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The content, consequence and likeness approaches to verisimilitude: compatibility, trivialization, and underdetermination

Graham Oddie

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Abstract Theories of verisimilitude have routinely been classified into two rival camps—the content approach and the likeness approach—and these appear to be motivated by very different sets of data and principles. The question thus naturally arises as to whether these approaches can be fruitfully combined. Recently Zwart and Franssen (Synthese 158(1):75–92, 2007) have offered precise analyses of the content and likeness approaches, and shown that given these analyses any attempt to meld content and likeness orderings violates some basic desiderata. Unfortunately their characterizations of the approaches do not embrace the paradigm examples of those approaches. I offer somewhat different characterizations of these two approaches, as well as of the consequence approach (Schurz and Weingartner (Synthese 172(3):415–436, 2010) which happily embrace their respective paradigms. Finally I prove that the three approaches are indeed compatible, but *only just*, and that the cost of combining them is too high. Any account which combines the strictures of what I call the strong likeness approach with the demands of either the content or the consequence approach suffers from precisely the same defect as Popper's—namely, it entails the trivialization of truthlikeness. The downside of eschewing the strong likeness constraints and embracing the content constraints alone is the underdetermination of the concept of truthlikeness.

Keywords Verisimilitude · Truthlikeness · Closeness to truth · Likeness to truth · Distance from the truth · Distances between propositions

Is there any coherent notion of truthlikeness or verisimilitude underlying the diverse judgements about closeness to the truth that we commonly endorse, and if so, are there

G. Oddie (✉)
Department of Philosophy, University of Colorado at Boulder, Boulder, CO 80309, USA
e-mail: oddie@colorado.edu

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relatively clearcut and uncontroversial judgements or principles sufficient to enable us to zero in on and characterize a determinate concept? In this paper I give an affirmative but qualified answer to these questions.

1 Three approaches to the concept of closeness to the truth

For the last three decades theories of verisimilitude have routinely been classified into two apparently rival camps—the *content* approach and the *likeness* approach.[1] And much of the debate on the core problem has consisted in proponents of one or other program lobbing particular recalcitrant intuitions into the opposing camp. Recently Zwart and Franssen (2007) have raised the level of the debate by raising and addressing some interesting questions that this distinction begs.[2] The first is whether or not these approaches are exclusive. The assumption (at least my assumption) has always been that the approaches themselves, and not just particular realizations, are incompatible. But perhaps this is wrong. Perhaps there are accounts of verisimilitude that satisfy the strictures of both, thereby not only achieving a happy bipartisanship but also narrowing down the range of possible solutions to the problem considerably. But before the compatibility question can be settled in a definitive way, a prior question must be raised and answered—namely, what does this relatively informal distinction actually *amount* to? Before Zwart (2001) this question had never really been addressed systematically, or answered with any precision.[3] Clarifying the distinction would help answer a third important question—raised by Schurz and Weingartner (2010) in their recent response to Zwart and Franssen—namely, whether the classification is exhaustive, and whether it captures the field of promising proposals for closeness to truth in a salient manner. A third alternative might just cut through the Gordian knot.

Zwart and Franssen have explicitly addressed the first of these two questions, by offering precise, qualitative analyses of content and likeness orderings, and then answering the question of exclusiveness in the affirmative. They prove an analogue of Arrow's famous impossibility theorem in social choice theory: namely, that "no qualitative unifying procedure of a functional form can simultaneously satisfy the requirements of *Unanimity, Independence of irrelevant alternatives* and *Non-dictatorship*".[4] Their conclusion is that any attempt to meld content and likeness orderings will violate one or other of these three basic desiderata. Consequently, if their analysis of the two approaches is apposite, there is no good compromise between the likeness

[1] See Zwart (2001, Chap. 1).

[2] These issues were first raised systematically in Zwart (2001).

[3] In his (2001, pp. 24–27) Zwart characterizes the two approaches in a rather simpler and broader fashion than the characterization presented in Zwart and Franssen (2007). The earlier characterization yields somewhat different results but there is a consistent thread running through them as shown below.

[4] *Unanimity*: if the individual orderings all agree that $A \geq B$ then the unified ordering does too. *Independence of irrelevant alternatives*: in this context this is really just a consistency requirement on the unified ordering. *Non-dictatorship*: where two individual orderings differ, the unified ordering does not always side with one of the individual orderings.

and content approaches. This is not only an interesting result in itself, but their analysis of the approaches represents a welcome development in the debate, which has sorely lacked this kind of theoretical generality.[5]

Despite the value of Zwart and Franssen's novel and interesting framing of this issue, as well as their intuitively appealing conclusion, I am not completely happy. An unfortunate upshot of their analysis of the content approach is that it rules out its first actual articulation—namely, Popper's pioneering account of verisimilitude in terms of truth contents and falsity contents. Since it was Popper's initial assay on the problem that launched the content approach, and inspired subsequent variants of the approach, it should, I think, count as an *instance* of the content approach, even if it fails to be materially adequate as an account of verisimilitude. (An account does not have to be correct in order for it to count as an instance of a certain approach to a problem.) And an unfortunate upshot of their analysis of the likeness approach is that it rules out *its* first published articulation—namely the Tichý (1974) account of propositional truthlikeness in terms of the average likeness of worlds in a proposition to the actual world.[6]

These admittedly quirky results may be, in part, a consequence of Zwart and Franssen's focus on qualitative orderings, rather than quantitative measures. But it would clearly be desirable to have a comprehensive analysis of the content-likeness distinction that covers both qualitative and quantitative accounts—and one which also places Popper's content account firmly in the content camp, and the average-likeness account firmly in the likeness camp.

In a recent response, Schurz and Weingartner (2010) also find Zwart and Franssen's characterization of Popper's approach wanting, but for different reasons. They argue that, suitably refined and developed, Popper's approach does yield an ordering which accords with likeness-based intuitions. But Schurz and Weingartner's refinement is not an instance of the content approach as characterized by Zwart and Franssen. Schurz and Weingartner argue that that's as it should be, because the central notion of Popper's approach is really not *content* after all, but *consequence*. On the *consequence* approach, which is distinct from the content approach, the verisimilitude of a theory is an appropriate function of the size of the classes of its true and false consequences. Popper's mistake, according to Schurz and Weingartner, was not this consequence approach as such, but rather his failure to limit the consequences that count to those that are suitably *relevant*—i.e. those that make a real positive or negative contribution to verisimilitude.[7] They argue that the core of Popper's program is not to be found either in Popper's actual proposal, or in the possible-worlds development of Popper's approach exemplified in the content approaches of Miller (1978) and Kuipers (1987), or in the rival likeness approach advocated *inter alia* by Tichý, Hilpinen, Niiniluoto, Tuomela, and Festa. In other words, they argue that the content-likeness distinction is neither exhaustive, nor exclusive.

5 This point has been argued forcefully by Mormann in his (2005) and (2006).

6 From here on I will sometimes abbreviate *worlds in the range of A* to *worlds in A*.

7 Echoing Mortensen (1983). At the end of his paper Mortensen effectively abandoned the idea, but by pursuing a different account of relevance, Schurz and Weingartner have revived it.

These conflicting conclusions depend on contested interpretations of an intuitive distinction. On the content-consequence side, it is contested because Popper's theory, like any particular theory, is an instance of different theory *schemas*—different ways of generating particular theories—each of which might plausibly characterize Popper's heuristic. Taking a cue from Schurz and Weingartner (and taking issue with a former self) I argue that we do need to distinguish between *content* and *consequence* approaches—and that the traditional content-likeness dichotomy has ignored important differences between these.[8] Taking a cue from Zwart and Franssen, I give rather natural and precise characterizations of these two approaches, both of which have the virtue of embracing Popper's actual proposal. Importantly, they share a crucial, core Popperian principle, one that any plausible characterization of Popper's content approach must endorse—what I will call *the strong value of content for truths*.[9] The strong value of content for truths turns out to play a privileged role in the content approach, whereas Popper's overall ordering (which extends the strong value of content for truths) turns out to play a similarly privileged role in consequence approach. Finally, I will also give a range of characterizations of the likeness approach of increasing strength, all of which embrace the proposal of average likeness. These characterizations all follow the same general format.

Having characterized the three approaches in a unified way, I go on to answer the questions raised above about the compatibility and completeness of these approaches. While these meta-questions are certainly interesting in themselves, the cash value of pursuing them must lie in a contribution to the core first-order question with which I began. I argue that the balance of the evidence comes down in favor of what I will call the *strong likeness* approach, which can be economically characterized by just two highly plausible principles.

2 Popper's program

Scientific inquiry, like all inquiry, involves the pursuit of the true, but it is not the pursuit of just any old truths. Trivial truths are too easy to come by. Scientific inquiry is aimed at deep and interesting truth. As Popper noted (perhaps a little too often) scientists pursue *highly contentful* truth. As Popper was also rather fond of noting, one particular highly contentful theory, Newton's theory of motion and gravitation, was empirically very successful, but was eventually shown to be false. Still, it was undoubtedly a huge improvement on its predecessors, and it was also an improvement on any true but trivial proposition about motion. So maybe a false theory can improve both on its false predecessors, and its true but trivial predecessors, with respect to the goal of contentful truth.

[8] There is, however, significant overlap of the resulting classes of content and consequence based orderings. Theo Kuipers has suggested, in correspondence, that it would thus be more perspicuous to talk of *perspectives* rather than *approaches*.

[9] Which approach Popper was pursuing may be undetermined not only by the currently accessible facts (the texts he left behind) but also by all the facts about Popper's psychology at the time he was thinking about these matters. There may be no fact of the matter of what general approach he was pursuing.

Popper couched his account in terms of the truths and falsehoods expressible in a first-order language. Each theory A comes along with, or is identical to, its Tarskian consequence class which (for reasons that will become apparent) I will label A_P. A_P is the set of all sentences, in some appropriate language, entailed by A. The *truth* is the complete theory T every member of which is true in the actual world. The *truth content* of A, given that T is the truth, is the intersection of A_P and T: which we will label A_P^T. The *falsity content* of A is the intersection of A_P and the set F of all sentences false in the actual world: A_P^F. Popper thought that, amongst truths, truthlikeness increases monotonically with content. Amongst truths, the more true consequences, or the greater the truth content, the closer to the truth. This suggests that one indicator of a proposition's closeness to truth is the "size" of its truth content.

Let $A \geq_T B$ abbreviate: A is *at least as close to the truth* T as B. We assume that $\geq_T$ is a pre-ordering of propositions (i.e. it is reflexive and transitive). A and B are equally truthlike ($A \approx_T B$) just in case $A \geq_T B$ and $B \geq_T A$; and A *is* closer to the truth than B ($A >_T B$) just in case $A \geq_T B$ and *not*-$B \geq_T A$. Popper assumed that on any adequate of closeness to truth logical equivalents have the same degree of truthlikeness. This desideratum is a trivial consequence of identifying logically equivalent propositions, but whether or not they are identified we assume this holds necessarily:

Logical equivalents are equally truthlike
If A is logically equivalent to B then $A \approx_T B$.

Let us say that A *strictly entails* B if A entails B and they are not logically equivalent (i.e. B does not entail A). Now consider the following two principles:

The weak value of content for truths
If A is true and A entails B then $A \geq_T B$.

The strong value of content for truths
If A is true and A strictly entails B, then $A >_T B$.

Strong value entails *weak value*. For suppose *strong value* holds: and assume that A is true and A entails B. If B entails A then A and B are logically equivalent and so, by *logical equivalents,* $A \approx_T B$. Hence $A \geq_T B$. If B does not entail A then, by *strong value*, $A >_T B$, and again we have $A \geq_T B$. However, *weak* does not entail *strong*. Consider a trivial account of truthlikeness that puts all truths on a par with respect to truthlikeness. This implies that if A and B are true and A entails B, $A \approx_T B$—hence $A \geq_T B$. This trivial theory satisfies *weak* but not *strong*.

While *strong value* yields a significant range of judgements, it only yields an ordering of truths, and arguably a theory of truthlikeness is of most interest when it comes to comparing falsehoods. False propositions have both a truth and a falsity content, and Popper suggested that, other things being equal, the more false consequences a theory entails, the *further* it is from the truth. From here it is a short step to Popper's original proposal, which can be framed thus:

Popper's account
$A \geq_T B$ just in case $B_P^T \subseteq A_P^T$ and $A_P^F \subseteq B_P^F$.
(i.e. A has as much truth content as B and B has as much falsity content as A.)[10]

[10] Popper (1963).

It follows directly from this that $A \approx_T B$ just in case A is logically equivalent to B, and that $A >_T B$ just in case either:

$B_P^T \subset A_P^T$ and $A_P^F \subseteq B_P^F$ (more truth content, no greater falsity content), or
$B_P^T \subseteq A_P^T$ and $A_P^F \subset B_P^F$ (at least as much truth content, less falsity content).[11]

Popper's account has some apparently appealing features. It entails the strong value of content for truths. So the whole truth, T, is closer to the truth than any other proposition, and the logically weakest truths, tautologies, are further from the truth than any other truths. If A is true and B is false, then A is closer to the truth than B just in case A entails B's truth content. So if A is false then the strongest true consequence of A, its truth content A_P^T, is closer to the truth than A itself, as are all the truths that imply A_P^T.

The account also has some very grave shortcomings. Every falsehood has false consequences and no truth has false consequences, so on Popper's account no falsehood is as close to the truth as any truth, and so no falsehood is as close (let alone closer) to the truth than any tautology. Newton's theory, for example, is deemed no closer to the truth about motion than the completely trivial proposition that either something is moving or not moving. And if Popper is right about the likely falsity of all scientific theories that have been or are likely to be proposed, no actual or likely scientific theory is closer to the truth than the most trivial, and (according to Popper) least truthlike of all truths.

The absolute trivialization of truthlikeness for falsehoods
No falsehood is closer to the truth than the most trivial of truths (tautologies).

Furthermore, because it is impossible to add a true consequence to a false proposition without thereby adding false consequences, it follows that no falsehood is closer to the truth than any other. For example, Newton's theory of motion is no closer to the truth about motion than Parmenides' theory that nothing moves. This rather disturbing result was proved independently by Tichý (1974) and Miller (1974). This result also trivializes the relation of closeness to truth, but in a different way. We can call it:

The relative trivialization of truthlikeness for falsehoods
No falsehood is closer to the truth than any other falsehood.

Most commentators have concentrated their attention on the relative trivialization result, but absolute trivialization is just as disturbing. Any account that delivers either the relative or absolute trivialization of truthlikeness for falsehoods cannot vindicate the idea of progress towards the truth in any realistic inquiry.

Note that there are corresponding trivialization principles for truths, which would be just as bad: namely absolute trivialization for truths (that no truth is closer to the truth than a tautology); and relative trivialization for truths (that no truth is closer to the truth than any other). In the case of truths relative trivialization entails absolute trivialization. Note that Popper's account does not suffer from these defects, although some accounts considered below do.

[11] Left to right: suppose $A >_T B$. i.e. $A \geq_T B$ and *not*-$B \geq_T A$. Then $B_P^T \subseteq A_P^T$ and $A_P^F \subseteq B_P^F$ and either *not*-$A_P^T \subseteq B_P^T$ or *not*-$B_P^F \subseteq A_P^F$. So either $B_P^T \subset A_P^T$ or $A_P^F \subset B_P^F$. Right to left is obvious.

