

Objectivity Regained: Benacerraf's Dilemma and Intuitions in Mathematics, Logic, Morality, and Philosophy

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Although these principles [of mathematics], and the representation of the object with which this science occupies itself are generated in the mind completely *a priori*, they would still not signify anything at all if we could no always exhibit their significance in appearances (empirical objects). Hence it is also requisite for one **to make** an abstract concept **sensible**, i.e., display the object that corresponds to it in intuition (*Anschauung*), since without this the concept would remain ... without **sense**, i.e., without significance. Mathematics fulfills this requirement by means of the construction of the sensible form (*Gestalt*), which is an appearance present to the senses (even though brought about *a priori*). In the same science, the concept of magnitude seeks its standing and sense in number, but seeks this in turn in the shapes, in the beads of an abacus, or in the strokes and points that are placed before the eyes. The concept is always generated *a priori*, together with the synthetic principles of formulas from such concepts; but their use and reference to supposed objects can in the end be sought nowhere but in experience, the possibility of which (as far as its form is concerned) is contained in them *a priori*.

--I. Kant¹

Both in mathematics and in ethics we have crystal-clear intuitions from which we build up all that we can know about the nature of numbers and the nature of duty.

--W.D. Ross²

F.P. Ramsey once emphasized in conversation with me that logic was a “normative science.” I do not know exactly what he had in mind.

--L. Wittgenstein³

[T]he distrust of the “intuitional” basis of analytic philosophy... is rooted in nothing less than an imperfect understanding of scientific method.

--A. Pap⁴

Of course, some philosophers think that something’s having intuitive content is very inconclusive evidence in favor of it. I think it is very heavy evidence in favor of anything, myself. I really don’t know, in a way, what more conclusive evidence one can have about anything, ultimately speaking.

--S. Kripke⁵

[A]lthough we cannot speak of the absolute security of finitism, there is a sense in which we can speak of its *indubitability*. That is, any nontrivial reasoning about number will presuppose finitist methods, and there can be no preferred or even equally preferable method from which to launch a critique of finitism. In other words, it is simply pointless to doubt it.

--W. Tait⁶

Pure intuition as Kant understood it was evidently supposed somehow to get us across the divide between the fuzzy *Lebenswelt* with its everyday objects and the sharp, precise realm of the mathematical, in terms of which mathematical conceptions of the physical world are developed.

--C. Parsons⁷

I. Introduction

I.1

“3+4=7.” Few statements, even necessarily true statements, are objectively⁸ knowable in such a way that they are completely convincing, intrinsically compelling, or self-evident, but this is one of them. And I can prove it to you. Just look at this carefully and thoughtfully:

$$||| + |||| = |||||$$

Therefore—to use Descartes’s famous terminology—it is *clearly, distinctly, and indubitably* objectively known by you that 3+4=7.⁹ And although your cognition of it, via the stroke diagram, obviously *began* in human sensory experience, nevertheless its specific content and evidential character were not *derived from*—that is, were not *determined by*, or more precisely, were *underdetermined by*—either human sensory experience or contingent facts. So you also know it *a priori*.

This consistent combination, within objective a priori knowledge, of

- (i) the necessity of a sense-experiential and contingent starting point for all actual or possible human cognition, and
- (ii) the underdetermination of meaning, truth, and belief-justification by all actual or possible sense-experiences and/or contingent facts,

is closely related to Immanuel Kant’s equally famous and very deep remark in the B or 1787 Introduction to the *Critique of Pure Reason* about the subtle modal relationship between the necessary empirical origins of all human cognition, and the existence and specific character of the a priori:

Although all our cognition commences **with** experience, yet it does not on that account all arise **from** experience.... It is therefore a question requiring closer investigation, and one not to be dismissed at first glance, whether there is any such cognition independent of all experience and even of all impressions of the senses. One calls such **cognitions a priori**, and distinguishes them from **empirical** ones, which have their sources *a posteriori*, namely in experience. (*CPR* B1-2)

It is also closely related to David Hilbert's slightly less famous, but equally deep, remark about the "intuitively present" character of the basic objects of finitary mathematical reasoning:

[A]s a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication.¹⁰

I will come back to the consistent combination of empirical starting points and empirical underdetermination within objective a priori knowledge, to Kant's very deep remark about this combination, and also to Hilbert's equally deep remark about the basic objects of finitary reasoning, a little later. For the moment, I am only highlighting the manifest fact that " $3+4=7$ " immediately presents itself to you as necessarily true and a priori. Moreover, it also immediately presents itself to you in such a way that neither its necessary truth and nor its apriority depends on anything merely subjective or idiosyncratic: *any* mature rational human animal could know this. And you are a mature rational human animal. So you have completely convincing, intrinsically compelling, or self-evident objective *a priori* knowledge that necessarily $3+4=7$. Furthermore, by means of your act of cognition, a strongly normative fact has also emerged. Precisely insofar as you are a rational human animal cognizer, you categorically (i.e., non-instrumentally and unconditionally) *ought* to believe that $3+4=7$. In that sense, arithmetic is a *robustly normative science*, that is, one of the *Moral Sciences* in the classical 19th century sense of *Geisteswissenschaften*. But how is all this possible?

I.2

This essay has four central topics. **First**, it is about the nature of mathematical truth and knowledge. So it is an essay in the philosophy of mathematics, with special reference to its semantics and epistemology. **Second**, it is about the nature of logical truth and knowledge. So it is also an essay in the philosophy of logic, with special reference to *its* semantics and epistemology. **Third**, it is about the nature and epistemic status of intuitions in mathematics, logic, and philosophy itself. So it is also an essay in the philosophy of philosophy, or *meta-philosophy*. **Fourth** and finally, I am also interested in developing some substantive analogies between an intuition-based epistemology of mathematics and logic on the one hand, and an intuition-based epistemology of *morality* and *philosophy* on the other, such that mathematics, logic, morality, and also philosophy itself, can all be shown to be *objective* robustly normative sciences for all actual and possible rational human animals, that is, objective rational *Moral Sciences*. So, finally, it is also an essay in *modal epistemology*, that is, an essay in the general theory of our a priori knowledge of necessity, essence, and possibility.

More precisely and positively now, I believe that mathematics, logic, morality, and philosophy all include and presuppose some basic and *authoritative*—i.e., completely convincing, intrinsically compelling, or self-evident—*a priori* rational intuitions that constitute authentic a priori knowledge of objectively necessary truths, such that those intuitions are *fully reliable*, and such that the rejection of them would be *self-stultifying* in the strongly normative sense that human rationality itself would be impossible without them, including also *skeptical* human rationality. Hence we *categorically ought not to reject them* insofar as we are rational human animals. In short, if I am correct, then these

fully reliable basic authoritative a priori rational intuitions, constituting authentic a priori knowledge of objectively necessary truths, are robustly normative conditions of the possibility of human rationality, and implicit even in every attempt to reject these intuitions for *any* intelligible or defensible reason whatsoever.

And that is not all. I also believe that, starting with these fully reliable basic authoritative a priori rational intuitions of objectively necessary truths, then mathematicians, logicians, moralists, and philosophers can also rationally construct *non-basic*, and *non-authoritative* (i.e., not completely convincing, not intrinsically compelling, and not self-evident), but still *fairly reliable* and *fairly convincing*, *fairly compelling*, or *fairly evident* a priori rational intuitions,¹¹ and thereby effectively extend their foundational base of authentic *a priori* knowledge to a *fairly secure non-foundational constructed corpus* of a priori knowledge, and make rational progress in mathematics, logic, morality, and philosophy.

Of course, an extreme “post-modern nihilist” skeptic could still choose to reject all of these intuitions, whether fully reliable and basic authoritative intuitions, or fairly reliable and non-basic constructed intuitions, for *no* intelligible or defensible reason whatsoever. But that would be perverse and pointless, except as a form of anarchic self-expression. Someone’s striking an attitude is almost always *interesting*—“human, all too human”—but an attitude struck is not an argument made.

I.3

Even more precisely, however, and following on from the four central topics I mentioned in sub-section **I.2**, in this essay I will address four hard philosophical problems.

The **first** hard problem I will address is *Benacerraf's Dilemma*, which seems to entail that objective mathematical necessary truth on the one hand, and rational human a priori knowledge of this sort of truth on the other hand, are mutually incompatible. In order to solve this problem, I think that we must also appeal directly and substantively to Kantian versions of *Mathematical Structuralism* and *mathematical intuition*.

Mathematical Structuralism says that mathematical entities are not independent substances of some sort, but instead are nothing more and nothing less than relational positions or roles in a larger mathematical theory-structure. Correspondingly, *a priori* mathematical intuitions, as I am understanding them, are intrinsically compelling, completely convincing, or self-evident (although not strictly infallible) conscious pattern-matching graspings of some proper parts of a larger mathematical theory-structure, via our direct conscious experience, in spatiotemporally-framed, diagrammatic, pictorial, structural, or schematic sense perception, memory, or sensory imagination, of—in effect—Hilbert's basic objects of finitary reasoning.

This kind of direct conscious experience is equivalent to what, in the first epigraph of this essay, Kant calls *the construction of a sensible form (Gestalt) in a priori intuition (Anschauung)*. It is also equivalent to what the cognitive psychologist Philip Johnson-Laird calls *mental models*.¹² We could also call it *mental diagrams*, *mental pictures*, *structural imagery*, or *schemata*. Whatever we call it, the main claim I am making is that mathematical necessary truths directly express proper parts of larger mathematical theory-structures, and a priori mathematical intuitions are self-evident (although not infallible) conscious pattern-matching graspings of some of *those* proper parts of *those* very structures, via the cognitive construction and manipulation of sensible

forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata. In turn, the ground of the necessary conformity between a priori mathematical intuitions in the rational human mind on the one hand, and mathematical structures in the manifest natural world outside the human mind on the other hand—a necessity conformity which suffices to close the gap between justification and truth, and thereby yield authentic a priori knowledge of necessary truth—will then be explained within a specifically Kantian metaphysical and epistemological framework.

The **second** hard problem I will address is what I call *The Extended Benacerraf Dilemma*, which generalizes the original Benacerraf Dilemma from mathematics to logic. In order to solve this extended version of the problem, I think that we must, correspondingly, appeal directly and substantively to Kantian versions of *Logical Structuralism* and *logical intuition*, as well as to the same specifically Kantian metaphysical and epistemological framework used for the solution of the original Benacerraf Dilemma.

