In general, inside any material, you'll have free currents (basically, wires running through it, or flowing ions...) and, as a result, \( \mathbf{B} \) fields appear which further polarize the material, adding in these bound currents. (Which in turn alter the field even more!)

All together, in "equilibrium",

\[
\mathbf{J} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{bound}} \]

This \( \mathbf{J} \) is real, it creates the total real \( \mathbf{B} \) field via Ampere's law,

\[
\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint \mathbf{J} \cdot d\mathbf{l}
\]

no exceptions in magnetoics, this is Ampere's law

So

\[
\nabla \times \frac{\mathbf{B}}{\mu_0} = \frac{1}{\mu_0} \mathbf{J}_f
\]

so we define

\[
\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}
\]

with \( \nabla \times \mathbf{H} = \mathbf{J}_f \) or \( \int \mathbf{H} \cdot d\mathbf{l} = I_f \), through

[Units of \( \mathbf{H} \) are \( \text{Amps/m} \), not \( \text{Tesla} \)!]

[Not really sure what to call \( \mathbf{H} \), it's just \( \mathbf{H} \)!]

[see notes p.4-1] The electric study was very similar, leading to \( \mathbf{E} \) [usually easy to measure]

[This is the current in wires, normally!]
Example  A long Al rod (radius \( r \)) carries uniform current \( J_{\text{free}} \), (total current \( I = J_{\text{free}} \cdot \pi r^2 \)) in the direction. Find \( H \) and \( B \) everywhere.

Like Griff Ex 6.2, but Al is paramagnet, his example was Cu = diamagnet, so compare to this with his example!

I expect \( \vec{B} = \bigcirc \) by Ampere's law (just like always, current is up, \( \vec{B} \) circulates)

I expect \( \vec{M} \) will be parallel to \( \vec{B} \) because inside it's a paramagnet so \( \vec{M} = \bigcirc \) too.

Of course, outside is vacuum, so \( M_{\text{outside}} = 0 \).

Since \( \vec{H} = \vec{B} - \vec{M} \), these "cancel", but I know \( \vec{M} \) is going to be small/weak for real materials, so I'm sure \( \vec{H} \) will still (also) go \( \bigcirc \) this way. Indeed,

\[ \oint \vec{H} \cdot d\vec{l} = I_{\text{free}} \text{, though proves that } \vec{H} \text{ "looks" like } \vec{B} \text{.} \]

Loop inside, shown dashed in direction at least
\[ \oint H \cdot dl = I_f \Rightarrow H \cdot 2\pi s = J_f \pi s^2 \]

\[ \vec{H} = \frac{I_f \pi s^2}{2} \hat{\varphi} = \frac{I}{2\pi R^2} \hat{\varphi} \] (since \( I = J_f / \pi s^2 \))

This is same as Griff, dir or para, makes no diff!

Outside \( \vec{H} = \frac{I}{2\pi s} \hat{\varphi} \) (same as Griff, doesn't matter) what material is

Outside \( \vec{M} = 0 \), so \( \vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi s} \hat{\varphi} \) usual old result, material independent.

Inside, we know direction of \( \vec{M} \), & know it's less than \( \frac{\vec{B}}{\mu_0} \), but we're stuck without knowing how \( M \) magnetizes. We'll get to that soon.

But I can argue qualitatively that if \( \vec{M} \) looks like \( \bigcirc \) then \( \vec{M} \times \hat{n} \) will run down the outside of cylinder (parallel to \( \vec{J}_f \)) (this is opposite Griff example!)

and \( \vec{A} \times \vec{M} \theta = J_f \) points up the cylinder (parallel to \( \vec{J}_f \), which is consistent with \( \text{para} \) magnetism, (and opposite Griff ex)
But to compute $J_B$ and $H_B$, we need $\mathbf{H}_{\text{inside}}$, and I can't get that from my (known) $\mathbf{H}_{\text{inside}}$ without knowing how $\mathbf{A}$ saturates. So...

**Linear Materials**

Many (common) materials magnetize proportional to $\mathbf{B}$.

Recall electric polarization was $\mathbf{P} = \varepsilon_0 \chi_E \mathbf{E}_{\text{tot}}$ (linear dielectrics).

