Parallel wires

\[ B \text{ due to } I_1 = \frac{\mu_0 I_1}{2\pi s} \]

\[ \text{onto page } (-z \text{ direction}) \]

\[ \text{For } I_2 \text{ due to } B \text{ of } I_1 = \int I_2 \frac{dl_2 \times \hat{b}}{l_1} = I_2 \cdot \frac{\mu_0 I_1}{2\pi s} \int dl_2 \left( -\hat{z} \right) \]

So

\[ \text{For } I_2 \text{ on } z = \frac{\mu_0 I_1 I_2}{\text{unit length}} \]

(towards \( I_1 \) if both are parallel)

- If current is spread out, Biot-Savart becomes

\[ \int dl \hat{b} = \iint \frac{\mu_0}{4\pi} \frac{\hat{r} \times \hat{\alpha}}{R^2} \text{ da} \]

for surface currents

\[ = \iiint \frac{\mu_0}{4\pi} \frac{\hat{r} \times \hat{\alpha}}{R^2} \text{ d}r' \]

for volume currents

- Don't try to use Biot-Savart to find \( \hat{b} \) from one single moving charge; that's not magnetostatics, wait till next semester!

- Superposition principle holds for \( \hat{b} \) just like \( \hat{E} \)
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One more example: Sheet of current $\vec{J}$.

(infinite in extent)

This is more common than you think, solenoids, solar wind...

we could set up the integral

breaking surface into patches $dA$ with current

$$\frac{\mu_0}{4\pi} \int \frac{(\vec{r} \times \vec{x} \times \vec{R}) \cdot \vec{R}}{R^2} dA$$

But there will be a much easier way coming soon! So let's hold off.

But by symmetry, convince yourself $\vec{B}$ must point:

towards you above, away (below) the sheet in the figure above.