The **third** hard problem I will address is *The Problem of the Epistemic Status of Intuitions*. Benacerraf's Dilemma, whether in the original version or in the extended version, is based on a logical, semantic, metaphysical, and epistemological clash between two fully reliable basic authoritative a priori *philosophical intuitions* about our natural-language semantics of truth on the one hand (i.e., "Truth is uniform and Tarskian"), and our causally-anchored sense-experiential epistemology on the other hand (i.e., "Human knowledge begins in causally-triggered sense experience"). My proposed solutions to the original and extended Dilemma not only preserve the objective necessity, apriority, and

fully reliable basic authoritative epistemic force of these two seemingly incompatible philosophical intuitions, but also include a substantive general theory of

- (i) basic and non-basic,
- (ii) fully reliable and fairly reliable, and
- (iii) authoritative and constructed

mathematical, logical, moral, and philosophical intuitions alike. Therefore, precisely to the extent that my Kantian solutions to the original Benacerraf's Dilemma and The Extended Benacerraf Dilemma are both cogent, then they will also jointly constitute an adequate vindication of what are classically known as *rational intuitions*, whether *clear and distinct* (i.e., fully reliable, basic, and authoritative) or *not clear and distinct* (i.e., fairly reliable, non-basic, and constructed). Or otherwise put, in explaining how we can objectively know a priori and with fully reliable epistemic authority and self-evidence via *mathematical* intuition that, e.g.,

$$3+4=7, \text{ i.e., } ||| + |||| = |||||$$

and also objectively know a priori and with fully reliable epistemic authority and self-evidence via *logical* intuition that, e.g.,

It is not the case that every meaningful sentence or statement whatsoever is both true and false., i.e., $\sim (\forall S) (S \ \& \ \sim S)$,

and also objectively know a priori and with fully reliable epistemic authority, via *moral* intuition, that, e.g.,

Arbitrarily torturing completely innocent people to death, i.e., like the Nazis did, is morally wrong, whatever the consequences.,

without at the same time falling into any inconsistency with respect to our fully reliable basic authoritative philosophical intuitions about the nature of truth and the nature of human knowledge, then I will also have effectively answered the radically skeptical

worries raised by proponents of Experimental Philosophy—a.k.a. “X-Phi”—in particular, but also by proponents of Scientific Naturalism in general, about the epistemic reliability of mathematical, logical, moral, or philosophical intuitions.

The usual strategy in contemporary meta-philosophy for determining the reliability of philosophical intuitions is to treat them as if they were somehow inherently separate, or at least *prima facie* separate, both from each other, and also from mathematical, logical intuitions, and moral intuitions. My idea, on the contrary, is that a correct treatment of philosophical intuitions *can flow only from correct treatments of fully reliable mathematical, logical, and moral intuitions as essential starting points*. This in turn is because, in my opinion, **first**, as Moral Sciences, mathematics, logic, first-order morality, and philosophy itself all ultimately have their foundations in what Kant called *the metaphysics of morals*, i.e., *a general theory of rational human normativity*,¹³ and **second**, philosophy is different from all the other forms of science, knowledge, freely-chosen self-conscious social practice, and freely-chosen self-conscious individual activity *only* in the maximally synoptic scope of its critical and reflective reach over *all and only* topics of fully natural and robustly normative relevance to us in our rational “human, all too human” predicament. In his famous paper, “Philosophy and the Scientific Image of Man,” Wilfrid Sellars glossed the nature of philosophy in the following way—

The aim of philosophy, abstractly formulated, is to understand how things in the broadest possible sense of the term hang together in the broadest possible sense of the term.¹⁴

I think that this formulation is *almost* correct, but still not *quite* right, and that what

Sellars should have written instead is—

The aim of philosophy, abstractly formulated, is to understand how things in the broadest possible sense of the term, **just insofar as they really matter to rational human animals or real human persons**, hang together in the broadest possible sense of the term.

The **fourth** and final problem I will address is *The Problem of Objectivity*, or the classical problem of how it is that truth and the intentional targets of all knowledge—especially including mathematical, logical, moral, and philosophical knowledge—can all be genuinely *mind-independent*, without also making them into what J.L. Mackie derisively called “Queer Facts,” i.e., supernatural items that are *humanly impossible* to know.¹⁵ Otherwise put, somehow objectivity must be the necessary conjunction of mind-independence *and* human knowability. Benacerraf’s Dilemma, whether original or extended, poses The Problem of Objectivity in a particularly sharp way. In order to solve this problem, I will argue that truths of all kinds, moral principles, and the other proper intentional targets of rational human knowledge are indeed objective, and furthermore that anything *X* which belongs to the manifest natural world is objective if and only if

(1) *X* is strictly underdetermined by any actual or possible contingent idiosyncrasies of individual minds and cultural or social agreements, i.e., *X* is inherently non-subjective and non-relative (**the moderate mind-independence thesis**), and also

(2) Necessarily, *X would be* directly cognized by rational human animals, at least to some extent, *were* rational human animals to exist (**the weak or counterfactual mind-dependence thesis**).

Claim (1), **the moderate mind-independence thesis**, entails the necessary presence of some a priori factors in the constitution of all truths and human knowledge about the manifest natural world. Claim (2), **the weak or counterfactual mind-dependence thesis**, entails that it is necessarily possible for rational human animals to cognize the manifest natural world directly, at least to some extent, and also that the manifest natural world basically contains some necessary *converse intentional properties* (a.k.a. “response-dependent properties”) including the general subjunctive conditional (a.k.a. “counterfactual”) modal converse intentional property such that necessarily, any of these manifest natural worldly properties *would be* directly cognized by rational human

animals, at least to some extent, *were* those rational human animals to exist. And this, again in turn, is equivalent to a version of Transcendental Idealism I call *Weak or Counterfactual Transcendental Idealism*. But in any case, the upshot of the two claims is that *objectivity is non-subjective, non-relative, necessary counterfactual universal rational human intersubjectivity*.

Bounded in a nutshell, then, the main thesis of this essay is that mathematics, logic, and by the very same token, morality and philosophy, are all *rational human constructions* in the quite specific sense that they are all objective robustly normative sciences for all actual and possible rational human animals, i.e., objective rational Moral Sciences, which is *why* we can authentically know them intuitively, but that

(i) The primitive procedural rules by which we construct mathematical, logical, moral, and philosophical systems of principles are innately-specified and non-optional, non-empirical or a priori, and strictly universal across all actual and possible rational human animals., and

(ii) Necessarily the manifest natural world structurally conforms to the innately-specified, non-optional, non-empirical or a priori, and strictly universal mental rule-structures of our rational human conceptual capacities, perceptual capacities, mathematical capacities, logical capacities, moral capacities, and philosophical capacities.

Or in other and even fewer words, the main thesis of this essay is that *objectivity has a human face, with robustly normative rationality written all over it*.

II. Objectivity Lost: Benacerraf's Dilemma

I who erewhile the happy garden sung,
 By one man's disbedience lost, now sing
 Recovered Paradise to all mankind,
 By one man's firm obedience fully tried
 Through all temptation, and the Tempter foiled
 In all his wiles, defeated and repulsed,
 And Eden raised in the waste wilderness.

--J. Milton¹⁶

As an account of our knowledge about medium-sized objects, in the present, this is along the right lines. [A reasonable epistemology] will involve, causally, some direct reference to the facts known, and, through that, reference to those objects themselves.... [C]ombining *this* view of knowledge with the "standard" view of mathematical truth makes it difficult to see how mathematical knowledge is possible. If, for example, numbers are the kinds of entities they are normally taken to be, then the connection between the truth conditions for the statements of number theory and any relevant events connected with the people who are supposed to have knowledge cannot be made out.

--P. Benacerraf¹⁷

II.1

Benacerraf's Dilemma, or BD for short, as originally formulated by Paul Benacerraf in 1973,¹⁸ is about the apparent impossibility of reconciling a *standard*, *uniform* semantics of truth in natural language with a *reasonable* epistemology of cognizing true statements, when the relevant kind of true statement to be semantically explained is *mathematical truth* and the relevant kind of cognition to be epistemologically explained is *mathematical knowledge*.

A "standard, uniform" semantics of truth, in Benacerraf's terminology, is a Tarskian satisfaction-theoretic and model-theoretic semantics¹⁹ applying across natural language as a whole, whereby each meaningful indicative sentence or statement *S* in the language conforms to the simple "disquotational" T-schema:

'*S*' is true if and only if *S*.

The fully generalized version of the T-schema includes, on its left-hand side, a *structural description* of a meaningful sentence or statement, and on its right hand side, a

translation of that sentence or statement into the meta-language.²⁰ In any case, this is what Tarski says by way of an informal explication of the semantic conception of truth:

a true sentence is one which says that the state of affairs is so and so, and the state-of-affairs indeed is so and so.

He then says, by way of qualification:

From the point of view of formal correctness, clarity, and freedom from ambiguity of the expressions occurring in it, the above formulation leaves much to be desired. Nevertheless its intuitive meaning and general intention seem to be quite clear and intelligible.²¹

I take this Tarskian thesis to be authoritative (i.e., completely convincing, intrinsically compelling, or self-evident), a priori (i.e., underdetermined by all actual or possible sense-experiential and/or contingent facts), and objectively necessarily true, in just the way that “3+4=7” is. For example, looking carefully and thoughtfully at the simple disquotational version of the T-schema, i.e.,

‘S’ is true if and only if S

has *precisely* the same sort of high-powered semantic, metaphysical, and epistemic force as looking carefully and thoughtfully at the Hilbert-style stroke diagram

||| + |||| = |||||||

Hence:

(I) Truth is uniform and Tarskian.

This standard, uniform Tarskian semantics of truth, together with some natural assumptions about standard mathematical linguistic practices, very plausibly, smoothly, and jointly yield classical Platonism about mathematics.

Now a “reasonable” epistemology is any epistemology that ties human linguistic knowers causally and perceptually to the known objects themselves. I also take this thesis to be fully reliable basic authoritative, apriori, and objectively necessarily true. Hence:

(II) All human knowledge begins in causally-triggered sense experience.

This reasonable epistemology, together with some equally reasonable assumptions about causation and its inherently spatiotemporal character, very plausibly, smoothly, and jointly yield the *denial* of classical Platonism about mathematics. Hence BD.

II.2

In sections **VI**, **VII**, and **VIII**, I will spell out a new solution to BD. I call this new solution *a positive intuition-based solution* for three reasons:

(1) It accepts Benacerraf's preliminary philosophical assumptions about the nature of truth and human knowledge as basic authoritative philosophical intuitions, as well as accepting all the basic steps of BD, and then it shows how we can, consistently with those very assumptions and premises, *still* reject the skeptical conclusion of BD and *also* adequately explain mathematical knowledge.