Both trivialization results for falsehoods require the clause pertaining to falsity contents. So one tempting response (Popper was so tempted) would be to drop the troublesome falsity content clause, and measure closeness to truth by truth content alone: A is as close to the truth as B if the truth content of A entails the truth content of B(and closer if, in addition, the truth content of B does not entail the truth content of A). Call this:

The pure value of truth content
If the truth content of A entails the truth content of B then $A \geq_T B$.[12]

The pure value of truth content entails both the *strong* and *weak* value of content for truths, but it also has consequences for false theories, since a falsehood can exceed another in truth content alone. Interestingly, this is what Zwart and Franssen identify as the defining feature of the content approach. They write: "All verisimilitude definitions that have the next definition as a necessary ingredient, we will call *content definitions*..." and then what follows is a definition of the relation of A's *having at least as much truth content* as B, which is just the left-hand side of the pure value of truth content viz.: A entails the truth content of B.

There are two senses in which this condition might be a "necessary ingredient" of content-based proposals. One possible reading of "necessary ingredient" is "necessary condition": i.e. that no content-based account of closeness to truth can deliver a richer ordering than that delivered by the pure value of truth content. This would capture Popper's original proposal (which delivers a subset of the judgments of the pure value of truth content) but it captures some hopelessly weak accounts as well. For example, a totally trivial concept of closeness to truth—that no proposition is closer to the truth than any other—would fall within the content approach. The more plausible reading is that any content-based ordering delivers at least this much ordering by truth content (though it may deliver more). And this is in fact what Zwart and Franssen intend. They write: "we call the ordering of a set of sentences generated by Definition 1, *the* content order of these sentences, and any order of which the content order is a subset *a* content order."[13] As they effectively acknowledge in their footnote 6, however, on this construal Popper's definition itself is ruled outside the content approach. "This does not mean that the content relation ... is a subset of Popper's ordering relation of sentences, because Popper's definition fails to order false sentences."

How does Zwart and Franssen's account fare with another intuitively content-based account—namely the symmetric-difference account proposed by Miller (1978)? Let's assume that the class of propositions forms a complete Boolean algebra with operations $\wedge$, $\vee$, $\neg$, and the relation $\vdash$ of entailment. (I will use $\models$ for the relation of strict entailment i.e $A \models B$ if $A \vdash B$ and *not*-$B \vdash A$.) A proposition picks out (and is sometimes identified with) a class of worlds—the class of worlds in which the proposition comes out true. Call this the *range* of the proposition. $A \vdash B$ if the range of A is a subset of the range of B. A proposition is *true* in w just in case w is in its range. Relative to

[12] Note that the antecedent here ($B_P^T \subseteq A_P^T$) is equivalent to ($B_P^T \subseteq A_P$) i.e. *A entails the truth content of B*—i.e. A entails all the truths that B entails. Since $A_P^T \subseteq A_P$ left to right is trivial. Assume $B_P^T \subseteq A_P$. Since $A_P = A_P^T \cup A_P^F$ and $A_P^T \cap A_P^F = \emptyset$, $B_P^T \subseteq A_P^T$.

[13] That this is the intended reading was confirmed in private communication.

world w, the *truth* (T) is the conjunction (or meet) of the set of all propositions true in w. The truth content of A in w, the strongest true consequence of A, is the disjunction (or join) of A and T: $A \vee T$.[14] Miller defines the *distance* between A and B as their symmetric difference, $A \Delta B$, which is equivalent to $(A \wedge \neg B) \vee (\neg A \wedge B)$. A is as close to Cas B is just in case $A \Delta C \vdash B \Delta C$ (i.e. $A \Delta C$ entails $B \Delta C$) and is closer provided $A \Delta C \models B \Delta C$ (i.e. $A \Delta C$ strictly entails $B \Delta C$). The distance of A from the truth is $A \Delta T$, and so A is *as close to the truth as* B just in case $A \Delta T \vdash B \Delta T$, and *closer* just in case $A \Delta T \models B \Delta T$. Miller's proposal, like Popper's, delivers the strong value of content for truths but it does *not* deliver the pure value of truth content.[15] So on Zwart and Franssen's characterization neither Popper's nor Miller's accounts fall within the content fold.

Note that the pure value of truth content is afflicted by absolute trivialization but not relative trivialization. However, it avoids the latter only by entailing:

The strong value of content for falsehoods:
If A and B are false, and A strictly entails B then $A >_T B$.

Thus it follows from the pure value of truth content that one can increase the truthlikeness of a false proposition (e.g. that *the moon is made of green cheese*) simply by conjoining to it some other falsehood (e.g. that *all the planets are made of mozzarella*). This has seemed to every investigator in this domain—with the notable exception of David Miller—unacceptable. Miller has mounted some feisty defenses of the strong value of content for falsehoods which follows from his symmetric difference account.[16] But even granting that Miller's symmetric difference account lies legitimately within the content approach—so that the strong value of content for falsehoods *may* be a component of a genuine content-based account—it is clearly better not to characterize the content *approach* in such a way that it is saddled with such a highly contested principle.

What the Zwart and Franssen (2007) analysis gets dead right is this: *any viable characterization of the content approach must incorporate the strong value of content for truths*. This is clearly a necessary (though insufficient) condition for an ordering to count as content-based, whatever other conditions a content-based account must satisfy.

In his (2001) Zwart classifies as content-based all and only those orderings that deem $\neg T$ (the weakest falsehood) to be the least truthlike proposition on offer.[17] Call this criterion *Weakest*. While *Weakest* does indeed capture Miller's, Popper's and Kuiper's orderings within the content fold (since they all deem $\neg T$ the least truth-

[14] In a complete Boolean algebra, for every subset of elements S both the *meet* of S and the *join* of S are also elements of the algebra. We could identify worlds with the maximal elements in a complete algebra of propositions, but this is something we can bracket here.

[15] If A and B are both true in w, $A \Delta T \vdash B \Delta T$ just in case $A - T \vdash B - T$ which in turn holds just in case $A \vdash B$. However a falsehood A cannot be as close to the truth as any truth B (because if A is false it cannot be the case that $A \vee T \vdash B - T$). In particular, if A is false, A is not as close to the truth as A's truth content $A \vee T$. On the value of truth content, since $A \vee T \vdash (A \vee T) \vee T$, it follows that A is as close to the truth as its truth content $A \vee T$).

[16] For a recent vigorous defense see Miller (2006), p. 185.

[17] This is a generalization from Definition 1.7 of Zwart (2001, p. 25) .

like) it doesn't ensure the strong value of content for truths. Further it rules out the possibility that amongst falsehoods truthlikeness decreases with logical content, which should at least be an option within the content approach.

3 The content approach

To achieve a unified account, we need to characterize both *kinds of orderings* and *kinds of measures* within each of the three approaches. We can capture the range of admissible orderings within each approach by first capturing the range of admissible measures. In each case we start by specifying what it takes for a numerical measure to count as, say, a content measure, and then proceed to characterize content orderings in terms of the relevant measures. For example, a *content-based measure* of closeness to truth is characterized as any appropriate function of truth and content, and a *content-based ordering* as any ordering that is appropriately realized by a content-based measure. The same schema will be used to characterize of consequence-based and likeness-based measures and orderings. Once measures of kind K have been characterized, an ordering of kind K is one that is *realized* by a measure of kind K, and (to rule out totally trivial orderings) one that delivers all *core* K-comparisons—namely, all comparisons delivered by all measures of kind K.

Assume that the class of propositions forms a complete Boolean algebra. Let $\geq$ be any pre-ordering of the elements of this algebra: $\geq$ is transitive and reflexive. The relations $\approx$ and $>$ can be defined in the usual way. A function ψ *realizes* $\geq$ if, whenever $A \geq B, \psi(A) \geq \psi(B)$ (hence whenever $A \approx B, \psi(A) = \psi(B)$ and whenever $A > B$, $\psi(A) > \psi(B)$). Note that the realizing function ψ may order more pairs than the relation $\geq$ does. Being *as close to the truth* is not a pre-ordering of propositions as such. Rather, each complete way the world might be induces such a pre-ordering. So, where w ranges over complete states (however characterized), we consider world-dependent orderings of propositions ($\geq_w$), where $A \geq_w B$ holds just in case (according to $\geq$) A is at least as close to the truth as B is, given that w is the actual way of the world, and a world-dependent function ψ realizes $\geq$ just in case for every w, ψ_w realizes $\geq_w$.

The particular feature of both Popper's account and Miller's symmetric difference account that lends support to the content characterization, is that they both hold verisimilitude to supervene, in different ways, on two factors: content and truth value. The respective contents of A and B, together with their truth values, jointly fix their ordering with respect to closeness to truth. This suggests an informal characterization of the content approach: namely that it renders closeness to truth a function of the truth factor and of the content factor.

It is easy to characterize the content factor within the spirit of Popper's own program. As Popper himself was wont to note, the amount of content in a proposition is a decreasing function of its probability, a countably additive measure on the Boolean algebra. Since content increases with logical strength, a measure of content should realize the entailment relation in the above sense, and this means that the underlying logical measure μ has to be *strictly positive*: if A strictly entails B then $\mu(B) > \mu(A)$. So let

μ be any countably additive, strictly positive measure, and call the associated function $(1 - \mu(A))$ the *standard* realization of content relative to measure $\mu(Content^{\mu}(A))$.

The truth factor also induces a simple ordering of propositions: *being at least as true as* (not to be confused with being at least as close to the whole truth). A two-valued function $Truth_w(A)$ realizes this provided $Truth_w(A) > Truth_w(B)$ if and only if A is true in w and B is false. The standard realization of the truth factor is set at 1 if A is true in w and at 0 if A is false in w.

To count as a content-based measure of closeness to truth, ψ must be some function of truth and content satisfying numerical analogues of Popper's truth and falsity content conditions: i.e. amongst equally true theories, the more content the better, and amongst equally contentful theories, the truer the better. This yields:

Content-based measures of closeness to truth
ψ is a content-based *measure* of closeness to truth iff there is a countably additive, strictly positive measure μ, and a function f such that, for all w:

i. $\psi_w(A) = f(Truth_w(A), Content^{\mu}(A))$,
ii. for any $c > d$, $f(1, c) > f(1, d)$, and
iii. for any $t > f$, and any c, $f(t, c) > f(f, c)$.

For an *ordering* $\geq$ to count as content-based $\geq$ has to be *realized by some content-based measure* ψ. That is insufficient, however, since even the totally trivial ordering is realized by such a measure. In addition we require that each content-based ordering capture all the *core* content judgments. $A \geq_w B$ is a *core content-based judgment* $(A \geq_w^{Ct} B)$ if for every content-based measure ψ, $\psi_w(A) \geq \psi_w(B)$. The collection of all core-consequence based judgments $\geq^{Ct}$ comprises the *core content ordering*. The class of content-based orderings are those that both extend the core content ordering and are realized by some content-based measure.

$\geq$ is a *content-based ordering of closeness to truth* iff:

i. $\geq$ is an extension of the core content ordering $\geq^{Ct}$, and
ii. $\geq$ is realized by some content-based measure of closeness to truth.

By virtue of clause (ii) of the definition of content-based measures, the strong value of content for truths is clearly part of the core content ordering, but in addition we can show that the strong value of content for truths actually *constitutes* the core content ordering (proofs in the Appendix):

The strong value of content for truths is the core content ordering. (1)

Thus any content-based ordering extends the strong value of content for truths in a content-based way (i.e. realizable by some content-based measure) as do both Popper's and Miller's orderings (a corollary to (1)). It is important to stress here that not any old extension of the strong value content for truths counts as content-based. It must extend the core content judgments *in the appropriate way*. That is to say, the entire extension must be realizable by a content-based measure, a function of truth and content obeying the two constraints—that amongst truths those with greater measurable content are superior, and that amongst theories that have the same degree of measurable content, truths are superior to falsehoods.

4 The consequence approach

Schurz and Weingartner argue, rather plausibly, that even if Popper's account counts as content-based in this sense, the real key to his approach lies elsewhere—namely, in his entirely overt comparison of true and false consequence classes. Schurz and Weingartner argue further that what the Tichý-Miller trivialization results show is that not just any old consequence classes will do. Popper's mistake was his failure to restrict consequences to those that are suitably *relevant*.[18] For example, if it is hot, rainy and windy, one relevant true consequence of the false proposition that *it is hot and dry* $(h \wedge \neg r)$ is the proposition that *it is hot* (h). Other consequences that follow from this, like $(h \vee r)$, $(h \vee w)$ and $(h \vee \neg w)$ do not further enhance the value of $(h \wedge \neg r)$. Since $(h \wedge \neg r)$ implies nothing about windiness as such, w is otiose—it could be replaced by any proposition at all. To hold that the true consequences like $(h \vee \neg w)$, also enhance the truthlikeness of $(h \wedge \neg r)$ would be to double-count the contribution of h to the truthlikeness of $(h \wedge \neg r)$. This suggests that there is some salient notion of the *relevance* of consequences to the determination of verisimilitude, and that A's *truth content* should be identified as the class of A's *relevant* true consequences, A's falsity content as the class of A's *relevant* false consequences.

An important constraint on relevance is, as Schurz and Weingartner note, that each proposition be logically equivalent to the conjunction of all its relevant consequences. Perhaps there should be other constraints but we will leave it at that. Schurz and Weingartner don't follow Popper's original definition of truthlikeness in terms of *containment* of relevant consequence classes, but substitute *entailment* instead—viz. A is as close to the truth as B if A's truth content entails B's truth content, and B's falsity content entails A's falsity content. (In my characterization of the consequence approach I will follow this.) Schurz and Weingartner show that a qualitative account based on these ideas can be extended to a quantitative measure which is "a genuine truthlikeness definition in the spirit of Tichý and Niiniluoto" but which, because it is effectively an extension of the qualitative account, is in accord with "a genuine qualitative verisimilitude definition based on the comparison of true and false content-parts in the spirit of Popper."[19]

We now characterize consequence-based measures and orderings in the spirit of Schurz and Weingartner's proposal. First, let A_R be the set of consequences of A that satisfy relevance criterion R. What counts as a relevant consequence will vary from one account to another, but whatever the relevance condition R is, the conjunction of members of A_R must be equivalent to A itself. We also require, of course, that if A is equivalent to B then $A_R = B_R$. On certain accounts, like Schurz and Weingartner's, the members of A_R are logically independent. On others, like Popper's, they are not. Let:

$A^T_{R,w}$ (*the R-relevant truth content of A in w*) = subset of A_R true in w; and
$A^F_{R,w}$ (*the R-relevant falsity content of A in w*) = subset of A_R false in w.

[18] Mortensen suggested this in his (1983) but by his own account of relevance Popper's account still entailed trivialization, so by the end of the paper he had cast significant doubt on the idea. Schurz and Weingartner resurrected it with a quite different account of relevance.

[19] Schurz and Weingartner (2010, p. 415).