We define $\chi_m$ for non-linear magnetic materials as

$$\mathbf{M} = \mathbf{\chi}_m \mathbf{H}$$

Note the lack of "symmetry", in the case of dielectrics, $\chi_E$ is defined looking at $\mathbf{E}$, but for magnetic materials, we use $\mathbf{H}$, not $\mathbf{B}$. Why? Because $\mathbf{H}$ is easy to compare in many cases!

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{free}}$$

That's what you measure, not $I_{\text{free}}$.

$\chi_m$ is "magnetic susceptibility".

Unless, typically small.
\( \chi_m \) is + for paramagnets, \( \vec{M} \) lines up with \( \vec{H} \).

\( \chi_m \) is - for diamagnets, \( \vec{M} \) opposes \( \vec{H} \).

Of course, \( \vec{H} = \frac{\vec{B} - \vec{M}}{\mu_0} \) so \( \vec{B} = \mu_0 (\vec{H} + \vec{M}) \)

\( = \mu_0 \vec{H} (1 + \chi_m) \)

so \( \vec{H} \) and \( \vec{B} \) point in the same direction, if \( \chi_m > 0 \) and even if \( \chi_m < 0 \), as long as \( \chi_m \) is small.

For normal materials, \(|\chi_m| \) is like \( 10^{-9} \) to \( 10^{-5} \).

For superconductors, \( \chi_m = -1 \), (which means \( \vec{B} = 0 \) inside, total "shielding")

Summary:

\[
\begin{align*}
\vec{M} &= \chi_m \vec{H} \\
\vec{B} &= \mu_0 (1 + \chi_m) \vec{H} = \mu \vec{H} \\
\vec{M} &= \frac{\chi_m}{\mu} \vec{B}
\end{align*}
\]

Note: free space \( \vec{B} = \mu_0 \vec{H} \), so \( \mu_0 = \frac{4\pi \cdot 10^{-7}}{\chi_m} \) is

"permeability of free space"
Back to our Al rod example: $\chi_{Al} = +2 \cdot 10^{-5}$

we had found (p. 4-12) $\vec{H}_{\text{inside}} = \frac{I}{2\pi n R^2} \hat{g}^\wedge$

so $\vec{M}_{\text{inside}} = \chi_n \frac{I}{2\pi n R^2} \hat{g}^\wedge$ very small

$\vec{B}_{\text{inside}} = \mu \frac{I}{2\pi n R^2} \hat{g}^\wedge$ which is almost identical to what we got in ch. 5 without knowing about magnetization.

since $\mu = \mu_0 (1 + \chi_m)$, however, $\mu > \mu_0$ (a bit)

tiny! so $\vec{B}_{\text{inside}}$ is "enhanced" a bit.

That's paramagnetism. A copper wire has $\chi_m = 10^{-5}$, so $\vec{B}_{\text{inside}}$ is ever so slightly reduced

(Outside, none of this matters.)
Example: Two sheets carry current $I$,

Top in $+x$ direction

Bottom in $-x$.

Both are same 1K1.

Between sheets is "linear material" (slab with susceptibility $\chi_m$).

What's $\vec{B}, \vec{H}, \vec{M}$ in slab? First, $\vec{\nabla} \cdot \vec{H} = I_{\text{free}} \Rightarrow \text{what's } "\vec{M}\"$

By symmetry, I know $\theta \vec{H}$ will points into page ($+y$) between slabs, will cancel to zero outside (think back to this same example in free space with no slab!).

So this Amperean loop gives $\oint \vec{H} \cdot d\ell = \vec{H} \cdot \ell$

$\text{Amperean Loop}$

$\Rightarrow \vec{H} = K y^\bot$ in between.

$= 0$ outside

$\Rightarrow \vec{M} = \chi_m \vec{H} = \chi_m K y^\bot$

$\Rightarrow \vec{B} \text{ inside} = \mu_0 \vec{H} = \mu_0 K y^\bot = \mu_0 (1 + \chi_m) K y^\bot$

Pretty much same as free space, just slight enhancement for paramagnetic reduction for diamagnetic.
What do Bound currents look like in that example?