(2) The standard, uniform Tarskian semantics of mathematical truth that I offer is based on Kant's philosophy of arithmetic, especially including his theory of *pure intuition*, as interpreted by Charles Parsons and by me.²²

(3) The reasonable (or causally-anchored) epistemology of mathematical knowledge that I offer is based on a theory about mental content that I call *Kantian Non-Nonconceptualism*, together with a critical appropriation of the phenomenology of logical and mathematical self-evidence and rational intuition developed by early Husserl in *Logical Investigations*, by early Wittgenstein in *Tractatus Logico-Philosophicus*, and also of Parsons's theory of Mathematical Structuralism and mathematical intuition—drawing on basic Kantian ideas, Brouwer's intuitionism, and Hilbert's finitist epistemology—as developed in *Mathematical Thought and its Objects*.

More precisely, however, what I will argue is that we can solve BD in three stages:

First, I explicitly accept Benacerraf's preliminary philosophical assumptions about the the nature of truth and human knowledge as basic authoritative philosophical intuitions, as well as explicitly accepting all the basic premises of BD.

Second, I hold that mathematical truth is adequately explained by accepting the following three claims:

(1) The natural numbers are essentially positions or roles in the mathematical natural number structure provided by Peano arithmetic, especially including the finitist sub-structure of Primitive Recursive Arithmetic.,

(2) The mathematical natural number structure provided by Peano arithmetic, especially including the finist sub-structure of Primitive Recursive Arithmetic, is abstract only in the sense that it is *transcendentally ideal*, which is to say that this

structure is identical to the formal structure of time as we directly or essentially non-conceptually cognize it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human comprehension of Peano arithmetic by means of conceptual understanding or thinking., and

(3) In our actual world, the unique, intended model (i.e., the one and only real truth-maker) of the abstract natural number structure provided by Peano arithmetic, especially including the finitist sub-structure of Primitive Recursive Arithmetic, is nothing more and nothing less than *an immanent structure* that is fully embedded in the set of manifest directly or essentially non-conceptually perceivable spatiotemporal causally-efficacious material objects—the natural inhabitants of Parsons’s “fuzzy *Lebenswelt* with its everyday objects”—insofar as they are the role players of the Peano-arithmetic-and-Primitive-Recursive-Arithmetic-specified natural number roles in the abstract formal structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception, together with all the formal concepts and other logical constructions, especially including specific logical inference patterns such as mathematical induction, needed for an adequate rational human comprehension of Peano arithmetic by means of conceptual understanding or thinking.

Third, I hold that mathematical knowledge is grounded on *a priori mathematical intuition*, by which I mean:

(1) a rational human animal’s *veridical sensible-form-in-Kantian-a priori-intuition-or-mental-model-or-mental-diagram-or-mental-picture-or-structural-imagery-or-schema-constructing-and-manipulating* abilities, which are innately specified in her mind and also inherently present, as necessary ingredients, in all rational human sense perception, and which entail her self-conscious and self-reflective cognition of phenomenologically self-evident formal structures of rational human sense perception, together with

(2) a rational human animal’s *logic-and-language-constructing-and-manipulating* abilities, which are innately specified in her mind and also inherently present, as necessary ingredients, in all rational human empirical conceptualizing and perceptual judgment, and which entail her self-conscious and self-reflective cognition of phenomenologically self-evident formal conceptual contents and specific patterns of logical inference in classical or non-classical logics.

The **second** stage of this argument invokes what I call *Kantian Structuralism* about the nature of numbers and mathematical truth. And the **third** stage invokes what I call *Kantian Intuitionism* about mathematical a priori knowledge. The basic idea behind Kantian Intuitionism is that mathematical intuition can be construed in such a way as to preserve both the abstractness and *causal inertness* of **the truth-makers** of mathematical

statements, and also the *causal relevance* of **the intentional targets** of mathematical intuition, as well as the *causal efficacy* of **the evidential verifiers** of mathematical beliefs or judgments.

In bold-facing these phrases, I want to emphasize specifically the point that truth-makers, intentional targets, and evidential verifiers can be *distinct sorts of things*, even if they are *essentially connected*. Suppose, e.g., that

(i) **The truth-maker** is a *mathematical immanent structure in the natural world*,

(ii) **The intentional target** is a *veridical sensible form in Kantian a priori intuition, mental model, mental diagram, mental picture, structural image, or schema* of a proper part of that very structure., and

(iii) **The evidential verifier** is a *natural worldly fact, picked out by direct or essentially non-conceptual sense perception*, which *implements* the immanent structure and thereby *satisfies* the abstract mathematical structure, and also strictly *conforms* to the veridical sensible forms in Kantian a priori intuition, mental models, etc.

These are all obviously distinct from one another, and also obviously essentially connected with one another. I will come back to these crucial points again later.

Odd as it might at first seem, I think that there is an interesting and important parallel between Benacerraf's Dilemma and Milton's epic poetry. Milton's *Paradise Lost* and *Paradise Regained*, as I read them, are about the necessary transition from the impossibly super-human objective conception of moral virtue embodied in pre-lapsarian Adam and Eve, and our consequent tragic Fall and expulsion from the Garden of Eden, towards a fully realistic and objective knowledge of our own "human, all too human" moral limits and of our inescapably finite, mortal role on this desperately nonideal Earth. Correspondingly, the philosophical story I am telling in sections **VI**, **VII**, and **VIII** is about the necessary philosophical transition from the impossibly super-human objective conception of mathematical truth and knowledge offered by classical Platonism, and our

consequent tragic Fall and collapse into BD, towards a fully realistic and objective but also inescapably *anthropocentric* conception of mathematical truth and knowledge, based on the idea that objectivity is non-subjective, non-relative, necessary counterfactual universal rational human intersubjectivity, but without either strict finitism or nominalism. In short, this is *objective necessarily true a priori knowable mathematics for rational human animals*. And essentially the same argument goes, in sections **IX** to **XIII**, with suitable changes made for differences in context, for *logic, morality, and philosophy itself*. So if my argument is sound, then the result will be, in effect, a mathematical, logical, moral, and philosophical Paradise Regained—with Kantian bells on.

II.3

Here is a fully explicit rational reconstruction of Benacerraf's Dilemma:

- (1) Natural language requires a standard, uniform semantics of truth. Hence: Truth is uniform and Tarskian. (Preliminary assumption I.)
- (2) A reasonable epistemology of cognizing true (mathematical) statements should be modelled on human sense perception. Hence: All human knowledge begins in causally-triggered sense experience. (Preliminary assumption II.)
- (3) Mathematical knowledge in a classical sense (i.e., as a priori knowledge of objectively necessary truth) exists as a basic feature of standard mathematical linguistic practices, so mathematical truth in a classical sense (i.e., as objectively necessary truth) also exists as a basic feature of those standard practices.
- (4) Given (1) and (3), our standard, uniform semantics of truth in natural language, as applied to mathematical truths, commits us to a truth-making ontology of abstract mathematical objects and also to the non-empirical knowability of true mathematical statements.
- (5) On the one hand, given (2), that fact that a reasonable epistemology of cognizing true (mathematical) statements should be modelled on human sense perception entails that knowledge involves causally efficacious, contact-involving or efficient, directly referential, non-inferential, and inherently spatiotemporal relations between human linguistic knowers and the known objects themselves.
- (6) But on the other hand, given (4), and since all abstract objects are causally isolated and inert, it then follows that all abstract mathematical objects are causally isolated and inert.

(7) So if we accept all of (1) – (6), then mathematical knowledge in the classical sense is both possible and impossible, which is absurd.

I will say that any proposed solution to BD is *negative* or *skeptical* if it rejects either of Benacerraf's preliminary philosophical assumptions about a standard, uniform semantics of truth and a reasonable epistemology or else rejects one or more of steps (3) to (6). Then there are at least six different categories of possible negative or skeptical solutions to BD. The first two categories I will call *pre-emptive* negative or skeptical solutions, since they consist in pre-emptively rejecting at least one of the two preliminary assumptions.

Pre-Emptive Negative or Skeptical Solutions

(1) *Reject the preliminary assumption (I) that natural language requires a standard, uniform semantics of truth, i.e., reject the assumption that truth is uniform and Tarskian.*

This in turn arguably entails either

(1.1) rejecting the Tarskian semantics of truth, or

(1.2) accepting a multiform semantics of truth in natural language.²³

(2) *Reject the preliminary assumption (II) that a reasonable epistemology of cognizing true (mathematical) statements should be modelled on human sense perception, i.e., reject the assumption that all human knowledge begins in causally-triggered sense experience.*²⁴

This in turn arguably entails either

(2.1) asserting that at least some human knowledge is noncausal and modelling the epistemology of cognizing true (mathematical) statements on human conceptual reasoning and concept-possession, or human judgment,²⁵

(2.2) asserting that at least some human knowledge is noncausal and modelling the epistemology of cognizing true (mathematical) statements on human self-consciousness,²⁶ or

(2.3) asserting that at least some human knowledge is noncausal and modelling the epistemology of cognizing true (mathematical) statements on the human imagination.²⁷

The other four categories I will call *concessive* negative or skeptical solutions, since they involve conceding both of the preliminary assumptions I and II, and then rejecting at least one of the other steps leading to the unacceptable conclusion.

Concessive Negative or Skeptical Solutions

(3) *Reject the classical necessity or apriority of mathematical truth.*

This in turn arguably entails accepting either

- (3.1) the contingency of mathematical truth, or
- (3.2) the aposteriority of mathematical truth.

(4) *Reject the truth-making ontology of abstract mathematical objects.*²⁸

This in turn arguably entails accepting either

- (4.1) empirical or phenomenal idealism (whether communal or solipsist),
- (4.2) intuitionism,
- (4.3) formalism,
- (4.4) conventionalism,
- (4.5) fictionalism or some other form of nominalism,
- (4.6) non-cognitivist anti-realism, or
- (4.7) pragmatic/practical realism.

(5) *Reject the thesis that human sense perception involves causally efficacious, contact-involving or efficient, referentially direct, non-inferential, and inherently spatiotemporal relations between human cognizers and the cognized objects.*

This in turn arguably entails accepting either

- (5.1) the replacement of causal efficacy by causal relevance,
- (5.2) the counterfactual theory of causation,
- (5.3) the probability-raising theory of causation,
- (5.4) a non-causal theory of perception,

(5.5) an indirect causal theory of perception (whereby a perceptual subject *S* can sense perceive a universal *U* or type *T* just by standing in a direct causal sense perceptual relation to an instance of *U* or a token of *T*),

(5.6) referential descriptivism, or

(5.7) cognitive inferentialism.