Since, for any w, $A^T_{R,w} \cup A^F_{R,w} = A_R$, and the conjunction of A_R is equivalent to A, the conjunction of $A^T_{R,w}$ and $A^F_{R,w}$ is also be equivalent to A. If A is true in w, $A^T_{R,w}$ is empty, and $A^T_{R,w} = A_R$. For Popper, of course, R is the vacuous condition (labeled P above), because Popper effectively deemed every consequence relevant. Hence $A^T_{P,w}$ is just the set of all A's true consequences in w (i.e. what we called A^T_P) and $A^F_{P,w}$ is the set of all A's false consequences in w (A^F_P). ψ is consequence-based provided it is a function of the relevant consequence classes, ψ increases with the increasing strength of relevant truth content, and decreases with increasing strength of relevant falsity content:

Consequence-based measures of closeness to truth
ψ is a *consequence-based measure of closeness to truth* iff there is a criterion of relevance R and a (possibly partial) real-valued function f on pairs of classes of propositions, such that, for any proposition A:

i. $\psi_w(A) = f(A^T_{R,w}, A^F_{R,w})$; and (wherever f is defined),
ii. if $E \models E^*$ and $F^* \vdash F$ then $f(E, F) > f(E^*, F^*)$,
iii. if $E \vdash E^*$ and $F^* \models F$ then $f(E, F) > f(E^*, F^*)$.[20]

On analogy with core content-based judgements we have *core consequence-based judgments*: $A \geq^{Cn}_w B$ if and only if for every consequence-based measure ψ, $\psi_w(A) \geq \psi_w(B)$. The collection of all core consequence-based judgments ($\geq^{Cn}$) is the *core consequence ordering*.

We now characterize consequence-based orderings in general.

Consequence-based orderings of closeness to truth:
$\geq$ is a *consequence-based ordering of closeness to truth* iff:
i. $\geq$ is an extension of the core consequence ordering $\geq^{Cn}$, and
ii. $\geq$ is realized by some consequence-based measure.

It follows immediately from clauses (ii) and (iii) of the definition of consequence based measures that if both A and B are true (i.e. $A^F_{R,w} = B^F_{R,w} = \emptyset$) and A is logically stronger than B (i.e. $A^T_{R,w} \models B^T_{R,w}$) then $A >^{Cn}_w B$. That is: the core consequence ordering, like the core content ordering, delivers the strong value of content for truths. Unlike the content approach however, the strong value of content for truths does not exhaust the core of the consequence approach. Rather, we have:

$$\textit{Popper's ordering is the core consequence ordering.} \tag{2}$$

Popper's ordering plays the same privileged role within the consequence approach as the strong value of content for truths plays within the content approach. This result is a rather striking vindication of Schurz and Weingartner's claim that it is the consequence, rather than the content, approach that best characterizes what Popper was up to.

[20] Note that f here need not be defined for every pair of classes of propositions. So long as for every A, f is defined for the R-relevant truth and falsity consequence classes of A, ψ will be defined for A.

As one would expect, Schurz and Weingartner's proposed measure induces an ordering that is consequence-based. Their criterion of relevance is, roughly speaking, the property S of being picked out by a disjunction of atomic propositions or their negations where no disjunct is redundant. (If we use pro positions rather than sentences, and individuate propositions by necessary equivalence, then there is no need to use their lexicographic ordering of sentences—something which has (misleadingly) made their proposal seem syntactic.) Suppose it is hot (h), rainy (h) and windy (w)—i.e. h, r, and w are all true. Then, given Schurz and Weingartner's relevance criterion S:

$$(h \wedge \neg r \wedge \neg w)^T_{S,w} = \{h\};$$
$$(h \wedge \neg r \wedge \neg w)^F_{S,w} = \{\neg r, \neg w\};$$
$$(h \wedge r \wedge \neg w)^T_{S,w} = \{h, r\};$$
$$(h \wedge r \wedge \neg w)^F_{S,w} = \{\neg w\}.$$

Schurz and Weingartner's measure ψ is clearly realized by an additive, consequence-based measure f, which assigns positive value to each relevant truth, and the corresponding negative value to each relevant falsehood. Thus:

$$\psi_w(h \wedge r \wedge \neg w) > \psi_w(h \wedge \neg r \wedge \neg w).$$

Similarly,

$$(h \wedge \neg w)^T_{S,w} = \{h\},$$
$$(h \wedge \neg w)^F_{S,w} = \{\neg w\}.$$

Since Schurz and Weingartner's measure also accords with condition (iii), it yields:

$$\psi_w(h \wedge \neg w) > \psi_w(h \wedge \neg r \wedge \neg w).$$

And it delivers a range of similarly intuitively pleasing likeness-based judgments in a propositional framework. However, although it delivers particular likeness-based judgments it is too quick to conclude that their measure also falls within the likeness approach. That clearly requires a precise, general, and materially adequate characterization of what the likeness approach amounts to—a task to which I now turn.

5 Likeness between worlds and the underdetermination of truthlikeness

The range of a proposition is the class of those worlds with which it is compatible, and that range either contains the actual world or it doesn't. That is pretty much all that a content or consequence theorist has to work with, and it yields three factors to juggle—the size of the range of a proposition (its content), whether the actual world lies within the range (its truth value), and its true and false consequence classes (suitably restricted when supplemented by some criterion of relevance). But what if the logical space has some additional structure, and worlds are ordered according to their differing *likeness to* or *distance from* each other?

Likeness is a form of *closeness*, and likeness functions can be characterized in terms of corresponding distance functions, for which there is a well-developed mathematical theory. A distance function on a logical space Ω is any real valued function δ satisfying the three metric conditions: for all $u, v, w, \delta_{uv} = 0$ iff $u = v$; $\delta_{uv} = \delta_{vu}$; $\delta_{uv} + \delta_{vw} \geq \delta_{uw}$. A normalized distance function is one such that the maximal distance between worlds is 1. Let a *likeness* function on Ω be any function λ such that for some normalized distance function δ on Ω, $\lambda_{uw} = 1 - \delta uw$. (Where λ and δ are so related we may make that explicit by writing λ^{δ} and δ^{λ}.) It follows that for any likeness function $\lambda : 0 \leq \lambda_{uw} \leq 1$; $\lambda_{uw} = 1$ if and only if u is identical to w, and $\lambda_{uw} = 0$ iff u is maximally distant from (unlike) w. For the purposes of determining compatibility of approaches, we will leave the range of permissible underlying likeness functions open, thus maximizing the possibility for compatibility.

With a couple of notable exceptions, accounts that have traditionally been placed within the likeness approach start either by defining a λ-function that measures likeness between worlds or by postulating such a function, and then going on to extend λ to propositional likeness. A likeness-based measure of truthlikeness is an extension of some λ-function to a measure of likeness of a proposition to a world. Where λ is likeness between worlds, and σ an extension function, $\sigma(\lambda)$ (or σ^{λ} for short) is the *σ-extension of* λ from worlds to propositions: thus $\sigma^{\lambda}_{w}(A)$ is the degree of likeness or closeness of proposition A to world w, given that λ measures likeness between worlds, and σ is used to extend λ. Provided σ^{λ} is a *legitimate* extension of λ then the ordering $\geq^{\lambda}$ that σ^{λ} realizes is a likeness-based ordering. A number of different ways of extending λ to propositions have been proposed.

One rather simple way of ordering propositions is by considering only those worlds in the range of a proposition that are closest to the actual world.

$$min^{\lambda}_{w}(A) = \text{the } \lambda\text{-likeness of worlds in } A \text{ of minimal distance from } w;$$
$$A \geq^{min}_{w} B \text{ iff } min^{\lambda}_{w}(A) \geq min^{\lambda}_{w}(B).$$ [21]

While falsehoods can differ in their *min*-value, truths clearly cannot. Since every true proposition contains the actual world in its range, it follows that all truths have the same *min*-value. So according to $\geq^{min}$ all truths are on a par, and all truths are superior to all falsehoods. As an ordering of closeness to the truth, $\geq^{min}$ thus exhibits the defect of absolute trivialization for falsehoods (no falsehood closer to the truth than a tautology) and absolute trivialization for truths (no truth closer to the whole truth than any other). $\geq^{min}$ is thus clearly inadequate as a truthlikeness ordering. Rather, as is now generally acknowledged, min^{λ} is a measure of *closeness to being true*, and $\geq^{min}$ an ordering of propositions according to their degree of *approximate truth*. All

[21] This is at the basis of the proposal in both Weston (1990) and Teller (2001). Note that I am framing the likeness approach in terms of the fundamental notion of *likeness* rather than of *distance*. So I am deliberately taking *min* to measure the degree of *likeness* of the closest world, rather than the distance of the closest world. Throughout I preserve the well-known nomenclature—like *minmax* and *minsum*—even though the underlying functions are likeness rather than distance functions.

truths are equally close to being true, and all are at the top of the hierarchy of being approximately true.[22]

max^λ, by contrast, is a measure of the likeness of a world furthest from actuality. And the ordering $\geq^{max}$ ranks a proposition higher the closer the farthest world in its range is to the actual world.

$$max^\lambda_w(A) = \text{the } \lambda\text{-likeness of the world in } A \text{ of maximal distance from } w;$$
$$A \geq^{max^\lambda}_w B \text{ iff } max^\lambda_w(A) \geq max^\lambda_w(B).$$

Note that both truths and falsehoods have differing max^λ values, so that $\geq^{max^\lambda}$ is not subject to the charge of trivializing truthlikeness either for truths or falsehoods. As far as truths go, the whole truth does best on max^λ-value, and a tautology does worst on max^λ-value. More generally, ranking propositions by their max^λ-value yields the *weak* value of content for truths: if A and B are both true, and A entails B then A is at least as close to the truth as B. However, it also yields a corresponding principle of the weak value of content for falsehoods: if A and B are both false, and A entails B then A is at least as close to the truth as B. Thus one might be tempted to think of max^λ as a kind of measure of content, and that is precisely the way that Hilpinen introduced max^λ (or rather, it's qualitative counterpart).

Hilpinen's qualitative proposal postulates a likeness ordering of worlds in the shape of a nested set of similarity circles, and derives from this a partial ordering of propositions. Hilpinen's extension of λ, hil^λ, is the following:

$$A \geq^{hil^\lambda}_w B \quad \text{iff } min^\lambda_w(A) \geq min^\lambda_w(B) \text{ and } max^\lambda_w(A) \geq max^\lambda_w(B).$$

It follows that[23]:

$$A >^{hil^\lambda}_w B \quad \text{iff } A \geq^{hil}_w B \text{ and either } min^\lambda_w(A) > min^\lambda_w(B) \text{ or } max^\lambda_w(A) > max^\lambda_w(B).$$

Because the min^λ-factor is the same for all truths, and the max^λ-factor satisfies the weak value of content for truths, $\geq^{hil^\lambda}$ also satisfies the weak value of content for truths. Unlike max^λ however, it does not yield the strong value of content for falsehoods, since strengthening a falsehood might eliminate closest worlds, ensuring a lower min^λ-factor.

Suppose λ realizes the structure of the ordering of similarity spheres on which $\geq^{hil^\lambda}$ is based. Then $\geq^{hil^\lambda}$ itself can be realized by many extensions of λ, including weighted averages of min^λ and max^λ. Let $minmax[\mu]^\lambda$ be the weighted average likeness of

22 Hilpinen had noted this already in his (1976). There is an important exception to the observation that min^λ captures approximate truth. Consider an infinite space in which there is no world closest to the actual world. Then σ^{min^λ} is not defined for many false propositions (including, for example, $\neg T$). We could amend it to yield the greatest lower bound of distances in the range from the actual world. But then σ^{min} will yield some anomalous results for approximate truth: $\neg T$, for example, might have the same degree of approximate truth as T itself.

23 Strictly speaking, in Hilpinen's ordering $min^\lambda_w(A)$ and $max^\lambda_w(A)$ pick out the smallest and largest spheres associated with a proposition A, and $\geq$ and $>$ are set-theoretical orderings.

closest and farthest worlds, where $\mu (0 \leq \mu \leq 1)$ is the weight assigned to the likeness factor, and $(1-\mu)$ the weight assigned to the information factor[24]:

$$minmax[\mu]^{\lambda}_{w}(A) = \mu min^{\lambda}_{w}(A) + (1-\mu) max^{\lambda}_{w}(A).$$

For each non-extreme value of μ, $minmax[\mu]^{\lambda}$ (a mixture of the closeness factor and the content factor) realizes $\geq^{hil^{\lambda}}$. The extreme values ($\mu = 1$ and $\mu = 0$) return us to the simple procedures *min* and *max*. On all values of μ, $minmax(\mu)$ yields the weak value of content for truths.

Although Hilpinen characterized max^{λ} as a content factor, it is rather coarse-grained for that purpose. For example, both the tautology and a complete proposition whose range is the furthest from the actual world have the same *max*-value. So A may be logically much stronger than B while $max^{\lambda}_{w}(A) = max^{\lambda}_{w}(B)$. To remedy this coarseness in the information factor Niiniluoto replaced max^{λ} with a far more sensitive measure. Let $sum^{\lambda}_{w}(A)$ be the normalized sum of the distances of worlds in A from w. That is:

$$sum^{\lambda}_{w}(A) = \left[\sum_{u \in A} \delta^{\lambda} uw\right] \Big/ \left[\sum_{u \in \Omega} \delta^{\lambda} uw\right].$$

A tautology has maximal *sum* value, 1; if A and B are incompatible then $sum^{\lambda}_{w}(A \vee B) = sum^{\lambda}_{w}(A) + sum^{\lambda}_{w}(B)$; and (for the most part) if A is logically stronger than B then $sum^{\lambda}_{w}(A) < sum^{\lambda}_{w}(B)$. sum^{λ} thus looks like a regular measure, and $(1-sum)^{\lambda}$ looks like the corresponding measure of content, albeit one in which degree of content is determined by distance from the actual world. The more distant a world is from actuality, the more it undermines the content-value of the proposition. Thus the presence of distant worlds in the range of a proposition weakens that proposition more than does the addition of the same number of close worlds. Consequently $(1-sum)^{\lambda}$ is likeness-favoring. That is to say, if two propositions have the same number of worlds in their ranges, then $(1-sum)^{\lambda}$ favors that proposition the worlds of which are, on average, closer to the actual world. For this reason $(1-sum)^{\lambda}$ might be advanced as a measure of closeness to truth, and $\geq^{(1-sum)^{\lambda}}$ the associated ordering, at least by a content-favoring Popperian. $(1-sum)^{\lambda}$ clearly delivers the strong value of content for truths, since amongst truths, $(1-sum)^{\lambda}$ increases with logical strength. Less happily, it delivers the strong value of content for falsehoods.

$(1-sum)^{\lambda}$ does not *quite* realize entailment, because sum^{λ} is not a strictly positive measure. For example, while a tautology has the maximal sum^{λ}-value (1), $\neg T$, which differs from a tautology only in that it rules out the actual world, also has the maximal sum^{λ}-value. So although $\neg T$ is logically stronger than a tautology, $sum^{\lambda}_{w}(\neg T) = sum^{\lambda}_{w}(T \vee \neg T)$. Indeed for any false proposition A the sum^{λ}-value of A is the same as that of its truth content: $sum^{\lambda}_{w}(A) = sum^{\lambda}_{w}(A \vee T)$.[25] But sum^{λ} is *almost* a

[24] Niiniluoto introduced the extension proposal $minmax[\mu]^{\lambda}$ in his (1977).

