Homework Set 7, Physics 3320, Spring 2012
Due Wednesday, March 14, 2012 (start of class)

1. Some Derivations.

A) Prove that, for any vector field $\mathbf{E}$, $\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$

B) Prove that $\nabla \times \mathbf{E} = i \mathbf{k} \times \mathbf{E}$, for a complex plane wave $\mathbf{E} = \mathbf{E}_0 e^{i(k \cdot \mathbf{r} - \omega t)}$, where $\mathbf{E}_0$ is a constant (independent of position and time).

C) Prove that, in free space, the magnetic field obeys the wave equation: $\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$

2. Energy in a wave on a string.

In section 9.1.1, Griffiths discusses a wave on a string and shows that the wave speed $v$ is given by $v = \sqrt{\frac{T}{\mu}}$ where $T$ is the tension in the string and $\mu$ is the mass density (mass per length) of the string.

In this problem, you will show that the energy in a sinusoidal string wave is proportional to the amplitude-squared ($A^2$) of the wave. Let’s consider a (real) sinusoidal traveling wave given by $y(x, t) = A \cos(k x - \omega t)$. Let’s say $x$ is the horizontal direction and $y$ is the vertical direction. This traveling wave contains kinetic energy because each atom of the string is moving up and down. The wave contains potential energy because the string is stretched when the wave is present, compared to the straight horizontal string when no wave is present.

We will assume “small amplitude” motion so that:

- each atom in the string moves vertically only,
- each portion of the string experience a very small fractional change in the length as the wave goes by,
- the tension $T$ in the string remains constant, independent of position and time,
- the slope of the string is always very small, $\frac{\partial y}{\partial x} \ll 1$ (Note that the figure below is greatly exaggerated.)

Only in this small amplitude limit is the wave speed independent of amplitude. If the amplitude is large, the string is stretched so much that the tension changes locally, which affects the wave speed.
A) Derive a formula for the time-averaged kinetic energy per length in the string when the wave is present. Your answer should be in terms of $A$, $\omega$, and $\mu$.

B) Show that, in this small amplitude limit, the amount by which a small segment $\Delta x$ of the string is stretched is

$$\Delta s = h - \Delta x \simeq \frac{1}{2} \left( \frac{\partial y}{\partial x} \right)^2 \Delta x,$$

where $\frac{\partial y}{\partial x}$ is the slope of the string.

C) Derive an expression for the time-averaged potential energy per length in the string. Your answer should be in terms of $A$, $T$, and $k$.

D) Show that the average potential energy in the string is equal to the average kinetic energy in the wave.

3. Wave on a string. Two points on a string are observed as a traveling wave passes. The points are at $x_1 = 0$ and $x_2 = 1$ m. The points are known to be less than one wavelength apart. The transverse motions of the two points are observed to be

$$y_1 = 0.2 \cos(3\pi t)$$
$$y_2 = 0.2 \cos(3\pi t + \pi / 8)$$

where all numbers are in SI units.

A) What is the frequency of this wave in hertz?

B) If we write the wave in the format $y = A \cos(kx \pm \omega t + \delta)$, what is the smallest possible value of $\delta$?

[Since the cosine function is periodic with period $2\pi$, the phase constant $\delta$ is only determined modulo $2\pi$.]

C) What is the wave length?

D) What is the wave speed?

E) Can you tell if this wave is moving right or left? If so, which way is it moving?
4. **Intensity from a distant light source.** Consider an isotropic light source which emits EM radiation with wavelength $\lambda$ with power $P_0$. (Isotropic = uniformly in all directions) And consider a detector with area $a$, facing the source which is a long distance $R$ away.

A) Derive a formula for amplitude of the electric field at the detector.

B) Derive a formula for the number of photons per unit time that strike the detector. What is the relationship between the rate (number per second) at which photons hit the detector, and the amplitude of the E-field at the detector?

C) Suppose the source is a 1.5-watt green LED bulb (that’s about the brightest single LED bulb commercially available). And suppose the detector is a human eye, 1 mile from the source on a dark night. About how many photons per second enter the eye? What is the amplitude of the E-field at the eye.

5. **Radiation pressure.**

A) On earth, the time-averaged flux of electromagnetic energy $\langle S \rangle$ from the sun is about 1400 W/m$^2$ (this is called the solar constant). Assume that the earth absorbs all that radiation. What is the size of the force per square meter due to the solar radiation. How does the radiation pressure from this light compare to atmospheric air pressure. If the earth reflected the sunlight, how would that affect the radiation pressure (qualitatively)?

B) What is the net force on the earth from this radiation pressure, assuming the earth absorbs all the EM energy? Compare this force to the gravitational force of the sun on the earth, and comment.

C) If I made a 100 kg spacecraft with a 10,000 m$^2$ large mirrored sail to reflect the sunlight and propel me away from the sun, what would its acceleration be? (Don’t forget to include the gravitational pull of the sun in your calculation. This is called a solar sail, and there are serious proposals to build such a craft!) What are some advantages and disadvantages over conventional spacecraft?
D) Fine particles of dust in interplanetary space are pushed out of the solar system by radiation pressure from the Sun. That’s why the night sky is nice and dark and transparent. Derive an expression for the size (the radius) of a particle that is at the critical size where the outward radiation pressure balances the inward pull of gravity. (You will need to think about what variables are involved here – I recommend writing the answer in terms of the luminosity $L$ (in watts) of the sun, and the mass density $\rho$ of the dust.) Insert reasonable numbers in your expression and estimate this critical size.

6. Read a paper. Look at the list of journal articles on the course website and decide which paper you wish to read and write a 1-page review of. This week, all you have to do is select your paper from this list, tell us which paper you have chosen and start reading it. In next week’s HW assignment, you will be asked to write a brief summary and review of the paper.