(6) *Reject the thesis that abstract objects are causally isolated and inert.*

This in turn arguably entails accepting both

(6.1) the causal relevance of abstract objects, and

(6.2) the causal efficacy of abstract objects.

Some caveats and qualifications are obviously required here.

First, it is very important to note that each of the possible negative or skeptical solutions I just mentioned is preceded by the qualifier ‘arguably’. I certainly do not intend to suggest that my taxonomy of negative or skeptical solutions is complete or exhaustive.²⁹ No doubt there are other ways of carving up the logical space of possible solutions that I have not considered. And it also strikes me as probably impossible to provide a principled procedure for generating a total list of possible solutions. I am just trying to provide a relatively orderly *indication* of how *some* other philosophers *might* go about attempting to solve BD in a *non-positive* way, as illuminating contrasts to the positive intuition-based solution that I am going to attempt to work out.

Second, even fully *granting* my taxonomy of possible negative or skeptical solutions, it remains obvious that some of these logically entail or logically exclude others, while at the same time, many of the them are also consistent with others—all of which gives rise to a large number of distinct possible *combined* negative or skeptical solutions. This in turn makes the project of proving the falsity of *all* the possible negative or skeptical solutions, one by one, highly strenuous and even simply unfeasible, given the

usual limits on human time, energy, and patience. And if on the other hand it turns out that my taxonomy is *incorrect*, then even if I *were* to succeed in refuting all the negative or skeptical solutions I have surveyed, together with all their combinations, obviously it still would not follow that I have fully cleared the field of relevant opposing views. In order to rule out this problem, then I would have to have a sound demonstration of the completeness of my taxonomy, which, as I have already conceded, I do not have in hand, and which is probably impossible.

Third, as a consequence of the first and second points, I am hereby making the following executive decision about philosophical strategy: In the rest of this essay, I will attempt to work out a positive or anti-skeptical solution to BD, but will *not* explicitly attempt to criticize or defeat the many possible negative or skeptical solutions, each of which would require a separate book-length treatment on its own—for that would make this long-ish essay into *a very big book*—and in any case it would presuppose a sound demonstration of the completeness of my taxonomy of these possible solutions, which, again, I have already conceded I do not have in hand and is probably impossible.

Fourth and finally, as I have said already, I call my solution to BD a “positive” or anti-skeptical one because it accepts Benacerraf’s preliminary philosophical assumptions I and II about the nature of truth and knowledge as basic authoritative philosophical intuitions, as well as accepting all the basic premises of BD—captured in steps (1) to (6), under plausible interpretations of those premises—and then attempts to show how we can, consistently with those very assumptions and premises, under those plausible interpretations, still *reject* the skeptical conclusion of BD—captured in step (7)—and also adequately explain mathematical knowledge. Now on the face of it, *any positive or anti-*

skeptical solution should have a distinct rational edge over any negative or skeptical solution, because only a positive or anti-skeptical solution will adequately preserve the rational force of the basic authoritative philosophical intuitions that generated the dilemma in the first place. If any of these philosophical intuitions did not have *basic authoritative* rational force, then BD would not be a *genuine* dilemma. In other words, BD would simply dissolve if either

(I) Truth is uniform and Tarskian., or

(II) All human knowledge begins in causally-triggered sensory experience.,

turned out to be *other than* fully reliable basic authoritative rational intuitive claims. But both (I) and (II) *do* seem to me to be fully reliable basic authoritative rational intuitive claims. I simply cannot see how, if logic is to be possible after the discovery of the semantic paradoxes and after Gödel's incompleteness theorems, truth can be *other than* uniform, fully in conformity with Tarski's semantic conception, and thereby such as to satisfy universally either the simple version of the T-schema:

'S' is true if and only if S,

or the fully generalized version. And I simply cannot see how *human* knowledge could be *other than* causally anchored in sensory experience. For this expresses only a *minimal* Empiricism, which says that, as sensory experiencing human animals, *cognitively we belong to the causally efficacious natural world*. How could that be denied? It also fully concedes that *not* all our knowledge is strictly determined by causally-triggered sensory experience, given the rock-solid starting point that *some* of our knowledge is objectively necessary and a priori—e.g.,

$3+4=7$, i.e., ||| + |||| = |||||||

It is not the case that every meaningful sentence or statement whatsoever is both true and false., i.e., $\sim (\forall S) (S \ \& \ \sim S)$., and

Arbitrarily torturing completely innocent people to death, i.e., like the Nazis did, is morally wrong, whatever the consequences.

So if those points are correct, the fact that we can and do take BD seriously clearly entails that if there really is a positive or anti-skeptical solution, then other things being equal it will trump any of the negative or skeptical solutions. This line of reasoning, in turn, is a specific expression of what I call *Preservationism about Rational Intuitions*, which I should now say something about before advancing to my positive intuition-based solution to BD. But even before I do that, we need to know what intuitions *are*.

III. What are Intuitions?

Of course, we are not clueless on the factors relevant to our cognitive reliability. We know, for example, that the reliability of our eyesight suffers when it is too dark or too foggy, or when the object seen is too far or too small. We more easily introspect headaches than many of our attitudes or emotions. And we know that simple propositions of arithmetic, geometry, and logic are prime candidates for reliable intuition. The more systematic our knowledge of the conditions within which a faculty is reliable, the better our epistemic perspective on that faculty, and the better our knowledge deriving from that faculty. These are matters of degree, however, and here intuition seems not inferior to introspection or perception.

--E. Sosa³⁰

III.1

Epistemic appeals to intuitions go at least as far back as Plato's *Republic* and *Seventh Letter* and Aristotle's *Nicomachean Ethics*, and can also be found in Descartes's *Rules for the Direction of the Mind* and *Meditations on First Philosophy*, and in Spinoza's *Ethics*, as well as in Leibniz's epistemological writings, in Kant's *Critique of Pure Reason* and his *Logic*, in Bolzano's *Theory of Science*, in Husserl's *Logical Investigations* and his later phenomenological writings, in Brentano's *Origin of the Knowledge of Right and Wrong*, in G.E. Moore's *Principia Ethica*, in Russell's *Problems of Philosophy*, in Brouwer's and Hilbert's writings on the foundations of mathematics, in W.D. Ross's *The Right and the Good*, in Kurt Gödel's later philosophically-oriented writings on the foundations of mathematics and logic, in Arthur Pap's *Semantics and Necessary Truth*, and also in the work of recent or contemporary post-Quinean epistemologists, post-Rawlsian ethicists, metaphysicians, and philosophers of logic or mathematics including Robert Audi, George Bealer, Lawrence Bonjour, Albert Casullo, Michael Huemer, Frances Kamm, Saul Kripke, Derek Parfit, Charles Parsons, John Rawls, Ernest Sosa, and Judith Jarvis Thomson.

Obviously there are important differences between appeals to intuitions by classical Platonists and Aristotelians, classical Rationalists, Kantians, neo-Kantians, post-

Kantians, post-Quineans, and post-Rawlsians. But formulated at a suitably high level of generality, here is the classical theory of intuitions shared by all (or at least most) of those philosophers:

- (1) An intuition is always a *rational* intuition, in that it directly expresses the operations of our specifically rational cognitive capacities.
- (2) A rational intuition is a *noninferential* rational cognition.,
- (3) Rational intuition can be either (i) *rational intuition-that* some proposition *P* is (necessarily) true (and a priori), or (ii) *rational intuition-of* special abstract or non-empirical objects of some sort.,
- (4) Rational intuition-that presupposes rational intuition-of., and
- (5) Rational intuitions can fully justify claims to objective a priori knowledge and also explain the cognitive process by means of which objective a priori knowledge occurs.

According to the classical theory of intuitions, then, there are two different basic types of rational intuitions, namely

- (i) rational intuitions-*that*, which are noninferential *propositional* cognitions aimed at objective a priori knowledge of necessary truth, and
- (ii) rational intuitions-*of*, which are noninferential *directly referential* cognitions aimed at objective a priori knowledge of necessary truth.

This is the difference, e.g., between propositionally intuiting the necessarily true arithmetical statement that $3+4=7$ via the cognitive construction and manipulation of a Hilbert-style stroke diagram for that proposition or statement, i.e.,

$$||| + |||| = |||||||$$

and directly intuiting the number 7 via a Hilbert-style stroke diagram for that number, i.e.,

$$||| |||||$$

It is also the difference between propositionally intuiting the necessarily true logical statement that it is not the case that every meaningful sentence or statement whatsoever is

both true and false via the cognitive construction and manipulation of a perspicuous formal translation of that proposition or statement into a standard system of logical symbols, i.e.,

$$\sim (\forall S) (S \ \& \ \sim S)$$

and directly intuiting the logical constant Negation via a standard logical symbol for Negation such as the tilde, i.e.,

~

And, finally, it is also the difference between propositionally intuiting the absolutely universal categorically normative statement that arbitrarily torturing completely innocent people to death, i.e., like the Nazis did, is morally wrong, whatever the consequences, via watching a brilliant documentary film like Alain Resnais's *Night and Fog*, and directly intuiting the absolute moral wrongness of Nazi torture via looking at one of the well-known horrifyingly stark black-and-white US Army photographs of piles of human bones found in Nazi death camps at the end of World War II. So the ultimate cognitive goals of rational intuitions-that and rational intuitions-of are the same—objective a priori knowledge of necessary truth—but both their immediate *intentional targets* and also their individuating *intentional contents* are importantly different.

Now rational intuitions-that might also be called *essentially conceptual* intuitions, because they imply our joint possession of the cognitive capacities involved in *conceptualization* generally, self-consciousness in the sense of possessing a concept of oneself and the capacity to make psychological self-reports, logic, logical inference, practical deliberation, epistemic and practical justification, and “reasons-responsiveness.”

By contrast, intuitions-of could also be called *essentially non-conceptual* intuitions,³¹ because they imply our joint possession of the cognitive capacities involved

in *directly referential cognition* generally, self-consciousness in the sense of pre-reflectively conscious embodied egocentric centering in space and time, and spatiotemporal cognition of all kinds, including minimal episodic memory,³² the location of objects, the tracking of objects, representing events, representing motion, representing direction, representing orientation, and representing abstract spatial or temporal local displays, or global systems of spatial or temporal relations. Kantians are particularly interested in essentially non-conceptual intuitions, whether empirical or non-empirical, both in view of Kant's own theory of empirical and pure spatial and temporal "intuitions" or *Anschauungen* in the Transcendental Aesthetic, and in view of his spatiotemporal intuition-based philosophy of mathematics,³³ but also in view of his theory of the role of essentially non-conceptual "productive imagination" or *productktive Einbildungskraft* in mathematical reasoning. Other philosophers in the intuitionist tradition like Plato, Descartes, Russell, Husserl, Brouwer, Hilbert, and Parsons have also talked about what I am calling essentially non-conceptual intuitions under the rubrics of "acquaintance" (*Kennen*), "seeing essences" (*Wesensschau*), "insight" (*Einsicht*), "the perception of a move of time," "immediate experience prior to all thought," and so-on.