[25] If λ is flat then $sum^{\lambda}_{w}(A) = |A - T_w|/|Taut - T_w|$, which is close to the standard measure $|A|/|Taut|$. However, sum^{λ} does not distinguish between A and $A - T_w$.

likeness-based measure of logical weakness and $(1 - sum)^{\lambda}$ is almost a likeness-based measure of content.

Niiniluoto proposed the extension procedure $minsum[\mu]^{\lambda}$ which is just like $minmax[\mu]^{\lambda}$ except that $(1 - sum)^{\lambda}$ replaces max^{λ} as the measure of information[26]:

$$minsum[\mu]^{\lambda}_{w}(A) = \mu min^{\lambda}_{w}(A) + (1 - \mu)(1 - sum^{\lambda}_{w}(A)).$$

For each λ-function and each possible value of μ, $minsum[\mu]^{\lambda \cdot}$ yields an ordering $\geq^{minsum[\mu]^{\lambda}}$. Provided $\mu \neq 1$ (i.e. the content factor gets some weight) $\geq^{minsum[\mu]^{\lambda}}$ delivers the strong value of content for truths. Further, $\geq^{minsum[\mu]^{\lambda}}$ avoids both kinds of trivialization for falsehoods.

$minmax[\mu]^{\lambda}$ looks like an approximation to $minsum[\mu]^{\lambda}$. But as a straight average of the closeness to the actual world of extreme worlds in A $minmax[1/2]^{\lambda}$ also looks like rough approximation to the overall average likeness of worlds in a proposition to the actual world:

$$average^{\lambda}_{w}(A) = \text{ average } \lambda\text{-likeness to } w \text{ of worlds in } A.$$

Tichý (1974) proposed $average^{\lambda}$ as the right extension of λ but he gave no argument for it.[27] It is not obvious, for example, that straight averaging should be preferred to weighted averaging, with the weights supplied by a logical measure function μ:

$$\begin{aligned} wtdave[\mu]^{\lambda}_{w}(A) &= \mu\text{-weighted average } \lambda\text{-likeness to} w \text{ of worlds in } A \\ &= \sum_{v} \mu(v|A)\lambda vw. \end{aligned}$$

average is the special case of *wtdaver*[μ] where μ is the equal weighting of worlds.

There are thus quite a number of proposals that explicitly extend likeness between worlds to a putative truthlikeness ordering.[28] So far we have *min*, *max*, *minmax* $(1 - sum)$, *minsum*, *average* and *wtdaverage*, and there are obviously infinitely many such extension functions out there waiting for an advocate. Indeed Schurz and Weingartner's central criticism of the likeness approach (and indirectly their argument for their consequence approach) is precisely this:

> the problem of extending truthlikeness from possible worlds to propositions is intuitively *underdetermined*...[29]

[26] Niiniluoto proposed this weighted average of the two factors, with the weights left open and not necessarily normalized, in papers delivered at conferences in 1983 and 1984. See Niiniluoto (1987, p. 228 ff.).

[27] This is not strictly accurate, since Tichý worked with distances between worlds, and distances from the truth, rather than likenesses between worlds and likeness to the truth. But what he proposed is tantamount to this likeness-based characterization. Later, in his (1976) and (1978) he generalized his approach to first-order structures.

[28] For example: Tichý (1974, 1976, 1978), Hilpinen (1976), Oddie (1978, 1986), Niiniluoto (1977, 1987), Tuomela (1978), Teller (2001) and Weston (1990).

[29] Schurz and Weingartner (2010, p. 423).

They argue both that the underdetermination problem strongly favors the consequence approach and that "…it seems that this extension problem is the root of the conflict between content- and likeness-considerations as diagnosed by Zwart and Franssen (2007)."

It is not clear that the consequence theorist is in a superior position on the score of underdetermination, since she leaves the relevance criterion *R* wide open. But from the brief survey above it is clear that underdetermination of extension procedures is indeed a major problem for the likeness theorist. Even given a determinate likeness function on worlds, we have many candidates for extending that to propositions. Using particular intuitive judgments as data points for deciding between rival extension proposals can only take us so far.

What the likeness theorist needs to solve the underdetermination of extension procedures are some clear and intuitively compelling general principles that constrain the extension of likeness between worlds to likeness of propositions to the truth. To solve the problem definitively the constraints, in addition to being individually compelling, should be relatively small in number, and single out a unique extension procedure.

6 The likeness approach

Zwart (2001) proffered the *Worst* and *Weakest* criteria (outlined below) for distinguishing likeness and content orderings. In his (2004) Niiniluoto responded with an analysis of his own *minsum* proposal (intended to capture both content and likeness factors) in terms of these criteria. In their (2007) Zwart and Franssen offered a somewhat stronger criterion for likeness orderings, one based on Hilpinen's qualitative proposal, which under suitable assumptions yields *Worst* as a special case. In this Sect. 1 attempt to further this enterprise in a series of characterizations of the likeness approach, each of which arises from its predecessor by placing an additional constraint on extension procedures.

The kernel of the likeness approach is that truthlikeness supervenes on likenesses between worlds: roughly, no difference in the relative ordering of propositions without some difference in the likenesses of worlds to the actual world. While supervenience comes in a range of strengths we have good reason here to adopt a strong supervenience thesis, one in terms of the *likeness profile* of a proposition. For example, consider the following:

$$\begin{aligned} A: &\quad (h \wedge r \wedge w) \vee (\neg h \wedge \neg r \wedge \neg w) \\ B: &\quad (\neg h \wedge r \wedge w) \vee (h \wedge \neg r \wedge \neg w) \\ w: &\quad hot, rainy, windy \\ z: &\quad cold, dry, still. \end{aligned}$$

A's range embraces the two worlds w and z. If w is actual A contains both the world closest to the actual world (w) and the world furthest from the actual world (z). Of course, if z is actual A also contains both the worlds closest to and furthest from the actual world, albeit with switched roles. So even though the worlds in A are of different distances from actuality in w and in z, the likeness *profile* of A is the *same* in both. And it is clear that is what matters for the truthlikeness of A. It should be the same in both worlds. Similarly, whether w or z is actual, on the standard measure one world in B has degree of likeness of 2/3 to the actual world and the other is degree of likeness 1/3. So B has the same likeness profile in both worlds, and so by supervenience its likeness to both should be the same. Finally, whether A has the same degree of truthlikeness as B in w (as *average* yields) or whether A is closer to the truth than B in w (as *minsum*[μ] yields, provided $\mu > 1$), that ordering should remain the same in v.

Sameness of likeness profile, then, should ensure sameness of truthlikeness. A proposition A has the *same* likeness profile in u that B has in v ($A_u \overset{\lambda}{\approx}_v B$) iff there is a 1–1 mapping f from A to B such that the likeness of w to u is the same as the likeness of $f(w)$ to v.

Supervenience
There can be no difference in the truthlikeness of a proposition without a difference in a proposition's likeness profile (i.e. if $A_u \overset{\lambda}{\approx}_v B$ then $\sigma_u^\lambda(A) = \sigma_v^\lambda(B)$).

Let $L_1 = Supervenience$. L_1 embraces all of the extension procedures considered in the previous section, and many more besides. For example, consider a function σ such that for any λ, A, B, and w : $\sigma_w^\lambda(A) = \sigma_w^\lambda(B)$. The procedure σ satisfies supervenience, but whatever the λ-function is, σ^λ not only trivializes the relation of truthlikeness for falsehoods, but trivializes truthlikeness for truths as well. However, since σ satisfies supervenience, for each λ, σ^λ counts as a likeness-based measure according to L_1. Or again, suppose that λ is the correct likeness-ordering on worlds and that in addition σ satisfies *supervenience*. Then $(1 - \sigma)$ also satisfies supervenience, and $(1 - \sigma)^\lambda$ realizes the inverse relation $\leq$ (where $A \leq_w B$ if and only if $B \geq_w A$). Now suppose that $\geq$ is among the best candidates for truthlikeness. Then $\leq$ is among the worst candidates. Almost all the judgments it makes will be incorrect (all save the equality judgments). So supervening on likeness profile is a purely formal constraint (like universalizability in ethics), one which is necessary but grossly insufficient to remove underdetermination. More conditions are required.

In his (2001) Zwart noted that well known likeness and content theories disagree over what is the worst, or least truthlike, proposition on offer. The proposition that is closest to the truth in w is the truth itself (A_w), the complete proposition that contains the world closest to the actual world w. What is the worst proposition, the one furthest from the truth? Suppose that there is exactly one world, z (or *Zwart's world*), of maximal distance from the actual world w. Then A_z is the "complete falsehood" in w, just as A_w is "the complete truth". The proposition whose range is the singleton of Zwart's world seems like the best candidate for *least truthlike proposition*. And all the likeness-based extension procedures considered so far assign A_z their lowest value.[30] This is encapsulated in the Zwart (2001) criterion:

[30] Provided, of course, that A_z exists. If there is no worst world relative to the actual world there is no complete proposition whose range contains the world maximally unlike the actual world.

Worst
The proposition whose range contains just that world maximally unlike the actual world is the least truthlike of all propositions (i.e. for all $B \neq A_z, \sigma^\lambda_w(A_z) < \sigma^\lambda_w(B)$).

Let L_2 be $L_1 + Worst$. L_2 is an improvement on L_1. It rules out various trivial extensions of λ, and it rules out the possibility that both σ^λ and $(1-\sigma)^\lambda$ are legitimate extensions of λ. (If $\sigma^\lambda_w(A) = 0$ then $1 - \sigma^\lambda_w(A) = 1$, so *Worst* cannot be true on both measures.) By the same token *Worst* rules out any extension that yields the negation of the truth $\neg T$ (i.e. the logically weakest falsehood) as the worst proposition on offer.[31] (So *sum* fails while $(1 - sum)$ survives.)

As noted, Zwart (2001) identifies content orderings with precisely those that deem the weakest falsehood $\neg T$ to be the least truthlike proposition on offer.[32] *Worst* combined with *Weakest* guarantees that likeness and content approaches are strictly incompatible. No extension procedure can satisfy both.[33] This dual criterion—*Worst* for likeness orderings and *Weakest* for content orderings—helps illuminate the nature of Niiniluoto's proposed extension procedure $minsum[\mu]^\lambda$.[34] Recall that each concrete *minsum*[μ] extension of λ is the sum of a likeness factor (μmin^λ) and a content factor $(1-\mu)(1-sum^\lambda)$. The proposition A_z always has minimal likeness factor ($min^\lambda_w(A_z) = 0$) while the weakest falsehood $\neg T$ always has the minimal content factor ($(1 - sum^\lambda_w(\neg T)) = 0$). Let's assume that as the logical space Ω expands to infinite complexity (abbreviated: $\Omega \to \infty$) the worlds of minimal distance from the actual world tend toward maximal likeness (1), so that the closest worlds to actuality in $\neg T$ also tend to 1 (as $\Omega \to \infty$).[35] Similarly, since the normalized weight of world z tends to 0 as $\Omega \to \infty$, the content factor of A_z also tends to 1.

So we have the following values throughout:

$$min^\lambda_w(A_z) = 0, (1 - sum^\lambda_w(\neg T)) = 0.$$

And, in the limit, (as $\Omega \to \infty$) we have:

$$(1 - sum^\lambda_w(A_z)) = 1, \quad and \quad min^\lambda_w(\neg T) = 1$$

31 Except in a degenerate logical space with just two worlds, in which A_z and $\neg T$ are the same.

32 This is a generalization from Definition 1.7 of Zwart (2001, p. 25). My concept of content-based ordering given in Sect. 2 allows for the failure of *Weakest*.

33 See foot note 31 for the one exception. Niiniluoto (2003) notes a possible modification of Zwart (2001) for those cases in which there is more than one maximally distant world: that the worst proposition W is the one whose range contains all and only the most distant. This is what *minsum* delivers quite generally—and that any complete proposition that entails W is deemed more truthlike than W. Further, if the distance function is flat ($\delta uv = 1$ iff $u \neq v$) then on *minsum*, W collapses into $\neg T$, thereby delivering the compatibility of *Worst* and *Weakest* in this limiting case.

34 What follows is inspired by the incisive reflections on content and likeness orderings in relation to Zwart's *Worst* and *Weakest* criteria developed in Niiniluoto (2003).

35 This holds if n is the number of atomic propositions, for example, but it will also presumably hold for other logical spaces.

Consequently we have the following limiting values for $minsum[\mu]^{\lambda}$:

$$minsum[\mu]^{\lambda}_{w}(A_z) = (1-\mu), \text{ and } minsum[\mu]^{\lambda}_{w}(\neg T) = \mu.$$

So, in the limit, according the ordering induced by $minsum[\mu]^{\lambda}_{w}$:

$$(\neg T >_w A_z) \quad \text{iff } \mu > 1/2, \text{ and}$$
$$(A_z >_w \neg T) \quad \text{iff } \mu < 1/2.$$

In other words, for *minsum*[μ] to satisfy *Worst* (A_z least truthlike) the likeness factor has to receive more weight than the content factor. And for *minsum*[μ] to satisfy *Weakest* ($\neg T$ least truthlike) the content factor has to be given more weight than the likeness factor. Weighting the two factors equally violates both *Worst* and *Weakest*, yielding neither a content nor a likeness ordering. Given different weightings, *minsum*[μ] can generate both likeness and content orderings (by the *Worst* and *Weakest* criteria) but there is no weight μ which secures both a content and a likeness ordering.[36] And that is exactly what one would expect given that the $minsum[\mu]^{\lambda}$ procedure blends content and likeness factors in a weighted compromise.

Worst is a compelling necessary condition on the likeness approach, but a closely related constraint on all complete propositions draws its plausibility from the same source. Suppose λ assigns worlds a wide range of degrees of likeness to the actual world w and consider an extension σ^{λ} of λ that assigns 0 to A_z and 1 to every other proposition:

$$\sigma^{\lambda}_{w}(A) = 0 \text{ if } A = A_z \quad \text{and} \quad \sigma^{\lambda}_{w}(A) = 1 \quad \text{if } A \neq A_z.$$

σ^{λ} satisfies both *Supervenience* and *Worst* but obviously falls short of being appropriately sensitive to the structure of λ. σ^{λ} trivializes truthlikeness for truths, and almost trivializes it for falsehoods, despite the fine-grained likeness distinctions amongst worlds. At the very least the relations between the complete propositions should track the relations between their members. Where A_u is a complete proposition whose range contains u, *Worst* tells us that the A_z bears to A_w the same degree of likeness that z bears to w—namely, minimal. Generalizing this, the likeness of A_u to the truth A_w should be identical to the degree of likeness of u to w. Call this:

Complete
The likeness to the truth of a complete proposition is the same as the likeness of its sole member to the actual world (i.e. for all u, $\sigma^{\lambda}_{w}(A_u) = \lambda uw$).

Complete rules out all L_2-extensions that yield the relative trivialization of truthlikeness for falsehoods. Any *Complete* extension of a likeness function has to induce as fine-grained an ordering of falsehoods that are complete as the λ-function induces on the worlds themselves. And it has to do that in a specific way, by assigning the same degree of truthlikeness to A_u as the degree of likeness to the actual world λ assigns to u.