Inside
\[ \nabla \times \vec{H} = 0 \quad \text{(since } \vec{M} \text{ is uniform)} \]
So, no Bound \( \vec{J}_B \).

But \( \vec{M} \times \hat{n} = +\vec{K} \times \hat{n} \) at top
\[ +\vec{K} \times \hat{n} \text{ at bottom} \]
This is parallel to \( K_{free} \), but very small, if \( X_M > 0 \)

Makes sense, that's the mechanism to enhance \( \vec{B} \) inside.

If \( X_M < 0 \), this opposes \( K_{free} \), reducing \( \vec{B} \) inside a little.

---

**Boundary conditions**

Since \( \oint \vec{H} \cdot d\vec{e} = I_{free} \), \( \vec{H} \) can "jump" at boundaries.

\[ \vec{H}_{above} \cdot \hat{n}_{above} \cdot L = \vec{H}_{below} \cdot \hat{n}_{below} \cdot L = K_f \cdot L \]

Or, in vector notation (since \( \vec{H} \) in fact has two possible components

we should really write:

\[ \vec{H}_{above} - \vec{H}_{below} = K_f \times \hat{n} \]

Convince yourself this gives right answer for "skeer" example above!"
Meanwhile, \( \vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \)

so \( \nabla \cdot \vec{H} = -\nabla \cdot \vec{M} \) which says

\[ H^\text{above}_\perp - H^\text{below}_\perp = -(M^\text{above}_\perp - M^\text{below}_\perp) \]

This will vanish if \( \vec{M} \) is continuous. Since \( \vec{M} = \mu_0 \vec{B} \) for linear materials, and \( \vec{B}_\perp \) is always continuous by \( \nabla \cdot \vec{B} = 0 \),

17. means \( H_\perp \) is always continuous everywhere except where \( \chi_m \) suddenly changes (edge of a material)

\[ \Box H \cdot dA = I_{\text{free}} \text{ looks simple, and for cases of "high symmetry" lets you deduce } \vec{H} \text{ easily (a la Ampere's law). But if symmetry is not high, beware.} \]

E.g. if \( I_{\text{free}} = 0 \) everywhere, you cannot conclude \( H = 0 \) everywhere! Think of a toy magnet! Just because \( \nabla \times \vec{H} = 0 \) everywhere does not mean \( \vec{H} = 0 \) everywhere (unless some symmetry arguments can be invoked, like our infinite sheet example)
Example:

Big chunk of material with uniform $\vec{B}_0$ throughout.

That $\vec{B}_0$ is the total $\vec{B}_0$ field, arising (perhaps) from external $\vec{B}$ and the magnetization of the material, superposed.

So that material has $\vec{M}_0 = \frac{x}{\mu} \vec{B}_0$ (uniform)

$\vec{H}_0 = \frac{1}{\mu} \vec{B}_0$ (uniform, up)

(no matter what $x$ is, $\mu > 0$.

Then I dig a "needle shaped" hole, as shown. What's $\vec{B}$, $\vec{H}$ inside the hole? (Clearly $\vec{M} = 0$ in there, it's a hole!)

Inside the center

There are no free currents at this boundary,

so $H$"above" - $H$"below" = 0

which says $H = H_0 \hat{\imath}$ inside the hole

But that means $\vec{B} = \mu_0 \vec{H} = \mu_0 H_0 \hat{\imath}$ inside the hole, at center.

In terms of $B_0 + M_0$, this is $\mu_0 H_0 = B_0 - \mu_0 M_0 = \frac{B_0}{1 + x \mu}$

$\mu_0$ is reduced in there, if $x \mu > 0$

Due to bound currents on walls, presumably!

"Metal" has high $x \mu$, shields $B$ inside!

$B$ is enhanced if $x \mu < 0$.
"μ metal" = Ni_{0.93} Fe_{0.16} Cu_{0.05} Cr_{0.02} has

\[ \frac{M}{M_0} = 10^5 \quad (\chi = +10^5, \text{unlike most paramags with } \chi \text{ typical } \chi \approx 10^{-3}) \]

inside μ-metal, B should be high; it's a super paramagnet.

Inside the "hole", B is quite weak.