But most recent and contemporary philosophers who are interested in intuitions have focused solely on *propositional* rational intuitions, and have either just neglected or else outright rejected essentially non-conceptual rational intuitions. I think that this is an important mistake. But for the rest of this section and the next section as well, in order to keep things relatively simple, I will follow the lead of the majority and focus only on propositional rational intuitions. Rational intuitions-of will return, however, and play a co-starring role in sections **VI-X**.

III.2

According to Bealer and many other contemporary epistemologists, rational intuitions are “intellectual seemings,”³⁴ i.e., non-inferential, self-conscious mental events in which we are *appeared-to intellectually*. But on my view, rational intuitions are *not* intellectual seemings, mainly because

(i) This characterization falsely assimilates the *conceptual* and *propositional* content of rational cognitions to the *perceptual* content of empirical cognitions., and because

(ii) This characterization also falsely suggests that rational intuitions are *passive* mental states rather than *active* intentional performances.

Moreover, it is precisely at this point that a fundamental error arises in recent and contemporary epistemology of intuitions. Under the rubric of the “intellectual seemings” approach to intuitions, it is *also* widely implicitly assumed, or explicitly held, that intuitions are merely spontaneous (i.e., unreflective, pre-theoretical) conscious non-inferential, or non-conscious inferential, non-calibrated or untested judgments—“a.k.a. armchair judgments”—about thought experiments and actual-world topics of actual or possible concern to philosophers,³⁵ perhaps with a further minimal requirement that these topics be “abstract.”³⁶

But, crucially, *that is nothing like what the classical theorists of intuition meant by “intuition.”* No epistemologist ever seriously held that there is anything epistemically special, or especially reliable, about ordinary shoot-from-the-hip philosophical opinions, e.g., in introductory philosophy classes or more advanced courses or seminars, in the debating periods after conference presentations or departmental philosophy colloquia, or in hallway philosophical discussions, or in philosophical discussions in coffee shops or pubs, just as no epistemologist ever seriously thought that there is anything epistemically special, or especially reliable, about ordinary shoot-from-the-hip *mathematical* opinions,

ordinary shoot-from-the-hip *logical* opinions, or ordinary shoot-from-the-hip *moral* opinions. Why would anyone ever think that any special logical, mathematical, or moral credence should be given to what people—all the way from undergraduate students, to graduate students, to professors, but also including amateur aficionados or casual discussants of logic, mathematics, and morality—more or less spontaneously assert in logic classes or seminars, mathematics classes or seminars, or moral philosophy classes or seminars, or in other more or less formal or informal academic settings, including coffee shops and pubs? Correspondingly, then, why should anyone ever think that any special philosophical credence should be given to what people—all the way from undergraduate students, to graduate students, to professors, but also including amateur aficionados or casual discussants of philosophy—more or less spontaneously assert in philosophy classes or seminars, or in other more or less formal or informal academic philosophical settings, including coffee shops and pubs? In short, the combined “intellectual seemings” and “armchair judgments” approach to intuitions falsely degrades or reduces *rational intuitions* to *ordinary shoot-from-the-hip opinions*. No wonder, then, that skeptically-minded epistemologists then “discover” that there is a problem about the reliability of philosophical intuitions. That would be like “discovering” that there is a similar problem about the reliability of logical, mathematical, or moral intuitions. *Of course* there is a problem. Yet it is nothing but *the problem of the reliability of ordinary shoot-from-the-hip opinions about these matters*, and has little or nothing to do with the question of the reliability of *rational intuitions*, whether in logic, mathematics, morality, or philosophy.

And one other thing in this connection. I also think that the combined “intellectual seemings” and “armchair judgments” approach is needlessly concessive to Quinean post-Empiricism and naturalism in that it *usually* assumes, without any argument, that there are no—or anyhow need not be any—genuinely *objective* facts about necessity or necessary truth, that any ontological appeal to real *universals* or other real abstracta is false, or worse, just crazy and anti-scientific, and that apriority is nothing but “armchair,” or *anti-empirical*, inquiry. But why should we take what some, or even many, Anglo-American philosophers have held since 1950 *more* seriously, other things being equal and on the face of it, than what Plato, Aristotle, Descartes, Leibniz, Kant, Bolzano, Brentano, Husserl, G.E. Moore, Russell, Brouwer, Hilbert W.D. Ross, and Gödel held? The philosophical charge of anachronism is a critical blade that cuts in two directions—on the one hand, towards undue reverence for the philosophical past merely because it is found in old books by The Mighty Dead, and on the other hand, towards undue reverence for the philosophical present *merely because it is the conventional wisdom we internalized in graduate school*.

As against the combined “intellectual seemings” and “armchair judgments” approach to intuitions, then, on my view intuitions are specifically *rational* intuitions in the classical sense, i.e., *noninferential beliefs or thoughts, insofar as they are actively conceptually presented or taken as candidates for a priori necessary truth and knowledge*. In intentionally performing a rational intuition, we actively conceptually present or take certain noninferential beliefs or thoughts not merely as *true*, but also as *if-true-then-necessarily-true*. Even more precisely, in intentionally performing a rational

intuition, we actively conceptually present or take certain noninferential beliefs or thoughts as:

- (i) if-true-then-necessarily-true, hence underdetermined by all actual or possible *empirical facts*, i.e., sensory experiences and/or contingent facts,
- (ii) objectively knowable by our rational faculties in a way that is underdetermined by all actual or possible empirical facts,
- (iii) inherently open to further critical reflection, and also
- (iv) fully reliable under suitably favorable cognitive conditions.

This raises another crucial point. By saying that the beliefs or thoughts targetted by rational intuitions are “non-inferential,” I do *not* mean that these beliefs or thoughts *cannot* be cognized or justified by means of arguments and inferences, or that they *cannot* be critically reflected upon, but instead only that, as occurrent intentional performances, they *need not* be cognized or justified by means of arguments and inferences in that very performance, and that they *need not* be critically reflected upon in that very intentional performance, and therefore still *can* be known without argumentative or inferential mediation, or without critical reflection, in that very intentional performance. Indeed, necessarily and at least in principle, rational intuitions inherently can *also* be cognized or justified by means of arguments and inferences, whether deductive, inductive, abductive (i.e., by inference to the best explanation), or transcendental (i.e., by inference to an a priori necessary presupposition of some statement, belief, or thought such that, synthetically a priori necessarily, *were* this a priori necessary presupposition to hold, then this statement, belief, or thought *would be* fully meaningful, true, and/or justified³⁷), and also inherently can *also* be critically reflected upon.

So qualified, this general three-part description is intended to hold for all rational intuitions both in mathematics, logic, morality, and philosophy, but *not* for “intellectual seemings” or “armchair judgments” in these domains.

At the same time, however, I think that there is a crucial difference between

(i) *authoritative rational intuitions*, which are rational intuitions that are completely convincing, intrinsically compelling, or self-evident, which retain their maximal, thick epistemic value under critical reflection, and that we must believe if we are rational human animals, i.e., *fully reliable* intuitions,

(ii) *constructed rational intuitions*, which are rational intuitions that presuppose one or more authoritative intuitions as a generative basis, but also include some intuitional evidence that is context-sensitive, contingent, and not itself fully authoritative, which means that they possess a middle-range and moderately thick epistemic value, under certain critical restrictions, i.e., *fairly reliable* intuitions., and

(i) *prima facie rational intuitions*, which are rational intuitions that we have some sort of minimal, thin evidential warrant for, but can be discounted upon critical reflection, i.e., *fairly unreliable* intuitions.

According to my account, then, authoritative rational intuitions are inherently *robust under critical reflection*, constructed rational intuitions are *inherently robust under critical reflection if and only if some well-specified set of other things remains equal*, i.e., *inherently robust under critical reflection ceteris paribus*, and merely prima facie rational intuitions are *inherently non-robust under critical reflection*.

So, e.g., my rational intuition that $3+4 = 4+3$, i.e.,

$$||| + |||| = |||| + |||$$

is authoritative and fully reliable; my rational intuition that for all natural numbers x and y , $x+y = y+x$, is constructed and fairly reliable; and my off-the-cuff rational intuition that 43, 311 is a prime number is prima facie and fairly unreliable. To be sure, the generative basis for my constructed intuition that for all natural numbers x and y , $x+y = y+x$, includes a large set of basic authoritative intuitions such as my intuitions that $1+1=1+1$, that $1+2=2+1$, that $1+3=3+1$, ..., i.e.,

$$| + | = | + |$$

$$| + || = || + |$$

$$| + ||| = ||| + |$$

etc.

but it is also plainly true that neither my grasp of the concept of a natural number, nor my grasp of the structural system of the natural numbers, nor my grasp of the concept or structure of the commutativity of the operation of addition over the natural numbers, is *itself* basic authoritative.

It is important to note that authoritative rational intuitions can be either *basic* or *non-basic*. Basic authoritative rational intuitions, as a class, are axiomatic premises in mathematical, logical, moral, or philosophical reasoning. But if a statement S_2 follows immediately as a logical or mathematical consequence from a statement S_1 , and statement S_1 is authentically known by a basic authoritative rational intuition, then S_2 is inferentially authentically knowable a priori by means of a non-basic authoritative logical or mathematical intuition of the following strict conditional statement S_3 :

(S_3) Necessarily, if S_1 then S_2 .

So non-basic authoritative intuitions are rational intuitions of strict logical or mathematical conditionals with antecedents containing statements authentically known a priori by basic authoritative intuitions. In this way, then, non-basic authoritative rational intuitions are distinct from constructed rational intuitions, since non-basic authoritative rational intuitions are all fully reliable logical or mathematical intuitions of strict conditionals grounded on basic authoritative rational intuitions of axiomatic premises in logical, mathematical, moral, or philosophical reasoning, whereas constructed intuitions

are, at best, only fairly reliable and do not depend on basic authoritative rational intuitions plus non-basic logical or mathematical intuitions alone.