36 Except of course, in the degenerate two-world case.

The extension procedures *min*, *max*, *minmax* and *average* and *wtdave* satisfy *Complete*. $(1 - sum)$ and *minsum* do not. While these latter two do yield the same ordering of complete propositions induced by *Complete* they don't do so in such an obviously straightforward way.[37]

Zwart and Franssen (2007) also go beyond *Worst* with a more powerful constraint. They take the right hand side of Hilpinen's definition of truthlikeness (call it *Hilpinen*) to be both a necessary and sufficient condition for an extension of λ to lie within the likeness approach. And *Hilpinen*, together with the assumption that there is a worst world z, implies *Worst*, but it is stronger. It also implies the ordering of complete propositions effected by *Complete*. If *Hilpinen* is both necessary and sufficient for likeness, a likeness-based ordering can extend Hilpinen's ordering ($\geq^{hil^\lambda}$) but cannot contradict it. That is to say, if $A \geq_w^{hil^\lambda} B$, then $\sigma_w^\lambda(A) \geq \sigma_w^\lambda(B)$ for any likeness-based extension σ^λ of λ.

Any *Hilpinen* extension of λ yields the weak value of content for truths. For suppose that A and B are true and that A entails B. Given that both are true, they share the same closest world—the actual world itself. Since A is at least as strong as B, its range is a subset of the range of B. So A's farthest world must be at least as close to actuality as B's farthest world. (An extension does not necessarily satisfy the strong value of content for truths, because A may strictly entail B even while they share both closest and furthest worlds.) So it follows from *Hilpinen* that *no likeness-based ordering contradicts the weak value of content for truths*.

Popper (1976) first noted that averaging can generate a violation of the weak value of content for truths. To see this, suppose we have a framework with n atomic propositions $(P_1, \ldots, P_n)$ all of which are true in w, and consider the following pair of truths:

A *Either* all of $P_1, \ldots, P_n$ are true *or* all of $P_1, \ldots, P_n$ are false.

B *Either* all of $P_1, \ldots, P_n$ are true, *or* $P_1, \ldots, P_{n-1}$ are true and P_n is false, *or* all of $(P_1, \ldots, P_n)$ are false.

My own intuition, for what that is worth, is that B seems closer to the truth than A, but at this stage my intuitions are hardly untutored and I acknowledge that there might be other ways of presenting the ranges of these propositions which reverse the intuition.[38] What does averaging deliver? The range of A includes just the two extreme worlds—the world which is most like the actual world (likeness = 1) and the world least like the actual world (likeness = 0) with overall average likeness of 1/2. The range of B includes these two and an additional world that, amongst non-actuals, could not be more like the actual world (likeness $= (n-1)/n$). So average likeness of B to w is $(2n-1)/3n$. If $n > 2$ this lies between 1/2 and 2/3. But A and B are both true and A is

[37] Recall that $sum_w^\lambda(A_u) = \delta^\lambda uw/S_w = (1-\lambda uw)/S_w$ where $S_w = \sum_{u\in\Omega} \delta^\lambda uw$. So, like *min*, $(1 - sum_w^\lambda(A_u)) > (1 - sum_w^\lambda(A_v))$ just in case $\lambda uw > \lambda vw$, The same clearly holds for a weighted average of *min* and $(1 - sum)$.

[38] I use the term "intuition" in a highly fallibilist manner. One has an intuition that P just in case it seems, or appears, to one that P. Hence intuitions are defeasible appearances. Other things being equal it is better for a theory to save more rather than fewer appearances, but not all appearances can be or ought to be saved.

the logically stronger. Weakening a proposition by adding worlds sufficiently like the actual world increases average likeness to the actual world. So averaging can violate the weak (and strong) value of content for truths.

Popper considered this a knockdown refutation of average likeness as an adequate extension procedure, rather than acknowledge possible counterexamples to the strong value of content for truths. I disagree, and because of this I will use a consequence of *Hilpinen*. If *Hilpinen* is acceptable then so too is my weaker constraint, and it has the advantage that it does not banish averaging from the likeness approach at this stage.

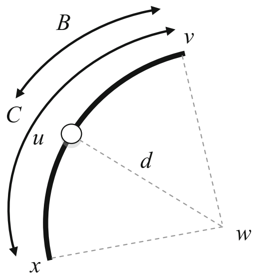
Let's say a proposition is *uniform* in w if all the worlds in its range are the same distance from w. What I propose is just *Hilpinen* restricted to uniform propositions. Like the points on the arc of a circle, the worlds in the range of a uniform proposition are all equidistant from the center world w. Call the distance, from the actual world, of worlds in a uniform proposition the *radius* of the proposition. In a uniform proposition both the *min*-value and the *max*-value of the proposition coincide with its radius, and so *Hilpinen* entails, for uniform A and B, if the radius of A is at least as great as that of B, B is at least as close to the truth as A.

Radius
If A and B are both uniform and B's radius is less than or equal to A's radius then B is at least as close to the truth as A (i.e. if $radius_w(B) \leq radius_w(A)$ then $\sigma_w^\lambda(B) \geq \sigma_w^\lambda(A)$).

It follows from *Radius* that if A and B are uniform, and have the same radius, then they are equally truthlike, and have the same degree of likeness to the truth as all the worlds in their range. In Fig. 1 three uniform propositions are depicted: A_u, B, and C. B is the arc $\overset{\bullet}{u}\,v$ and C is the arc $\overset{a}{x}\,uv$. By *Complete* the distance of A_u from w is d, and its likeness is λuw (where $\lambda uw = \lambda vw = \lambda xw$). By *Radius* the distance of B, C, and A_u from w are all the same (d) and their likeness to w is also the same.

Radius and *Complete* are jointly equivalent to *Uniform*:

Uniform
If A is a uniform proposition with radius $d = 1 - \lambda vw$ then the degree of likeness of A to the truth is λvw (i.e. if $radius_w(A) = 1 - \lambda vw$ then $\sigma_w^\lambda(A) = \lambda vw$).



$radius_w(A_u) = radius_w(B) = radius_w(C) = d$

$\delta^\lambda uw = d = 1 - \lambda uw$

$\sigma_w^\lambda(A_u) = \lambda uw$ (by *Complete*)

$\sigma_w^\lambda(B) = \sigma_w^\lambda(C) = \sigma_w^\lambda(A_u)$ (by *Radius*)

Fig. 1 Uniform equivalent to Radius + Complete

Let L_3 be L_2 + *Uniform*. The procedures *min*, *max*, *minmax*[μ], *average* and *wtdaver*[μ] all satisfy *Uniform*. However *sum*, $(1 - sum)$ and *minsum*[μ] clearly do not.[39] *Uniform* says that arcs with the same radius are the same distance from the center, whatever the length of the arc. The smaller of two arcs with the same radius is the logically stronger, and since $(1 - sum)$ delivers the strong value of content for falsehoods, it violates *Uniform.* And because *sum* delivers the disvalue of content for falsehoods it also violates *Uniform.* Since the *min*-value is the same for propositions with the same radius, *minsum*[μ] delivers the strong value of content for uniform falsehoods (provided $\mu < 1$). So *minsum*[μ], like *sum* and $(1 - sum)$, also violates *Uniform*.

That *sum*, $(1 - sum)$ and *minsum*[μ] are incompatible with *Uniform* means either that *Uniform* is not a genuine likeness-based constraint or that *sum*, $(1 - sum)$ and *minsum*[μ] are not *purely* likeness-based extensions of world-likeness to propositional truthlikeness. The latter horn seems clearly the right one to adopt for both *sum* and $(1 - sum)$, since the latter delivers the strong value, and the former the strong disvalue, of content for falsehoods. And for *minsum*[μ] this horn is suggested by the incompatibility result (relative to *Worst* and *Weakest*) derived above: *minsum*[μ] is transparently a likeness-content *compromise*, and like most compromises no party gets everything it wants.[40]

An anonymous referee gives the following interesting *penalty* argument against *Uniform*:

> If the true number of planets is 8, are the two rival theories "6" and "6 or 10" really equally truthlike, as [*Uniform*] requires? This principle might be plausible for a notion which excludes all content or information considerations (like approximate truth), but it violates the important idea that a theory has to pay a penalty for each mistake that it allows in its range. Thus, adding new mistakes, even when they happen to have the same size as some other mistakes, decreases the truthlikeness of a theory.

Consider the propositions mentioned here (where θ is *the number of planets*):

$$\begin{array}{ll} A & \theta = 6, \\ B & \theta = 6 \vee \theta = 10, \\ T & \theta = 8. \end{array}$$

The premise of the penalty argument is that "a theory has to pay a penalty for each mistake that it allows in its range", and the conclusion is that "… adding new mistakes, …, decreases the truthlikeness of a theory." Thus, B is further from the truth than A.

The notions of *mistake* and *paying a penalty* need a little clarification. When one says "theory A makes a mistake that B does not" what is often meant is that A has a certain implication P, *A endorses P* while B does not, and that endorsement is false. But this endorsement idea is rooted in the consequence approach and is clearly not

39 Not that Niiniluoto would regard *Uniform* as a desideratum. In his (1987, p. 233), Niinluoto's general adequacy condition M11 (*thin is better than fat*) is in direct conflict with *Uniform*.

40 cf. The *non-dictatorship* condition of Arrow's theorem.

the sense intended here—for B has strictly fewer implications than A and so makes fewer "mistakes" in this endorsement sense. A mistake, in the sense intended here, is any non-actual possibility that the proposition *permits* (rather than *endorses*). And the premise says that a proposition must *pay a penalty* for any such mistake—i.e. any non-actual possibility that it permits (i.e. any non-actual possibility in its range). A highly natural reading of this, which delivers the conclusion, is that the *addition* of a possibility to the range of a proposition must incur a penalty—the addition of a non-actual possibility must detract from a proposition's truthlikeness. The shift from A to B consists in the addition of a non-actual possibility, that $\theta = 10$, and so by the penalty premise that shift constitutes a move away from the truth. The extension procedure ($1 - sum$) embodies the conclusion here. Now consider the shift from A^* to B^*:

$$\begin{aligned} A^* \quad \theta &= 100, \\ B^* \quad \theta &= 100 \vee \theta = 10. \end{aligned}$$

The shift from A^* to B^* also involves the addition of a mistake ($\theta = 10$), the very same additional mistake that the shift from A to B involves, and, if the penalty argument is sound, a penalty should be paid, and so the shift is a move away from the truth. Those (like Miller) who consistently and rigorously endorse the strong value of content for falsehoods, would agree. Also, that is what is delivered by ($1 - sum$). But on *min*, *minmax*, *average,* and *wtdaverage* the shift from A^* to B^* is deemed a move towards the truth. It may also constitute an advance according to *minsum*, depending on whether the diminution of content (the *sum*-factor) is outweighed by an increase in likeness (the *min*-factor). So the penalty argument must be less general than this reading suggests, at least if it is to be used as an argument in favor of *minsum* and against *Uniform*. What isn't clear is how the argument could be modified, in a *non ad hoc* way to avoid this unwelcome corollary.[41]

Even with the *Uniform* constraint we still have several competing extension procedures, and so without further constraints the underdetermination objection remains. The final constraint to consider involves the impact on the truthlikeness of a proposition of modifying its likeness profile in a single respect: replacing a world in the range with another world. Let A contain world u and exclude world v, and let the range of $A^{v/u}$ differ from that of A only in that world u is replaced by world v. If v has the same degree of likeness as u to the actual world, then $A^{v/u}$ and A have the same likeness profile, and so by *Supervenience* they must have the same degree of truthlikeness. If v is closer to or further from the actual world than u is then $A^{v/u}$ has a different likeness profile from A, and there may well be a difference between the truthlikeness of A the truthlikeness of $A^{v/u}$. What could that difference depend on?

Given that the likeness profile of $A^{v/u}$ differs from A (because u and v are differing likeness to the actual world), the difference in truthlikeness of A and $A^{v/u}$ can certainly depend on those two factors: the likeness of u to the actual world w (λuw) and the likeness of v to the actual world (λvw). We already know what the impact will be when

[41] A proposal in this direction is Niiniluoto (1987) condition M8.

the two propositions are complete. Consider A_u and a world v distinct from u. $A^{v/u}$ is A_v. From *Complete* we know that the difference in truthlikeness $(\sigma_w^\lambda(A^{v/u}) - \sigma_w^\lambda(A))$ in this case is just $(\lambda vw - \lambda uw)$, exactly as one would expect. The difference here obviously depends just on λuw and λvw, but more generally it can depend on the size of A. Clearly the difference that replacing u with v could make to any proposition will be *largest* when the propositions concerned are smallest, or complete (like A_u and A_v). If A is larger then the impact of the substitution on A's overall likeness profile will be smaller. However, we don't have to stipulate any *particular* function of size here, or even that the difference is a decreasing function of the size of A. Rather, we merely allow that the difference between $\sigma_w^\lambda(A^{v/u})$ and $\sigma_w^\lambda(A)$ be *some function or other* of size:

Difference:
The difference in closeness to the truth of A and $A^{v/u}$ is some function or other of the following three factors:

i. the likeness of u to the actual world,
ii. the likeness of v to the actual world,
iii. the size of the range of A;

(i.e. there is some f such that $\sigma_w^\lambda(A^{v/u}) - \sigma_w^\lambda(A) = f(\lambda uw, \lambda vw, |rangeof A|)$).

Difference has considerable intuitive plausibility. Which of the extension procedures are compatible with it? Consider propositions E, F and G (Table 1): their ranges are the same size, they all share world u (*cold*, *dry*, *still*), and none contains v (*hot*, *rainy*, *still*).

Where λ is the standard propositional likeness measure, the values of min^λ, max^λ, $minmax^\lambda$, $average$ and *wtdave*[μ] for all the propositions, relative to the world w are tabled.