This is used on equipment where you want to eliminate external B fields passively.
Less do this for a wafer like cavity:

\[ \begin{array}{c}
\hat{B}_\text{above} = \hat{B}_\text{below}, \text{ so } \hat{B} = B_0 \hat{z} \\
\end{array} \]

This time, I can argue \[ \hat{B} \] is not modified!

But then \[ \hat{H} = \frac{\hat{B}}{\mu_0} \hat{M} = \frac{B_0}{\mu_0} \hat{z} \] is not \[ H_0 \]!! It's \[ \frac{\mu_0}{\mu} \] !

So \[ H_{\text{center}} = \frac{\mu_0}{\mu} H_0 = \frac{B_0}{\mu_0} = H_0 + M_0 \]

for paramag material, \[ H_{\text{center}} \] is enhanced \((\mu > 1)\)

dia \(\), \[ H_{\text{center}} \] is reduced \((\mu < 1)\)

(Remember, \[ H \] can jump at boundary, if \( M \) suddenly changes!)

\[ \begin{array}{c}
\text{Here, it's like we've "superposed" a disk of } \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{B \text{center} } \sim \frac{1}{R} \text{ will be small if disk has big } R, \text{ hence } \hat{B} \text{ unchanged.} \\
\text{In prev example, we superposed a solenoid which does change} \\
\end{array} \]
Ferromagnetism: I won't spend a ton of time on this, it's pretty complicated microscopically, but some materials have a unique property, due to QM electron-pair interactions, that favors spin aligned electrons. I can easily see why spins above and below like to align, but not why spins on sides would. It's not a classical effect!

This means $\bar{M}$ "locks in", it's not a response to $\bar{B}_{ext}$, so you cannot define $\chi_m$ or $\mu$ for ferromagnets. It's not a linear material!

It's a local effect, you find regions of parallel spins $\bar{m}$, called domains. See fig 6.26 in Griffiths. In your fridge, you can align those domains in presence of $\bar{B}_{ext}$ (Kitchen magnet) making strong pair like effects $\Rightarrow$ attraction. When pull magnet away, thermal relaxation randomizes domains quickly, so fridge doesn't remain magnitized. ("Magnets do\) it's a very special ferromagnet!\)
- Fe, Co, +Ni are about the only materials like this
- At high T, even permanent mags will randomize ("Curie Temp" is the critical value, above which they "melt their alignment"
- Typical Fe domains are ~ mm on a side.
- " " Magnetization in a domain is $M \approx 10^6 \frac{A}{m}$ which yields typical $\overrightarrow{B} = \frac{2}{3} \mu_0 M \approx 1 \ T$. So that's about 
- for spherical domains 
- the most you're likely to get from normal mags
- Applying $\overrightarrow{B}$ to ferromagnets merely re-aligns domains. So small $B_{ext}$ can radically (non-linearly!) alter $B_{tot}$.

For metal inside solenoid (Iron, e.g.)

\[ \overrightarrow{H}_{\text{inside}} = \pi \overrightarrow{I}_{\text{free}} \]

The unusual way, by Ampere: $\overrightarrow{H}_{\text{inside}} = n \overrightarrow{I}_{\text{free}}$

But $B_{\text{inside}}$ will be $\mu_0(\overrightarrow{H} + \overrightarrow{M}) + M$ can be huge (depends on $\overrightarrow{H}$, more field => more domains join in, influencing each other too)
1 Tesla

Huge

\[ \vec{B} \]

result of Bext + extra field from \( \vec{M} \)

\[ \vec{H} = \mu_0 n \vec{I}_{\text{free}} \]

you control this by currents in solenoid

Note that, by itself, this solenoid would only generate

\[ \vec{B}_{\text{vacuum}} = \mu_0 \vec{H} = 4\pi \times 10^{-7} \frac{T \cdot m}{A} \times 10^{-3} \frac{A}{m} \approx 4\pi \times 10^{-10} T \]

so we're getting billion-fold enhancement!

If you back off the current, \( B \) decreases, but some alignment remains! This is "hysteresis": memory

\[ \vec{B} \]

end here!

Permanent magnet.

\[ \vec{H} = n \vec{I}_{\text{free}} \]