To summarize so far, then, I think that there are three significant theoretical advantages of my account of intuitions as *rational intuitions*, with its three distinct types of rational intuition, over the “intellectual seemings” and “armchair judgments” approach to intuitions. These are, **first**, that my account lays down some fairly clear standards for what will count as an “intuition” in the specifically *philosophical* sense of that much abused and misused term, **second**, that my account connects directly and relevantly with the history of philosophy, and **third**, that my account does not deploy an overly simplified *univocal* theory of intuition. There seems to be no good reason to hold *either*

(i) that everything anyone casually or unreflectively calls an “intuition” (e.g., “I have an intuition that there is a big martini sitting on the kitchen table” or “I have an intuition that Obama will be a two-term President”) is going to count as an intuition in the specifically philosophical sense *or*

(ii) that the recent or contemporary use of the term “intuition” by philosophers is in any way relevantly or significantly continuous with what the classical theorists of intuitions were talking about *or*

(iii) that whatever we decide to call an “intuition” in the specifically philosophical sense must be of one kind only.

In this connection, it needs to be especially emphasized that according to my account, *all three* kinds of rational intuition are *defeasible*, i.e., *fallible*. Candidates for being objective a priori necessary truth and knowledge are never, as a matter of analytic, conceptual, or logical necessity, automatically *elected* to the status of being objective a priori necessary truth and knowledge. All candidates for election can, as a matter of analytic, conceptual, or logical possibility, fall short. Descartes was simply wrong about the infallibility of clear and distinct rational intuition, as is clearly and distinctly shown by his explicit appeal to the existence and non-deceitfulness of God as a required

mediating principle between clear and distinct rational intuition on the one hand, and necessary truth on the other. If either God does not exist or, assuming even that God exists and is a perfect being, if deceit is compatible with God’s perfect nature as an omnipotent, omniscient, and omnibenevolent (a.k.a. “3O”) being, then infallibility fails. But it *is* analytically, conceptually, and at the very least logically possible that God does not exist, and it is *also* analytically, conceptually, and at the very least logically possible that deceit is compatible with God’s 3O nature. *It is not a truth of logic* that God exists and is not a deceiver. Therefore, even given the fact of a clear and distinct rational intuition, neither its maximal, thick epistemic force nor its necessity-to-believe—which, when conjoined, yield its indubitability—itself *analytically, conceptually, or logically entails* either necessary truth or full justification. Otherwise put, all authoritative rational intuitions analytically, conceptually, or logically *can* be false.

Nevertheless, as it so happens, even in this fully natural, desperately nonideal, and “human, all too human” world, some authoritative rational intuitions *just are* objectively necessarily true and fully justified priori—e.g.,

$$3+4=7, \text{ i.e., } ||| + |||| = |||||$$

It is not the case that every meaningful sentence or statement whatsoever is both true and false., i.e., $\sim (\forall S) (S \ \& \ \sim S)$., and

Arbitrarily torturing completely innocent people to death, i.e., like the Nazis did, is morally wrong, whatever the consequences.

—which is as much as to say that for *those* rational intuitions, the connection between justification and truth is *intrinsic* and *synthetic a priori*, but not analytic, conceptual, or logical. Fallibilism about authoritative rational intuitions is not *skepticism* about them.

Before I go on, I should also say something more about the very idea of *apriority*.³⁸ As I noted in section I, in the first *Critique*, Kant says that

Although all our cognition commences **with** experience, yet it does not on that account all arise **from** experience.... It is therefore a question requiring closer investigation, and one not to be dismissed at first glance, whether there is any such cognition independent of all experience and even of all impressions of the senses. One calls such **cognitions a priori**, and distinguishes them from **empirical** ones, which have their sources *a posteriori*, namely in experience. (CPR B1-2)

I think that these remarks express a deep insight. Kant's deep insight is that apriority is in fact the *underdetermination* of the semantic content, truth, and justifiability of a statement *S* by all actual or possible empirical facts—i.e., inner or outer sensory experiences and/or contingent facts—or what is the same thing, *the failure of the strong supervenience of the content, truth, and justifiability of S on all empirical facts*, and

NEITHER

(1) the supposed fact of the *strict exclusion* of sense experience by the content, truth, or justifiability of *S* = The Classical Rationalist Conception of the A Priori, NOR

(2) the supposed fact that some beliefs—e.g., a belief expressed by *S*—are *armchair beliefs* in that a believer or community of believers resolves to hold those beliefs in such a way as to make them immune from empirical disconfirmation = The Pragmatic or Quinean Conception of the A Priori.³⁹

The obvious big problem with The Classical Rationalist Conception of the A Priori is that it stands in direct contradiction with the following basic authoritative rational intuition:

All human knowledge begins in causally-triggered sensory experience., i.e., with the second preliminary assumption of BD, i.e., with Minimal Empiricism. In other words, the obvious problem with The Classical Rationalist Conception of the A Priori is that it falsely *ignores* the truth in (Minimal) Empiricism.

Correspondingly but oppositely, the obvious big problem with The Pragmatic or Quinean Conception of the A Priori is that if it were true, then there is no inherent reason why any randomly chosen clearly crazy and false principles—e.g.,

The thought screen helmet scrambles telepathic communication between aliens and humans. Aliens cannot immobilize people wearing thought screens nor can they control their minds or

communicate with them using their telepathy. When aliens can't communicate or control humans, they do not take them.⁴⁰, or

$3+4\neq 7$

—could not be a priori, provided that a sufficiently resolute believer or community of believers held those statements to be immune from empirical disconfirmation, coherently with all the other beliefs in their Quinean “web of belief.” Or in other words, the obvious problem with The Pragmatic or Quinean Conception of the A Priori is that it falsely *overestimates* the truth in Empiricism.

Granting The Kantian Conception of the A Priori, then to say that a statement *S* is a posteriori is to say that the semantic content, truth, or justifiability of *S* is determined by or strongly supervenient on empirical facts, and NEITHER

- (i) merely that *S*'s content must bear some non-trivial relation to experience, NOR
- (ii) merely that the truth of *S* must be learned or confirmed by means of experience, at least in part, NOR
- (iii) merely that *S*'s justification must be supported by experiential evidence and established by experimental methods, at least in part.

More positively put, according to The Kantian Conception of the A Priori, it is perfectly possible for a statement *S* to be such that

- (i*) *S*'s content must bear some non-trivial relation to empirical facts.,
- (ii*) The truth of *S* must be learned or confirmed by means of empirical facts, at least in part., and
- (iii*) *S*'s belief-justification must be supported by sense-experiential evidence about empirical facts and established by experimental methods, at least in part.,

and *also* a necessary and priori.⁴¹ Here are three incontrovertible examples of a priori necessary truths such that their content must bear some relation to empirical facts, their truth must be learned or confirmed by means of empirical facts, at least in part, and their

belief-justification must be supported by sense-experiential evidence about empirical facts and established by experimental methods, at least in part:

It is not always true that it is the case that Socrates is mortal and also not the case that Socrates is mortal.

If Socrates is a bachelor, then Socrates is an unmarried male.

3 martinis + 4 martinis = 7 martinis, i.e.,

Y Y Y + Y Y Y Y = Y Y Y Y Y Y Y

Otherwise put, Kant's deep insight is that there is no such thing as semantic content, truth, or objective authentic a priori knowledge (i.e., fully justified true belief in a necessary truth) that *altogether excludes* empirical facts, which yields a *minimal* Empiricism, but that it does *not* follow from this that any version of *maximal* Empiricism is true—i.e., that the semantic content, truth, and justification of belief in such statements are either determined by/strongly supervenient on or (even more radically) reducible to empirical facts. That is clearly and simply a *non sequitur*.

Just to be perfectly clear and explicit about a familiar idea, strong supervenience⁴² is a strict determination-relation between sets of properties of different ontological “levels,” a relation that is weaker than strict property-identity, and is usually taken to be asymmetric, although two-way or bilateral supervenience is also possible. But assuming for the purposes of simpler exposition that supervenience is asymmetric, then, more precisely, *B*-properties (= the higher level properties) strongly supervene on *A*-properties (= the lower-level properties) if and only if

(i) for any property *F* among the *A*-properties had by something *X*, *F* necessitates *X*'s also having property *G* among the *B*-properties (upwards necessitation), and

(ii) there cannot be a change in any of *X*'s *B*-properties without a corresponding change in *X*'s *A*-properties (necessary co-variation).

It follows from strong supervenience that any two things *X* and *Y* share all their *A*-properties in common only if they share all their *B*-properties in common (indiscriminability). Facts are just actual or possible instantiations of properties. Hence strong supervenience for properties entails strong supervenience for facts, and failures of strong supervenience for properties correspondingly entails failures of strong supervenience for facts.

Intuitions, *as I am understanding them*, are therefore

- (i) *rational* intuitions, i.e., noninferential beliefs or thoughts, insofar as they are conceptually presented or taken as candidates for a priori knowledge of necessary truth, where
- (ii) apriority is the underdetermination of the semantic content, truth, and belief-justification of a statement *S* by all actual or possible empirical facts (i.e., sensory experiences and/or contingent facts), or what is the same thing, the failure of the strong supervenience of the content, truth, and belief-justification of *S* on all empirical facts, where
- (iii) these rational intuitions can be either authoritative (fully reliable), constructed (fairly reliable), or *prima facie* (fairly unreliable), and where
- (iv) all rational intuitions of any kind are fallible, although
- (v) some authoritative (i.e., completely convincing, intrinsically compelling, or self-evident, a.k.a. “clear, distinct, and indubitable”) rational intuitions *just are* fully reliable, objective, necessarily true, and a priori.

Here is an objection to my thesis about the relationship between apriority, aposteriority, and strong supervenience.⁴³ Sometimes it is claimed that since necessary truths hold in every logically possible world, *then they logically strongly supervene on everything*, including of course some (or all) actual or possible sensory experiences and contingent facts. So since—at least for Kantians—necessity and the a priori are necessarily equivalent,⁴⁴ then the a priori *also* logically strongly supervenes on everything, including some (or all) actual or possible sensory experiences and contingent

facts. This, in turn, would directly entail the a priori is in fact *a posteriori* by my criterion of aposteriority. But I think that this objection is wrong, for this reason.

Even if the *existence* of all necessary truths logically strongly supervened on everything, it would not follow that their *specific character* logically strongly supervened too. For although all *logically* necessary truths are necessarily equivalent, their *structural senses* are different in virtue of their inherently different logical forms. For example, “ $P \rightarrow P$ ” does not have the same structural sense as “ $P \vee \sim P$ ” because its logical form is inherently different. It is in virtue of *transformation* rules—e.g., De Morgan’s Equivalences—that we are able to move with logical spontaneity from one logical truth having a certain structural sense, to another logical truth having a distinct although necessarily equivalent structural sense. So their structural senses can, in a purely logical sense, spontaneously vary independently of their being logically necessarily true, and this intensional fact is made manifest by the application of transformation rules. In turn, therefore, their structural senses do not logically strongly supervene on whatever it is that their existence logically supervenes on, under the supposition that their existence logically strongly supervenes on everything. And that is true in every logically possible world: logically necessary truths with inherently different logical forms are all intensionally non-equivalent. So their specific character does not logically strongly supervene on *anything*, except of course on pure logic itself.