For a measure to satisfy *Difference* the substitution of v for u must make the same difference to each proposition with the same sized range. As Table 2 suggests, three of our extension procedures survive the *Difference* constraint: namely *sum*, $(1 - sum)$, and*average*. It is easy to check that *average* quite generally satisfies *Difference*. To see that *sum* and $(1 - sum)$ do also, recall that *sum* is the normalized sum of distances in the range of a proposition:

Table 1 Minimally different propositions

E	$(\neg h \wedge \neg r \wedge \neg w) \vee (\neg h \wedge r \wedge \neg w) \vee (h \wedge \neg r \wedge w)$
F	$(\neg h \wedge \neg r \wedge \neg w) \vee (h \wedge r \wedge w) \vee (h \wedge \neg r \wedge w)$
G	$(\neg h \wedge \neg r \wedge \neg w) \vee (\neg h \wedge r \wedge \neg w) \vee (h \wedge \neg r \wedge \neg w)$
u	$\neg h, \neg r, \neg w$
v	$h, r, \neg w$
w	h, r, w
$E^{v/u}$	$(h \wedge r \wedge \neg w) \vee (\neg h \wedge r \wedge \neg w) \vee (h \wedge \neg r \wedge w)$
$F^{v/u}$	$(h \wedge r \wedge \neg w) \vee (h \wedge r \wedge w) \vee (h \wedge \neg r \wedge w)$
$G^{v/u}$	$(h \wedge r \wedge \neg w) \vee (\neg h \wedge r \wedge \neg w) \vee (h \wedge \neg r \wedge \neg w)$

Table 2 Difference principle illustrated

Propositions	min_w^λ	max_w^λ	$minmax_w^\lambda[1/2]$	sum	$1-sum$	$minsum[1/2]$	$average_w^\lambda$
E	2/3	0	1/3	6/12	6/12	7/12	3/9
$E^{v/u}$	2/3	1/3	1/2	4/12	8/12	8/12	5/9
F	1	0	1/2	4/12	8/12	10/12	5/9
$F^{v/u}$	1	2/3	5/6	2/12	10/12	11/12	7/9
G	1/3	0	1/6	7/12	5/12	9/24	2/9
$G^{v/u}$	2/3	1/3	1/2	5/12	7/12	15/24	4/9
Differences							
$\sigma_w^\lambda(E^{v/u}) - \sigma_w^\lambda(E)$	0	1/3	1/6	10/12	2/12	1/12	2/9
$\sigma_w^\lambda(F^{v/u}) - \sigma_w^\lambda(F)$	0	2/3	1/3	10/12	2/12	1/12	2/9
$\sigma_w^\lambda(G^{v/u}) - \sigma_w^\lambda(G)$	1/3	1/3	1/3	10/12	2/12	3/12	2/9

Let s^λ be the sum of all distances from w in Ω: $\sum_{x\in\Omega} \delta^\lambda xw$. Then

$$sum_w^\lambda(A) = \left[\sum_{x\in A} \delta^\lambda xw\right] \Big/ s^\lambda.$$

Suppose A contains u but not v. Then:

$$\begin{aligned} sum_w^\lambda\left(A^{v/u}\right) - sum_w^\lambda(A) &= \left[\sum_{x\in A} \delta^\lambda uw\right] \Big/ s^\lambda - \left[\sum_{x\in A} \delta^\lambda xw + \left(\delta^\lambda vw - \delta^\lambda uw\right)\right] \Big/ s^\lambda \\ &= \left(\delta^\lambda uw - \delta^\lambda vw\right) \big/ s^\lambda = ((1-\lambda uw) - (1-\lambda vw)) \big/ s^\lambda \\ &= (\lambda vw - \lambda uw) \big/ s^\lambda. \end{aligned}$$

So there is an f such that: $sum_w^\lambda(A^{v/u}) - sum_w^\lambda(A) = f(\lambda vw, \lambda uw)$, which entails *Difference* for both *sum* and $(1 - sum)$.[42]

The procedures which obviously fail *Difference* are *min* and *max*, and the derivative procedures that depend on them—namely *minmax*[μ] and *minsum*[μ].

An anonymous referee for this journal has argued that a violation of *Difference* is admissible in a likeness-based extension procedure:

> … it can be claimed that the location (not only the number) of other worlds in the range of A is relevant: in the planet case, the shift from [14 or 15 or 20] to [13 or 15 or 20] is more dramatic than the shift from [10 or 14 or 15] to [10 or 13 or 15], since the former improves the best guess of the theory.

According to the objection, the former shift is larger than the latter, because the former increases the likeness of the proposition by increasing the likeness of the closest world to actuality (*min* value) while latter merely increases the likeness of its second closest

42 Note that *Difference* does not say that the difference *has* to vary with size of A, but that it may.

Table 3 Compatibility of measures with the likeness constraints

		sum	$1 - sum$	*minsum*	*min*	*max*	*minmax*	*average*
L_1	Supervenience (=Weak likeness)	Yes	Yes	Yes	Yes	Yes	Yes	Yes
L_2	L_1+ Worst (= Moderately Weak likeness)	No	Yes	Yes	Yes	Yes	Yes	Yes
L_3	L_3+ Uniform (= Moderately Strong likeness)	No	No	No	Yes	Yes	Yes	Yes
L_4	L_3+ Difference (= Strong likeness)	No	No	No	No	No	No	Yes

world. Now, *min* is undoubtedly an excellent likeness-based measure of *closeness to being true*, and there is every reason to expect that substitution of v for w can have dramatically different effects on that score. But unless truthlikeness is a smooth function of *closeness to being true* and some other factors (*content*, say), the fact that*min* violates *Difference* is not in itself a reason to think that *closeness to the whole truth* should also violate *Difference*. Unlike $(1 - sum)$ and *average*, *min* does not respond smoothly and continuously to changes in likeness profile, which is what *Difference* ensures.

We now have a series of increasingly strong characterizations of the likeness approach from L_1 (*Weak Likeness* constraint) through to the strongest, L_4 (*Strong Likeness* constraints). The two intermediate constraints, L_2 *and* L_3, are *Moderately Weak* and *Moderately Strong* respectively. We can show that *average* is not just the only extension procedure among those considered to satisfy the strong likeness constraint, but that it is the only extension procedure *simpliciter* to so qualify.

The only extension procedure satisfying the strong likeness constraint is average (3)

Note that the only two principles used explicitly in the proof of (3) are *Uniform* and *Difference*, which jointly entail both *Worst* and *Supervenience*.

Average likeness thus plays a privileged role with respect to the strong likeness approach. There are, of course, still a range of procedures satisfying the weak or moderate likeness-based constraints, which are thus part of the likeness approach weakly construed (Table 3).

7 Combining content-based, consequence-based and likeness-based constraints

We have characterized the three approaches in order to determine whether or not they can be combined—whether or not there are measures or orderings that satisfy the desiderata of say, both likeness and content approaches, or both likeness and consequence approaches, or all three—and if so what they would look like. At first blush the main result of Sect. 5 seems to answer negatively the question about the compatibility of the strong likeness approach with either of the other two approaches. For any content-based or consequence-based measure of closeness to truth delivers the strong value of content for truths, whereas the only strong likeness-based measures involve averaging likeness between worlds, and that, as Popper contended, apparently contradicts the strong value of content for truths. The two strong likeness constraints (*Uniform* and *Difference*) may be individually compatible with the strong value of content for truths, but it seems that together they rule it out.

In fact, perhaps surprisingly, this is not right. The three approaches are, strictly speaking, compatible after all, even on the strong likeness approach. This begins with the compatibility of the approaches at the level of the three kinds of measures:

$$\textit{There is a measure of closeness to truth that is simultaneously content-based, consequence-based and likeness-based.} \quad (4)$$

The measure which satisfies the strictures of all three approaches is actually rather simple. Call the following likeness function *flat*: for all u and $v : \lambda^{Flat}uv = 1$ if $u = v$, and $\lambda^{Flat}uv = 0$ if $u \neq v$. Let ψ^{Flat} be average likeness based on λ^{Flat}: (i.e. $\psi_w^{Flat}(A) = average_w^{\lambda^{Flat}}(A)$. ψ^{Flat} is a likeness-based, a content-based and a consequence-based measure. Since ψ^{Flat} satisfies *Strong,* it also satisfies the weaker likeness constraints.

While ψ^{Flat} is a likeness measure, it embodies a rather extreme *nihilism* about likeness—namely, that as a matter of brute necessity no world is more like the actual world than is any other. While this seems highly implausible, in fact this kind of likeness-nihilism seems to be precisely the upshot of a now famous line of argumentation against likeness-based accounts of truthlikeness—viz. the famous "translation argument"—first articulated by David Miller at the very birth of the likeness approach.[43]

Not only does likeness nihilism satisfy all three approaches, we can show quite generally that by requiring that a measure satisfy the strictures of all three approaches we thereby *guarantee* nihilism about likeness. That is to say, by combining the approaches we solve not only the underdetermination of extension procedures, but we uniquely pin down the underlying likeness function. This follows from a somewhat stronger result, that it is a perfectly general feature of any measure that satisfies both the strong likeness and content constraints:

$$\textit{If a measure of closeness to truth satisfies both strong likeness and content constraints then the likeness function between worlds is } \lambda^{Flat}. \quad (5)$$

(4) tells us that the three approaches are indeed compatible, and (5) tells us that together they uniquely determine the likeness function. However, the cost of requiring a measure of closeness to truth to satisfy both strong likeness and content approaches is very high—namely:

43 Miller's argument was first articulated in his (1974). See Oddie (1986, Chap. 6).

Any measure of closeness to truth that satisfies the content-based and strong likeness-based constraints entails both the relative and absolute trivialization of truthlikeness.

This is a straightforward implication of (5). If σ satisfies strong likeness, and σ^{λ} is content-based, then λ is flat, and $\sigma^{\lambda} = \psi^{Flat}$. If A is true in w then $\psi_w^{Flat}(A) = 1/|A| > 0$. So every truth has positive degree of closeness to truth. However if A is false in w then $\psi_w^{Flat}(A) = 0$: i.e. every falsehood has zero degree of closeness to truth. Any strongly likeness-based measure that is content-based deems every falsehood less truthlike than any truth (viz. absolute trivialization) and deems no falsehood more truthlike than any other (viz. relative trivialization).[44]

These trivialization results do not necessarily hold for the weaker likeness-based constraints. For example, *min*, *max* and *minmax* are moderately likeness-based measures. While *min* and *max* yield absolute trivialization for falsehoods, they do not yield relative trivialization. Further, *sum*, $(1 - sum)$ and *minsum* are all moderately weak likeness-based measures, and they do not yield either the relative or absolute trivialization results for falsehoods. The compatibility of the approaches to measuring truthlikeness yields compatibility at the level of orderings as well:

$$\textit{The content-based, consequence-based and likeness-based approaches to orderings of closeness to truth are compatible: there are orderings of closeness to truth that are content-based, consequence-based, and strongly likeness-based.} \quad (6)$$

Indeed it turns out, perhaps rather surprisingly, that Popper's original ordering satisfies all three approaches. As we have seen, it delivers both the core content-based judgments (viz. the strong value of content for truths) and the core consequence-based judgments. Popper's ordering is realized by ψ^{Flat} (which is a measure in all three approaches). Because we have placed so few constraints on measures of likeness between worlds—by allowing even the nihilistic flat function—the core likeness-based judgments are rather sparse indeed, and it turns out that Popper's ordering incorporates them all.[45]

It follows directly from (5) that if an ordering is realized by a measure that is both content-based and strongly likeness-based then the likeness measure must be based on a flat distance function. Hence the ordering must render falsehoods no closer to the

[44] The role that a flat or nihilisitic likeness (or distance) function can play in reconciling the approaches has been noted before in the literature. For example, Niiniluoto (1987) points out that for the trivial or "flat" distance function the *minsum* measure reduces to Levi's measure of epistemic utility, a content-based measure of closeness to truth (viz. a weighted average of truth value and content measured relative to even probability). This shows that the *minsum*$[\mu]$ extension procedure can deliver a purely content-based measure, but only in the extreme case of nihilism about likeness.

[45] This answers something that the reader might find puzzling. All but the weakest likeness-based constraints incorporate *Worst*, whereas Popper's ordering incorporates *Weakest* which seems incompatible with *Worst*. However, *Worst* and *Weakest* are compatible if the likeness function is flat. In that case there is no worst world, no complete proposition containing just the most distant worlds, and the proposition that contains all the worlds most distant from the actual is equivalent to the weakest falsehood—$\neg T$.

truth than any truth, and falsehoods no closer to the truth than any other.

> *An ordering of closeness to truth that can be realized by a measure that is both strongly likeness-based and content-based entails both the relative and absolute trivializations of truthlikeness for falsehoods.*

What about orderings that satisfy both strong likeness and content constraints in virtue of their realization by *distinct* content-based and likeness-based measures? It may be possible for the associated strongly likeness-based measure to escape flatness, and so the ordering may deem some falsehoods to be closer to the truth than others. Even if this is possible, such orderings are still less than ideal for vindicating the possibility of normal progress in an inquiry, since they still deem no falsehood closer to the truth than a trivial truth. Indeed, the following result applies to *any ordering at all that delivers the strong value of content for truths*, whether it is content-based, consequence-based or neither of those.

> *Any ordering of closeness to truth that delivers the strong value of content for truths and is strongly likeness-based yields the absolute trivialization of truthlikeness for falsehoods.*

The proof of this is both simple and instructive so I include it here rather than relegate it to the appendix. Suppose $\geq$ is strongly likeness-based and delivers the strong value of content for truths. Suppose, for the sake of a reductio, that there is a world w such that in w a falsehood A is closer to the truth than some truth or other. Since $\geq$ delivers the strong value of content for truths, amongst truths the furthest from the truth are the tautologies, like $(A \vee \neg A)$. Hence $A >_w (A \vee \neg A)$. Since $\geq$ is strongly likeness-based there is a likeness function λ such that $average^{\lambda}$ realizes $\geq$. So: $average^{\lambda}_{w}(A) > average^{\lambda}_{w}(A \vee \neg A))$. Let n be the total number of worlds in the space and let f be the number of worlds in F. $average^{\lambda}_{w}(A \vee \neg A)$ is the following weighted average of $average^{\lambda}_{w}(A)$ and $average^{\lambda}_{w}(\neg A)$:

$$average^{\lambda}_{w}(A \vee \neg A) = f/n\ average^{\lambda}_{w}(A) + (n-f)/n\ average^{\lambda}_{w}(\neg A).$$

Since $average^{\lambda}_{w}(A) > average^{\lambda}_{w}(A \vee \neg A)$ it follows that:

$$average^{\lambda}_{w}(\neg A) < average^{\lambda}_{w}(A \vee \neg A)). \qquad (*)$$

$\neg A$ is both true and logically stronger than $(A \vee \neg A)$, violating the strong value of content for truths. Hence no falsehood can be closer to the truth than any truth. Which is just to say that an ordering that satisfies both strong likeness and content yields the absolute trivialization of truthlikeness for falsehoods.

8 Generalizing the likeness approach: weighted averaging and straight averaging

Since *average* is the only extension procedure to satisfy *Uniform* and *Difference*, it follows that *weighted average* is a strong likeness-based extension procedure only where the weights are equal. As the critics of averaging have often pointed out, there is something arbitrary about restricting extensions to straight averaging. It is tantamount to stipulating that every world has the same logical weight in the evaluation of the closeness of a proposition to the actual world, an assumption that it is impossible to extend to denumerable spaces. Of course, that assumption is tacitly embodied in the difference principle in that the effect of replacing a world in a proposition is a function *inter alia* only of the bare *number* of worlds in the proposition. However a likeness-based theory of closeness to truth need not be committed to a particular account of how much weight worlds have, and it shouldn't rule out denumerable spaces.[46] The way for the likeness theorist to remain agnostic on this issue would be to entertain different measures, μ, and consider μ-*weighted* averages in general. This involves a generalization of the *Difference* principle to allow for different weighting functions.

Suppose that μ is a logical measure function on the logical space Ω. Let $A^{v/u}$ differ from A only in the substitution of a world v for a world u with the same μ-value.

Generalized Difference:

The difference in closeness to truth of A and $A^{v/u}$ is some function or other of the following three factors:

i. the likeness of u to the actual world;
ii. the likeness of v to the actual world;
iii. the μ-value of worlds *u and v*.

(i.e. there is some f such that $\sigma_w^\lambda(A^{v/u}) - \sigma_w^\lambda(A) = f(\lambda uw, \lambda vw, \mu(v))$).[47]

From this together with *Uniform* we can show that *weighted average* (or *expected likeness*) is the only likeness-based measure of closeness to truth[48]:

$$\sigma w(A, \lambda, \mu) = wtdavew(A, \lambda, \mu) = \sum_v \mu(v|A)\lambda(w, v).$$

Supplant *average* with *weighted average* and we can still demonstrate the central result of this paper: that any strongly likeness-based ordering that delivers the strong value of content yields the absolute trivialization of truthlikeness for falsehoods.[49]

46 See also Kieseppä (1996) for an illuminating treatment of the extension of distance measures to infinite, continuous, multi-dimensional spaces, including extensions by averaging.

47 σ thus depends not just on λ, but also on μ. So that *Generalized Difference* does not contradict *Supervenience* we also need to generalize the notion of likeness profile. $A \overset{u}{\underset{v}{\approx}} B$ iff there is a 1–1 mapping f from A to B, such that for every $x\lambda xu = \lambda\ f(x)v$ and $\mu(x) = \mu(f(x))$.

48 For continuous spaces we also require a modest continuity assumption—for example, that the difference function f reach its maximum on complete propositions. Finally we can lift the restriction on the finiteness of the logical space and subject to some modest conditions, and derive: $\sigma_w(A, \lambda, \mu) = \int_A \lambda_w\ d\mu$. See Oddie and Milne (1991) for details of a closely related proof.

49 Clearly $wtdave_w((F \vee \neg F), \lambda, \mu) = \mu(F)wtdave_w(F, \lambda) + \mu(\neg F))wtdave_w(\neg F, \lambda)$.

9 The essential dilemma: underdetermination versus trivialization

On my characterizations of content, consequence and likeness approaches they are, as Schurz and Weingartner maintain, strictly compatible. However, with Zwart and Franssen I have argued that there is a deep tension between the likeness approach and the content and consequence approaches. This tension can be cast in the form of a dilemma.

We have a limited number of data points—that is, uncontroversial clear-cut judgements on simple cases that narrow down the range of acceptable theories of truthlikeness. So a bottom-up data-driven approach to the problem leaves the concept of truthlikeness grossly underdetermined. What we need in addition to particular data points are a small number of compelling, top-down principles that narrow the range of admissible measures and orderings. For content and consequence theorists the strong value of content for truths is a constraint of this sort. But even with this constraint in hand we are left with a largely underdetermined concept of closeness to truth. The two weak likeness constraints (*Supervenience* and *Worst*) also leave the concept wide open, but the two strong likeness constraints (*Uniform* and *Difference*) jointly nail down the permissible extension procedures—ways of extending likeness between worlds to the truthlikeness of propositions—to just one, namely *averaging*. What they do not do (and are not intended to do) is narrow down the range of admissible likeness functions.

A natural thought, then, is to combine the strong likeness constraints with the content constraints. Together these do completely solve *both* underdetermination problems. But the sole likeness measure to satisfy both kinds of constraints is the nihilistic likeness function, λ^{Flat}. And likeness nihilism yields the very defects that made Popper's original account untenable—namely the relative and absolute trivialization of truthlikeness for falsehoods. So any ordering that satisfies both strong likeness constraints and the strong value of content for truths implies the trivialization of truth likeness.

Absent some compelling new principles which would solve the underdetermination problems without over-constraining the likeness function we are thus caught between the horns of underdetermination and trivialization. Pending the discovery of such principles, I myself am content to embrace the two strong likeness constraints and abandon the strong value of content for truths.

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Footnote 49 continued
Provided F is possible, $\mu(F) > 0$, and so this together with $wtdave_w(F, \lambda, \mu) > wtdave_w((F \vee \neg F), \mu)$ (value of content for truths) entails: $wtdave_w(\neg F, \lambda, \mu) < wtdave_w((F \vee \neg F), \lambda, \mu)$.

Appendix

(1) The strong value of content for truths is the core content ordering

$A \geq_w^T B =_{df}$ A and B are both true in w and $A \vdash B$; and $A \geq_w^{Ct} B =_{df} \psi_w(A) \geq \psi_w(B)$ for every content based measure ψ. We will show $A \geq_w^T B$ iff $A \geq_w^{Ct} B$.

Let **B** be a complete metric Boolean algebra of propositions. As is well known, for every such algebra there is a countably additive, strictly positive measure on **B**. (Throughout, abbreviate "countably additive, strictly positive measure on **B**" to "measure".) First, if A and B are logically independent (i.e. not $A \vdash B$ and not $B \vdash A$) then it is tedious but straightforward to show that there are measures $\mu_1 \mu_2$ and μ_3 such that $\mu_1(A) = \mu_1(B)$, $\mu_2(A) > \mu_2(B)$ and $\mu_3(A) < \mu_3(B)$. Let $Content^{\mu}(A) = 1 - \mu(A)$.

Define:

$$f^H : f^H(1, c) = c, \text{ and } f^H(0, c) = -c.$$

$$f^M : f^M(1, c) = c + m, \text{ and } f^M(0, c) = c - m \text{ (for some } m > 0)$$

$$f^P : f^P(1, c) = c, \text{ and } f^P(0, c) = 0;$$

let ψ^H, ψ^M and ψ^P be the corresponding content-based measures relative to measure μ (where $\psi_w(A) = f(Truth_w(A), Content^{\mu}(A))$).[50]

(1) Suppose A and B are both true in w. Suppose further that $A \vdash B$. Hence $A \geq_w^T B$. For all μ, $\mu(A) \leq \mu(B)$, $Content^{\mu}(A) \geq Content^{\mu}(B)$, and so by clause (ii) of content based measure, $\psi_w(A) \geq \psi_w(B)$ for any content-based ψ. So $A \geq_w^{Ct} B$. Now suppose not $A \vdash B$. Then *not*-$A \geq_w^T B$. Assume $A \geq_w^{Ct} B$ (for sake of a reductio). Clearly B is not logically equivalent to A, and not $B \vdash A$ (for then $\psi_w(B) > \psi_w(A)$—contradiction). So A and B are logically independent. But then there is a μ such that $\mu(A) > \mu(B)$, $Content^{\mu}(A) < Content^{\mu}(B)$, and so $f^M(1, Content^{\mu}(B)) > f^M(1, Content^{\mu}(B))$; hence $\psi_w^M(B) > \psi_w^M(A)$—contradicting $A \geq_w^{Ct} B$. Hence for true A and B, $A \geq_w^T B$ iff $A \geq_w^{Ct} B$. We now have to show that if either A or B is false, and they are not logically equivalent, not $A \geq_w^{Ct} B$.

(2) Let A and B be false in w. Suppose A and B are logically independent. Then there is a μ such that $\mu(A) > \mu(B)$: hence $Content^{\mu}(A) < Content^{\mu}(B)$, $\psi_w^H(A) > \psi_w^H(A)$ and $\psi_w^M(B) > \psi_w^M(A)$. Hence neither $A \geq_w^{Ct} B$ nor $B \geq_w^{Ct} A$. Suppose that A and B are not logically independent. Without loss of generality, assume $B \vdash A$ and not $A \vdash B$. Then $\mu(A) > \mu(B)$, $Content^{\mu}(A) < Content^{\mu}(B)$, $\psi_w^H(A) > \psi_w^H(A)$, $\psi_w^M(B) > \psi_w^M(A)$; and so neither $A \geq_w^{Ct} B$ nor $B \geq_w^{Ct} A$.

(3) Suppose A is false in w, and B is true. (a) $\psi_w^P(B)$ is positive and $\psi_w^P(A)$ is zero, so $\psi_w^P(B) > \psi_w^P(A)$: hence *not*-$A \geq_w^{Ct} B$. (b) B (true) does not entail A (false). Either (b1) $A \vdash B$ or (b2) not $A \vdash B$. If (b1) then (since not $B \vdash A$) $\mu(B) > \mu(A)$, $Content^{\mu}(B) < Content^{\mu}(A)$. If (b2) then A and B are logically independent, so there is a μ s.t. $\mu(A) > \mu(B)$, and Content $^{\mu}(B) < Content^{\mu}(A)$.

[50] H is for Hempel; M for Miller, and P for Popper. Hempel effectively proposed ψ^H as a measure of epistemic utility, and ψ^M and ψ^P realize, respectively, Miller's and Popper's orderings.

Let $m = (Content^{\mu}(A) - Content^{\mu}(B))/2$. Then:

$$\begin{aligned}\psi^M w(A) &= Content^{\mu}(A) - (Content^{\mu}(A) - Content^{\mu}(B))/2\\ &= (Content^{\mu}(A) + Content^{\mu}(B))/2\\ &> Content^{\mu}(B) = \psi^M_w(B).\end{aligned}$$

Since $\psi^M_w(A) > \psi^M_w(B)$, $not\text{-}B \geq^{Ct}_w A$.

Corollary *Popper's ordering, and the symmetric difference ordering, are both content-based.*

Popper's ordering, $\geq^P$, delivers all the core content-based judgements (1). Further, ψ^P clearly realizes $\geq^P$. This is obvious if A and B are true. If A and B are false, then neither $A >^P_w B$ nor $B >^P_w A$. (ψ^P extends $\geq^P$ by delivering $A \approx B$ for all false A, B.) If A is true and B is false, then $A >^P_w B$ whenever $A \vdash B \vee T_w$. In all such cases $\psi^P_w(A) > 0$, and $\psi^P_w(B) = 0$. (Again, ψ^P extends Popper's ordering.)

Miller's ordering, $\geq^M$, entails the strong value of content for truths and so delivers all the core content-based judgements (1). We show that if $A \geq^M_w B$ then provided $m = 1$, $\psi^M_w(A) \geq \psi^M_w(B)$. If A and B are true, and $A \geq^M_w B$ then $A \vdash B$, $Content^{\mu}(A) + 1 \geq Content^{\mu}(B) + 1$, and so $\psi^M_w(A) \geq \psi^M_w(B)$. If A and B are false, then $A \geq^M_w B$ iff $A \vdash B$: $Content^{\mu}(A) - 1 \geq Content^{\mu}(B) - 1$, and so $\psi^M_w(A) \geq \psi^M_w(B)$. If A is false and B is true, then not $A \geq^M_w B$, so no constraint imposed. Suppose A true, B false and $A \geq^M_w B$. Then (since $\geq^M$ yields that no falsehood is as close to the truth as any as a truth) in fact $A >^M_w B$.

Since A is true: $\psi^M_w(A) = Content^{\mu}(A) + 1 > 0$.
Since B is false: $\psi^M_w(B) = Content^{\mu}(B) - 1 < 0$.
Hence, $\psi^M_w(A) > \psi^M_w(B)$.

(2) Popper's ordering is the core consequence ordering.

$A \geq^P_w B$ = Popper's account entails that A is as close to the truth as B in w;
$A \geq^{Cn}_w B$ = for all consequence-based measures ψ, $\psi_w(A) \geq \psi_w(B)$.

We show that $A \geq^P_w B$ iff $A \geq^{Cn}_w B$.

(1) Assume $A \geq^P_w B$. (i) Suppose A and B true: then A $\vdash B$, and (since every consequence-based measure delivers the strong value of content for truths) $A \geq^{Cn}_w B$. (ii) If both A and B are false then by relative trivialization they are logically equivalent and the result follows. Suppose just one of A or B is false. Since $A \geq^P_w B$, A cannot be false (by absolute trivialization), so A is true, B false. A (being true) is equivalent to A_P, and hence to $A^T_{P,w}$ (since A is true in w, $A_P = A^T_{P,w}$). $A^T_{P,w} \vdash B^T_{P,w}$, and $B^F_{P,w} \models A^F_{P,w}$ (since $B^F_{P,w} \neq \emptyset$, and $A^F_{P,w} = \emptyset$). Clearly for every R, $A^F_{R,w} = \emptyset$ so A is equivalent to $A^T_{R,w}$. Since, $A^T_{P,w} \vdash B^T_{P,w}$ and $B^T_{P,w} \vdash B^T_{R,w}$, it follows that $A^T_{R,w} \vdash B^T_{R,w}$. Since, $B^F_{P,w} \neq \emptyset$, $B^F_{R,w} \models A^F_{R,w}$. Hence for every R, $A^T_{R,w} \vdash B^T_{R,w}$

and $B^F_{R,w} \models A^F_{R,w}$. It follows that for every f satisfying condition (iii) on consequence-based measures, $f(A^T_{R,w}, A^F_{R,w}) > f(B^T_{R,w}, B^F_{R,w})$, hence for every consequence-based measure ψ, $\psi_w(A) > \psi_w(B)$, so $A \geq^{Cn}_w B$.

(2) Now assume $A >^{Cn}_w B$. We show that $A >^P_w B$. (i) Let A and B both be true. If $A \models B$ then $A >^P_w B$. Suppose not $A \models B$. If $B \models A$ then $B >^T_w A$, hence not $A >^{Cn}_w B$ (because every instance of the strong value of content for truths is consequential); and if A and B are logically equivalent, not $A >^{Cn}_w B$. So A and B must be logically independent. Let criterion W be such that: for any A, $A_W = \{A \vee T_w, A\}$. Note that A is equivalent to A_W; that if A is true in w then $A^T_{W,u} = \{A \vee T_w, A\}$, $A^F_{W,w} = \emptyset$; if A is false in w then $A^T_{W,u} = \{A \vee T_w\}$, $A^F_{W,w} = \{A\}$. Let $f(E, F) = Content^\mu(E) - Content^\mu(F)$, and $\psi_u(A) = f(A^T_{W,u}, A^F_{W,u})$. Clearly ψ is consequence-based. If A is true in w then $\psi_u(A) = Content^\mu(A)$. Since A and B are logically independent, there is a μ such that $Content^\mu(A) > Content^\mu(B)$ and vice versa, so not $A >^{Cn}_w B$. (ii) Let A and B both be false. If C is false in w then $\psi_w(C) = Content^\mu(C \vee T_w) - Content^\mu(C) = 1 - (\mu(C) + \mu(Tw)) - (1 - \mu(C)) = -\mu(Tw)$. So $\psi_w(A) = \psi_w(B)$, and not $A >^{Cn}_w B$. (iii) Suppose A is false and B is true: then $\psi_w(A) < \psi_w(B)$, so not $A >^{Cn}_w B$. (iv) Finally, suppose that A is true and B is false. If $A \models (B \vee T_w)$ then $A >^P_w B$. So suppose not $A \models (B \vee T_w)$. Then there is a u such that $u \neq w$, u is in range of A and u is not in range of B. Let U be the condition such that for every $C : C_U = \{C \vee (B \vee T_w), C \vee \neg(B \vee T_w)\}$. Note that C is equivalent to C_U. Let $f(E, F) = Content^\mu(E) - Content^\mu(F)$, and $\psi_v(A) = f(A^T_{U,v}, A^F_{U,v})$. Clearly ψ is consequence-based. $A^T_{U,w} = \{A \vee (B \vee T_w)), A \vee \neg(B \vee T_w))\}$, $A^F_{U,w} = \emptyset$, and $\psi_w(A) = Content^\mu(A) = 1 - \mu(A)$. $B \vee (B \vee T_w)$ is equivalent to $B \vee T_w$; $B \vee \neg(B \vee T_w)$ is equivalent to $\neg T_w$; $B^T_{U,w} = \{B \vee T_w\}$; $B^T_{U,w} = \{\neg T_w\}$, and:

$$\begin{aligned}\psi_w(B) &= 1 - (\mu(B) + \mu(T_w)) - (1 - \mu(\neg T_w)) \\ &= 1 - \mu(B) - \mu(T_w) - 1 + 1 - \mu(T_w) \\ &= 1 - \mu(B) - 2\mu(T_w).\end{aligned}$$

$\psi_w(B) \geq \psi_w(A)$ iff $(1 - \mu(B) - 2\mu(T_w)) > 1 - \mu(A)$ iff $\mu(B) \leq (\mu(A) - 2\mu(w))$. Since A is true and B is false, B does not entail A. This inequality will hold provided $\mu(u) \leq \mu(B \vee \neg A) + \mu(w)$, and because the range of $(B \wedge \neg A)$ and $\{w, u\}$ are disjoint there is such a μ. It follows that not $A >^{Cn}_w B$. Since A and B are equivalent under all consequence based measures if and only if they are logically equivalent, the result clearly follows.