IV. Intuitions and the Irrelevance of Experimental Philosophy

Philosophical intuition is epistemologically useless, since it can be calibrated only when it is not needed. Once we are in a position to identify artifacts and errors in intuition, philosophy no longer has any use for it. Moreover, the most plausible account of the origins of philosophical intuitions is that they derive from tacit theories that are very likely to be inaccurate. There is a sense, then, in which philosophical intuitions can always be “explained away”: when a dispute arises, I can always, with some plausibility, suppose your intuitions are the artifacts of bad tacit theory. This is a game everyone can play, and I think we should all play it. We should, that is, dismiss philosophical intuitions as epistemologically valueless.

--R. Cummins⁴⁵

So ought we trust intuitions in philosophy? The first part of my answer is: no, when the intuitions are participating in practices that are hopeless, lacking any substantive means of error-detection and error-correction; and yes, when the intuition is embedded in practices that are hopeful. The second part of my answer is to suggest that [philosophers’ appeals to intuitions] falls into the first of those categories and thus ought be considered untrustworthy. But some uses of intuition, including those about logic and math and about epistemic principles whose merits can be partially tested in the laboratory of the history of science, can reasonably be placed in the second category, and we can trust them for establishing premises to use in our arguments—including (I hope!) my arguments here. In general, though, we can now see a way for the opponent to answer the question from the Sosa quote from §1: “Can intuition enjoy relative to philosophy an evidential status analogous to that enjoyed by perception relative to empirical science?” The opponent may now reply, “No, for intuition, as philosophers tend to appeal to it, lacks the hopefulness that perception has in science (and, indeed, in our ordinary lives). Once we learn *how* to be careful with our philosophical intuitions—that is, when our practices have been rendered hopeful—then we will have a successful analogy between [philosophers’ appeals to intuition] and scientist[s’] appeals to perception.”

--J. Weinberg⁴⁶

IV.1

Are intuitions epistemically reliable? So formulated, I think that there is *no philosophically relevant answer to this question*, precisely because the question radically underspecifies what is meant by the word ‘intuitions’. But once we have stated carefully what we take intuitions to be, i.e., *rational* intuitions, then it seems to me that there are at least three distinct views one could take about the reliability of intuitions in *this* sense:

- (i) Preservationism about Rational Intuitions (PARI),
- (ii) Radical Skepticism about Rational Intuitions (RSARI), and
- (iii) Radical Skepticism about Philosophical Intuitions Only (RSAPIO).

Preservationism about Rational Intuitions, or PARI, says that all rational intuitions are at least minimally reliable, although it also postulates a mutually exclusive categorization of rational intuitions into the three sub-classes of

- (i) authoritative (fully reliable) rational intuitions,
- (ii) constructed (fairly reliable) rational intuitions, and
- (iii) prima facie (fairly unreliable) rational intuitions,

and it also holds that at least some rational intuitions in mathematics, logic, morality, and philosophy are fully reliable and authoritative. Radical Skepticism about Rational Intuitions, or RSARI, says that all rational intuitions are completely unreliable and proposes the elimination of the very idea of a rational intuition. Finally, Radical Skepticism about Philosophical Intuitions Only, or RSAIO, says that that all philosophical rational intuitions are completely unreliable and proposes the elimination of the very idea of a philosophical rational intuition, but also accepts that at least *some* rational intuitions in mathematics, logic, or morality are *somewhat* reliable, and possibly some rational intuitions in mathematics, logic, or morality are even *fully* reliable.

Perhaps the most important thing to notice, again, about the way I have sliced things up is that I have explicitly narrowed the focus of all these views about the reliability of intuitions to *rational* intuitions. This means that issues about the reliability of “intellectual seemings” and “armchair judgments,” as such, are essentially not relevant to this categorization, and indeed, if I am correct, essentially not relevant to the correct epistemology of rational intuitions in mathematics, logic, morality, and philosophy. For it seems to me that the basic motivation amongst contemporary epistemologists for defending either RSARI or RSAPIO is the actual fact that “intellectual seemings” and

“armchair judgments” are all or mostly completely unreliable. But *that* actual fact is essentially irrelevant to the question of the reliability of *rational* intuitions.

IV.2

Experimental Philosophy, or X-Phi, is the contemporary fusion of Hume’s classical empiricism, Wilfrid Sellars’s scientific naturalism, and W.V. O. Quine’s epistemic psychologism,⁴⁷ as specifically directed at and focused on the study of intuitions. As such, all defenders of X-Phi explicitly or implicitly hold that

- (i) All human cognition and knowledge both begins in empirical facts (i.e., sensory experiences and/or contingent facts), and also derives from empirical facts, i.e., is strongly supervenient on empirical facts.,
- (ii) Natural science (and in particular fundamental physics, chemistry, and biology), tells us the ultimate truth about the world and ourselves, and all facts are strongly supervenient on the fundamental physical facts.,
- (iii) Empirical scientific psychology tells us the truth about human knowledge., and
- (iv) Empirical scientific psychology tells us the truth about intuitions.

Granting for the purposes of argument that my narrowing of focus to rational intuitions is philosophically appropriate and well justified, then the leading proponents of X-Phi—e.g., Cummins, Gendler, Goldman, Knobe, Nichols, Stich, and Weinberg⁴⁸—can *all* be classed as defenders of either RSARI or RSAPIO. Sometimes it is difficult to know precisely which grade of radical skepticism is being defended. But for my purposes, it does not matter. As Cummins very accurately and bluntly puts the RSAPIO thesis: “philosophical intuition is epistemologically useless.” Weinberg’s philosophical rhetoric, as encoded in his influential paper’s title, “How to Challenge Intuitions Empirically Without Risking Skepticism,” suggests that his view is non-skeptical or at least non-radically skeptical. But it is clear enough from the text I quoted above that, just like the blunter Cummins, although he rejects RSARI, Weinberg too holds RSAPIO.

It should be particularly emphasized that I do not have any quarrels with the empirical scientific psychological study of rational intuition as such. Empirical evidence about human cognition, or empirical evidence concerning what philosophers or non-philosophers say in response to various questionnaires, is *sometimes* philosophically *relevant* and *always* philosophically *interesting*.

But at the same time, I do have four serious worries about RSARI and RSAPIO. And if these worries are cogent, then X-Phi is irrelevant to the epistemology of rational intuitions.

First, in light of what I argued in section **III**, the fact that it can be empirically shown that most people's reported "intellectual seemings" or "armchair judgments" are not reliable has no more direct bearing on the epistemic value of rational intuitions, than the fact that it can be empirically shown that most people are not good at simple arithmetic, probability judgments, or logical deduction tests, has any sort of direct bearing on the epistemic value of arithmetic, probability theory, or logic.⁴⁹ It would be like arguing from the obvious fact that most people are not good at living up to their own moral principles, to the conclusion that morality is "ethically useless." The sharp difference between the categorical *ought* and the factual *is* is partially constitutive of the very idea of intuition-based epistemology in mathematics, logic, morality, and philosophy, not counter-evidence *against* it.

Second, and correspondingly, the sharp difference between the basic or non-basic authoritative rational intuitions we categorically ought to have and only sometimes do have, and the constructed and *prima facie* rational intuitions that we mostly actually do

have, is partially constitutive of the very idea of rational intuition, not counter-evidence *against* the epistemic value of intuitions.

Third, if either RSARI and RSAPIO were authentically known to be true, then *how* would they be known to be true, *except* by means of basic authoritative philosophical rational intuitions? Neither RSARI nor RSAPIO is itself an *empirical* claim. On the contrary, if they are authentically knowably true at all, then they are necessarily true and a priori knowable. This is clear enough from the fact that both RSARI and RSAPIO implicitly presuppose *Minimal Empiricism*, the second preliminary assumption of BD:

All human knowledge begins in causally-triggered sensory experience.

So it is clear that if RSARI or RSAPIO are authentically knowable at all, then it must be by means of at least some basic authoritative rational intuitions. It then directly follows that both RSARI or RSAPIO are a priori self-contradictory, and also rationally and strongly normatively self-stultifying. In the case of RSARI, how could the epistemic reliability of aprioristic human rationality be *radically* challenged or *definitively* rejected without presupposing the epistemic reliability of aprioristic human rationality? And in the more special case of RSAPIO, how could the epistemic reliability of aprioristic human *philosophical* rationality be radically challenged or definitively rejected without presupposing the epistemic reliability of aprioristic human *philosophical* rationality? So RSARI and RSAPIO are not only, in effect, cognitive suicide—they are categorically *impermissible*.

Fourth and finally, the most interesting and seemingly powerful argument in X-Phi for either RSARI or RSAPIO—Cummins’s Dilemma of Calibrating Intuitions, or The DCI for short⁵⁰—is in fact clearly unsound.

Here is Cummins's argument in a nutshell. To "calibrate" intuitions is to have an effective way of testing them for reliability. The DCI then says that *either* (i) intuitions *can* be calibrated, in which case philosophers do not need to appeal intuitions, *or else* (ii) intuitions *cannot* be calibrated, in which case philosophers should not appeal to intuitions. So no matter how you look at it, intuitions are "epistemologically useless."

But on the contrary, I think that The DCI is a *false* dilemma, and that correspondingly, Cummins's argument fails. This is because Cummins—or anyhow other defenders of The DCI, even if not Cummins himself—make at least eight unargued assumptions, each of which is also presupposed by The DCI, and each of which is independently plausibly challengeable:

- (i) There is one and only one kind of intuitions—that, or propositional rational intuitions, and this single kind is the class of "intellectual seemings" and/or "armchair judgments" [**the single kind assumption**].
- (ii) There is one and only one method of calibrating intuitions [**the single method assumption**].
- (iii) If *any* method of inquiry can calibrate intuitions, it must be a method belonging to the natural sciences [**the naturalistic assumption**].
- (iv) Natural science does not itself require calibration [**the no-fault-naturalism assumption**].
- (v) Intuitions cannot be used to calibrate other intuitions [**the no-meta-calibration assumption**].
- (vi) No intuitions are self-calibrating [**the no-reflexive-calibration assumption**].
- (vii) Intuitions are all cognitively generated by a distinct, encapsulated "intuition faculty" or "intuition module" [**the modularity assumption**], and
- (viii) An epistemology of intuitions must be either Foundationalist or Coherentist, and there are no other intelligible options [**the Founderentist assumption**].⁵¹

Nevertheless, if what I have already argued, and what I will later argue, in this essay is correct, then *all eight of these assumptions are false*.