(3) The only extension procedure to satisfy strong likeness is average.[51]

For each proposition $A = \{u_1, u_2, \ldots, u_n\}$ π is an *A-function* (of *degree n*) provided $\pi^\lambda_w i = \lambda u_i w$ for all $i \leq n$, and is undefined for $i > n$. The strong likeness constraint is then equivalent to the following restrictions on a functional σ, defined on all functions

[51] This is a special case of a more general result to the effect that value = expected value, proved in Oddie and Milne (1991). Here I provide a simplified proof of the more limited result.

of finite degree. (We assume supervenience, which guarantees that each A-function π has the same σ-value.)

Uniformity
If π is degree n, and for all $i \leq n,\ \pi(i) = s$, then $\sigma(\pi) = s$.
Difference
Suppose π and π^* are degree n and differ at most in that for some $j \leq n,\ \pi_j \neq \pi_j^*$; (i.e. for all $i \neq j \leq n, \pi_i = \pi_i^*$): then there is an f such that for all such $\pi,\ \pi^*,\ n$:

$$\sigma(\pi) - \sigma(\pi^*) = f(\pi_j, \pi_j^*, n).$$

Where π is degree n, *average*$(\pi) = \sum_{k=1}^{n} \pi_k/n$. Note that *average* obviously obeys *Uniform*. Since $average(\pi) - average(\pi^*) = \sum_{k=1}^{n} \pi_i/n - \sum_{k=1}^{n} \pi_k^*/n = (\pi_j - \pi_j^*)/n$ *average* also obeys *Difference*. Assume that σ obeys *Uniform* and *Difference*. Let π be any function of degree n and let π^0 be the function of degree n such that for all $i \leq n,\ \pi_i^0 = 0$. For each $k > 0$, π^k is the function defined thus: for all $j \leq k, \pi_j^k = \pi_j$, for all j such that $k < j \leq n, \pi_j^k = 0$; and for all $j > n,\ \pi_j^k$ undefined. Note that $\pi^n = \pi$ and each π^k differs from its predecessor π^{k-1} in at most that $\pi_k^k \neq \pi_k^{k-1}$(because $\pi_k^{k-1} = 0$ whereas $\pi_k^k = \pi_k$which may be larger than 0). By *Uniformity*, $\sigma(\pi^0) = 0$. By *Difference*, there is a function f such that:

	$\sigma(\pi^k) - \sigma(\pi^{k-1}) = f(\pi_k^k, \pi_k^{k-1}, n)$.
Since $\pi_k^{k-1} = 0$:	$\sigma(\pi^k) - \sigma(\pi^{k-1}) = f(\pi_k^k, 0, n)$.
Since	$\sigma(\pi^0) = 0$,
	$\sigma(\pi) = \sigma(\pi) - \sigma(\pi^0)$.
Note that	$\sigma(\pi) - \sigma(\pi^0) = \sum_{k=1}^{n} \sigma(\pi^k) - \sigma(\pi^{k-1})$.
Since	$\sigma(\pi^k) - \sigma(\pi^{k-1}) = f(\pi_k^k, 0, n)$ and $\pi_k^k = \pi_k$
it follows that	$\sigma(\pi) = \sum_{k=1}^{n} f(\pi_k^k, 0, n) = \sum_{k=1}^{n} f(\pi_k, 0, n)$.

If for all $k \leq n,\ \pi_k = s$ then by *Uniformity*, $\sigma(\pi) = s$, and so from this we have the following constraint on f:

	$s = \sum_{k=1}^{n} f(s, 0, n) = nf(s, 0, n)$.
i.e.	$f(s, 0, n) = s/n$ *for all* s *and* n.
Since	$\sigma(\pi) = \sum_{k=1}^{n} f(\pi_k, 0, n)$,
we have	$\sigma(\pi) = \sum_{k=1}^{n} \pi_k/n$.
That is:	$\sigma(\pi) = average(\pi)$.

(4) Compatibility of content, consequence, and strong likeness approaches: there is a measure of closeness to truth that is content-based, consequence-based and likeness-based

For simplicity assume a Boolean algebra **B** over a finite space Ω of worlds of cardinality n. For all all u and v: $\lambda^{Flat}uv = 1$ if $u = v$, and $\lambda^{Flat}uv = 0$ if $u \neq v$. $\psi_w^{Flat}(A) =_{df} average_w^{\lambda^{Flat}}(A)$.

ψ^{Flat} is clearly likeness-based. Let $|A|$ be the cardinality of the range of the meet of A. Note that:

if A is true, $\psi_w^{Flat}(A) = 1/|A|$,
if A is false, $\psi_w^{Flat}(A) = 0$.

Let $\mu(A) = |A|/n$, and let $Content^{\mu}(A) = 1 - \mu(A)$ (the standard measure of content w.r.t. μ). Let $f^n(c) =_{df} (1/(n(1-c)))$. Note that $f^n(1-\mu(A)) = 1/|A|$, and that if $c < d$ then $f^n(c) < f^n(d)$. Recall that $Truth\,(A) = 1$ if A is true, and 0 if A is false, so:

$[Truth_w(A) \times f^n(Content^{\mu}(A)] = 1/|A|$ if A is true, 0 if A is false. i.e.
$\psi_w^{Flat}(A) = [Truth_w(A) \times f^n(Content^{\mu}(A)]$.

Hence ψ^{Flat} is a function of truth and content factors, and since the function satisfies conditions (ii) and (iii) of the definition of a content-based measure, ψ^{Flat} itself is a content-based measure.

Let $\varepsilon(G)$ be an indicator function which returns 1 if $|G| < n$, and 0 if $|G| = n$ (recall that $n = |\Omega|$). That is, $\varepsilon(G)$ returns 1 if G is not tautologous and 0 if it is. Let P be the empty, Popperian criterion of relevance, and let g^{μ} be the following function:

if there is a proposition A such that $E = A^T_{P,w}$ and $G = A^F_{P,w}$ then

$$g^{\mu}(E, G) = f^n(Content^{\mu}(E) - Content^{\mu}(G)) - \varepsilon(G)/(n+1);$$

If there is no such A then:

$$g^{\mu}(E, G) \text{ is undefined.}$$

For any proposition A,

$$g^{\mu}\left(A^T_{P,w}, A^F_{P,w}\right) = f^n\left(Content^{\mu}\left(A^T_{P,w}\right) - Content^{\mu}\left(A^F_{P,w}\right)\right)$$
$$-\varepsilon\left(A^F_{P,w}\right)/(n+1).$$

If A is true in w, then $Content^{\mu}(A^F_{P,w}) = 0$ and $\varepsilon(A^F_{P,w}) = 0$. Where $a = |A|$:

$$g^{\mu}\left(A^T_{P,w}, A^F_{P,w}\right) = f^n\left(Content^{\mu}\left(A^T_{P,w}\right)\right)$$
$$= f^n(1-(a/n)) = 1/(n(1-(a/n)) = 1/a = \psi_w^{Flat}(A).$$

If A is false in w, then $\varepsilon(A^F_{P,w}) = 1$, $A^F_{P,w}$ is equivalent to A, $Content^{\mu}(A^F_{P,w}) = Content^{\mu}(A)$, $Content^{\mu}(A^T_{P,w}) = Content^{\mu}(A \vee T_w) = 1 - (\mu(A) + \mu(T_w))$, and so:

$$
\begin{aligned}
g^{\mu}\left(A^{T}_{P,w}, A^{F}_{P,w}\right) &= f^{n}\left((1-(\mu(A)+\mu(T_w)))-(1-\mu(A))\right)-(1/(n+1)) \\
&= f^{n}(-\mu(T_w))-(1/(n+1)) = f^{n}(-1/n)-(1/(n+1)) \\
&= 1/(n+1)-(1/(n+1)) \\
&= 0 \\
&= \psi^{Flat}_{w}(A).
\end{aligned}
$$

Clearly ψ^{Flat} satisfies the last two conditions for a consequence-based measure as well: clause (ii) is satisfied since for truths ψ delivers the strong value of content; clause (iii) is satisfied vacuously when the relevance criterion is the empty Popperian criterion P.

(5) If a measure of closeness to truth is both content-based and strongly likeness-based then the associated likeness function is λ^{Flat}

Suppose $average^{\lambda}_{w}(A) = \psi_w(A)$ for some content-based measure ψ. Since there are functions f and μ such that $\psi_w(A) = f(Truth_w(A), Content^{\mu}(A))$ it follows (since $Content^{\mu}$(A) is world-independent) that in any two worlds u and v such that $Truthu(A)$ = $Truth_v(A)$, $\psi_u(A) = \psi_v(A)$. Hence: $average^{\lambda}_{w}(A) = average^{\lambda}_{v}(A)$. Suppose A is any proposition whose range contains just one world w, and let u and v be two worlds distinct from u and distinct from each other:

$$
\begin{aligned}
average^{\lambda}_{u}(A_w) &= \lambda wu, \\
average^{\lambda}_{v}(A_w) &= \lambda wv.
\end{aligned}
$$

Since $average^{\lambda}_{u}(A_w) = average^{\lambda}_{v}(A_w)$, it follows that $\lambda wu = \lambda wv$. So for any two distinct worlds v and v, world w bears the same degree of likeness to both. By analogous reasoning, world v bears the same degree of likeness to both w and v, and world v bears the same degree of likeness to both u and w. Hence any three distinct worlds bear the same degree of likeness to each other. All worlds are maximally unlike each other, and since likeness is based on a normalized distance function, it follows that for any two distinct worlds u and v, $\lambda uv = 0$. So any likeness-based measure that is also content-based yields a flat measure of likeness between worlds.

(6) There are orderings of closeness to truth (e.g. Popper's) that are content-based, consequence-based, and strongly likeness-based

Consider Popper's ordering $\geq^{P}$. ψ^{Flat} realizes $\geq^{P}$. Recall that $\psi^{Flat}_{w}(A) = 1/|A|$ if A is true and is 0 if A is false. (i) Suppose A and B are both true. If $A \geq^{P}_{w} B$, then since both are true, $A \vdash B$, and $|A| \leq |B|$, so $\psi^{Flat}_{w}(A) \geq \psi^{Flat}_{w}(B)$. If $A >^{P}_{w} B$, then A is logically stronger than B, $|A| < |B|$ and so $\psi^{Flat}_{w}(A) > \psi^{Flat}_{w}(B)$. (ii) Suppose A is false and B is true. Then *not* ($A \geq^{P}_{w} B$). We also have $\psi^{Flat}_{w}(B) =$

$1/|B| > 0 = \psi_w^{Flat}(A)$. Hence not $\psi_w^{Flat}(A) \geq \psi_w^{Flat}(B)$. (iii) Suppose A is true and B is false. If $A \geq_w^P B$ then again we have $\psi_w^{Flat}(A) = 1/|A| > 0 = \psi_w^{Flat}(B)$. (*iv*) Suppose A and B are both false. It follows that *not*-$(A >_w^P B)$ and *not*-$(B >_w^P A)$. But also $\psi_w^{Flat}(A) = 0 = \psi_w^{Flat}(B)$. Hence the measure ψ^{Flat}, which is simultaneously content-based, consequence-based and strongly likeness-based realizes $\geq^P$.

We have already shown that $\geq^P$ delivers all the content-based and consequence-based judgments (1 and 2). Let $A \geq_w^L B$ iff for all likeness-based measures ψ, $\psi_w(A) \geq \psi_w(B)$. We show that if $A >_w^L B$ then $A >_w^P B$. Suppose $A >_w^L B$. (i) Let A and B both be true in w. Assume (for the sake of a reductio) that nor $A \vdash B$. Then A and B cannot be logically equivalent. Not $B \models A$ (for if $B \models A$ then $\psi_w^{Flat}(B) > \psi_w^{Flat}(A)$, contradicting) assumption that $A >_w^L B$). So A and B must be logically independent. Let $(A \wedge \neg B)$, $A \wedge B$, and $(A \wedge B)$ have respectively a, b, and c members, and let δ be a distance function with the following features:

if u and v are distinct $A \wedge B$ worlds, then $\delta uv = 1/2$,
if u is an $A \wedge \neg B$ world, then $\delta uw = 1$,
if u is a $\neg A \wedge B$ world, then $\delta uw = \varepsilon$.
Average δ-distance of A from w: $[(c-1)/2 + a]/(a+c)$.
Average δ-distance of B from w: $[(c-1)/2 + b\varepsilon)/(b+c)$.

By making ε as close to 0 as we like we can make the latter arbitrarily close to $[(c-1)/2]/(b+c)$.

So average δ-distance of A from w > average δ-distance of B from w

$$\text{iff } [(c-1)/2 + a]/(a+c) > [(c-1)/2]/(b+c)$$
$$\text{iff } a(b+c) > (a-b)[(c-1)/2].$$

But $a > (a-b)$ and $(b+c) > [(c-1)/2]$, so $a(b+c) > (a-b)[(c-1)/2]$. Since $\lambda^\delta uv = 1 - \delta uv$ the average λ^δ-likeness of A to w is less than that of B to w. This contradicts assumption that $A >_w^L B$. (ii) Suppose A is false and B is true. Then since $\psi_w^{Flat}(B) > \psi_w^{Flat}(A)$, not $A \geq_w^L B$. (iii) Suppose both A and B are false. If they are incompatible then its clearly possible to find likeness functions that render A further from w than B. Just make worlds in A far from w and worlds in B close to w—similarly if A and B are logically independent. If A is logically stronger than B then make the worlds in $(\neg A \wedge B)$ further from w than the worlds in A. If B is logically stronger than A, then make worlds in $(A \wedge \neg B)$ closer to w than worlds in A. So whatever their logical relations, if A and B are false in w, not $A \geq_w^L B$. (iv) Suppose A is true and B is false. $A \geq_w^P B$ if (and only if) w is the only world in A not in B. Suppose there are some non-actual worlds in A which are not in B. Let all the non-actual worlds in $A \wedge \neg B$ be distance 1 from w, and all worlds in B of distance ε from w. Since we can make ε arbitrarily small, we can make average λ-likeness of A-worlds to w less than average λ-likeness of B-worlds to w: not $A \geq_w^L B$. (Note that while $\geq^P$ is a likeness-based ordering it is not the core likeness-based ordering, since the value of content for truths (either weak or strong) is not a core likeness-based principle. $\geq^P$ extends the core likeness-based ordering, but in a direction which implies nihilism about likeness.)

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