As against assumption (i), **the single kind assumption**, if I am correct, then there are at least *three mutually distinct classes or kinds of rational intuitions*: namely, authoritative, constructed, and prima facie.

It is relevant to note here that many *contemporary enemies* of The DCI also hold **the single kind assumption**, e.g., Bealer, Huemer, and Sosa. So if it is plausibly arguable that **the single kind assumption** is false, then this suffices to refute both all the friends and also many of the contemporary enemies of The DCI.

It is also relevant to note here that *another* unargued assumption and presupposition of The DCI is that rational intuitions are neither already calibrated nor not in need of calibration. I will call this **the neither-nor assumption**. But I think that it is *not* plausibly arguable either that rational intuitions are already calibrated or that rational intuitions are not in need of calibration. This is obviously true of prima facie intuitions, since by hypothesis these are all fairly unreliable, hence they cannot either be already calibrated nor not in need of calibration. But if one holds **the single kind assumption**, as many contemporary enemies of The DCI do, then it is *also* not plausibly arguable either that rational intuitions are already calibrated or that rational intuitions are not in need of calibration. For “intellectual seemings” and “armchair judgments” are, at best, “super-weakly justified”⁵² in that they are not *completely* open to radical skepticism, i.e., not *completely* unreliable. But this on its own falls far short of showing that “intellectual seemings” and “armchair judgments” are either already calibrated or not in need of calibration, since calibration is just *an effective test for reliability*, and no “intellectual seemings” or “armchair judgments,” *as such*, can claim either to be already effectively tested for reliability or not in need of an effective test for reliability. So, ironically enough

for many contemporary enemies of The DCI, **the neither-nor assumption** is an unargued assumption and presupposition to which The DCI *is* actually entitled.

As against, assumption (ii), **the single method assumption**, if I am correct, then rational intuitions need to be calibrated by *three* co-basic and inherently complementary methods: namely, (1) by authoritative rational intuitions, (2) by natural science, and by Wide Reflective Equilibrium in the Rawlsian sense. I defend this important methodological point in section **XII**.

As against assumption (iii), **the naturalistic assumption**, if I am correct, then (as I just implicitly asserted in the immediately preceding paragraph) natural science is only *one* of three co-basic and inherently complementary ways of calibrating rational intuitions, and also that, in any case, natural science is *not* an entirely independent way of calibrating, since natural science presupposes, at the very least, some basic authoritative rational intuitions in mathematics, logic, and morality, e.g., the mathematical intuition that

$$3+4=7, \text{ i.e., } ||| + |||| = |||||$$

the logical intuition that

It is not the case that every meaningful sentence or statement whatsoever is both true and false., i.e., $\sim (\forall S) (S \ \& \ \sim S)$,

and the moral intuition that

Arbitrarily torturing completely innocent people to death, i.e., like the Nazis did, is morally wrong, whatever the consequences.

Natural science without simple arithmetic and minimal logical consistency would be either impossible full stop or at the very least crazy and self-stultifying. And although natural science, as a rational human practice, does not presuppose so *very* many robustly normative constraints, nevertheless faking experimental results, or using scientific

instruments and methods for the sole purposes of human degradation and torture, *are* inherently ruled out by the very nature of natural scientific practices—*Nazi* science is *not* natural science.

As against assumption (iv), **the no-fault-naturalism assumption**, if I am correct, then natural science *itself* needs to be calibrated, not only by some presupposed basic authoritative rational intuitions in mathematics, logic, and morality, like the ones I just mentioned, but also by Wide Reflective Equilibrium in the Rawlsian sense. If, to use Sellars's apt metaphors, *the Scientific Image* of the ourselves and the world turns out to be directly incompatible with *the Manifest Image* of ourselves and the world⁵³ with respect to the truth of some specific claim or set of claims, then it is at least as likely that *natural science* is mistaken about that claim or set of claims, as that Wide Reflective Equilibrium in the Rawlsian sense is mistaken, other things being equal. This is precisely because some basic authoritative rational intuitions in mathematics, logic, and morality, such as the ones cited above, are every bit as fully presupposed by, and also as epistemically and normatively autonomous in relation to, *natural science*, as they are fully presupposed by and epistemically and normatively autonomous in relation to Wide Reflective Equilibrium in the Rawlsian sense. At the end of the philosophical day, natural science may well prove to be epistemically and normatively privileged in various ways. But this cannot be reasonably asserted or assumed without any further argument, or calibration, *at the beginning of the day*. Wittgenstein expresses essentially the same thought in a characteristically vivid way:

The truly apocalyptic view of the world is that things do not repeat themselves. It isn't absurd, e.g., to believe that the age of science and technology is the beginning of the end for humanity; the idea of great progress is a delusion, along with the idea that the truth will ultimately be known; that there is nothing good or desirable about scientific knowledge and that mankind, in seeking it, is falling into a trap. It is by no means obvious that this is not how things are.⁵⁴

As against assumption (v), **the no-meta-calibration assumption**, if I am correct, then prima facie rational intuitions can be calibrated either by basic or non-basic authoritative rational intuitions, or by constructed rational intuitions, or by a combination of the two, in conjunction with the natural sciences and Wide Reflective Equilibrium in the Rawlsian sense.

As against assumption (vi), **the no-reflexive-calibration assumption**, if I am correct, then at least some basic authoritative rational intuitions, such as the ones cited above, by virtue of the specific modal character of their internal justificational structure, together with their intrinsic connection to the truth, constitute authentic a priori knowledge, and are thereby *self-calibrating*. Some *other* examples of these self-calibrating intuitions are the philosophical intuition that truth is uniform and Tarskian, the philosophical intuition that human knowledge begins in causally-triggered sensory experience, and the philosophical intuition (which will play an extremely important role in the next section) that at least some of the truths of Primitive Recursive Arithmetic, or PRA, are authentically knowable a priori by fully reliable basic authoritative rational intuitions on the basis of Hilbert-style “objects of finitary reasoning,” i.e., on the basis of the cognitive construction and manipulation of sensible forms in Kantian pure intuition, mental models, diagrams, structural imagery, or schemata. Such self-calibrating rational intuitions are *also* fully confirmed by the natural sciences and by Wide Reflective Equilibrium in the Rawlsian sense.

As against assumption (vii), **the modularity assumption**, if I am correct, then rational intuitions are in fact generated by the complete “central” or “global” innate human capacity for non-instrumental or categorically normative theoretical or practical

rationality, involving all of the other basic human psychological capacities, including consciousness, self-consciousness, sense perception, memory, imagination, conceptualization, essentially non-conceptual cognition, and judgment.

Finally, as against assumption (viii), **the Founderist assumption**, if I am correct, then the best overall epistemological explanation of intuitions is neither Foundationalist nor Coherentist, but in fact *Transcendental Idealist*. Foundationalism says that knowledge is grounded solely on some non-normative primitive facts (a.k.a. “the Given,” as it occurs in Sellars’s “Myth of the Given”⁵⁵), whether internal or external, that somehow fully justify corresponding foundational beliefs just by means of causing those beliefs. Coherentism says that knowledge is grounded solely on networks of consistency or entailment relations between beliefs. The classical problem with Foundationalism is that *non-normative* primitive facts cannot *normatively* support (i.e., justify, via reasons) beliefs, and the classical problem with Coherentism is that compatibility-relations and inferential networks on their own do not guarantee any sort of correspondence with the actual facts, i.e., they do not guarantee *truth*. By sharp contrast to Foundationalism and Coherentism alike, Transcendental Idealist epistemology says that knowledge is true belief that is fully justified by evidence which includes an intrinsic connection to the truth—an intrinsic connection that is inherently governed by categorically normative a priori theoretical and practical principles, and is also metaphysically guaranteed by the necessary conformity of the underlying formal or structural features of the manifest natural world to the underlying formal or structural features of the innately-specified cognitive faculties of rational human animals. Even if I am wrong that the Transcendental Idealist explanation of intuitions is the *best* overall

epistemological explanation, nevertheless I am still right that it constitutes a distinct and intelligible *third* kind of epistemological explanation which is fundamentally distinct from both Foundationalism and Coherentism.

Therefore, if I am correct about all of this, then The DCI is not a real dilemma at all, and furthermore whatever real epistemic issues are raised by it can plausibly arguably all be resolved in a way that entails the denial of both RSARI and RSAPIO, and the also irrelevance of X-Phi, together with the denial of each of the unargued assumptions or presuppositions of The DCI that I have spelled out—with the sole exception of **the neither-nor assumption**, which I take to be true.

So, if I am correct, then Preservationism about rational intuitions, or PARI, is strongly warranted, at least as a working hypothesis. In the next section, I will provide a direct argument for the falsity of RSARI, RSAPIO, and Experimental Philosophy alike.

V. Philosophical Intuitions, Scientific Naturalism, and The Mathematico-Centric Predicament

[H]ow does mathematical language function? Does it relate the world in the same ways as the language of natural science? What happens when human beings come to understand mathematical theories? How does mathematics work in various kinds of applications? And so on. To answer these questions, [the scientific-naturalist philosopher of mathematics] must face many of the metaphysician's concerns: do mathematical entities exist, and if so, what is the nature of that existence? Are mathematical claims true, and if so, how do humans come to know this? These are not detached, extra-scientific pseudo-questions, but straightforward components of our scientific study of human mathematical activity, itself part of our scientific investigation of the world around us.

--P. Maddy⁵⁶

V.1

As I pointed out in section IV, Experimental Philosophy, or X-Phi, is the contemporary fusion of classical Humean Empiricism, Sellarsian Scientific Naturalism, and Quinean epistemic psychologism, as specifically directed at and focused on the study of intuitions; and, as such, all defenders of X-Phi explicitly or implicitly hold that

- (i) All human cognition and knowledge both begins in empirical facts (i.e., sensory experiences and/or contingent facts), and also derives from empirical facts, i.e., is strongly supervenient on empirical facts.,
- (ii) Natural science (and in particular fundamental physics, chemistry, and biology) tells us the ultimate truth about the world and ourselves, and all facts are strongly supervenient on the fundamental physical facts.,
- (iii) Empirical scientific psychology tells us the truth about human knowledge., and
- (iv) Empirical scientific psychology tells us the truth about intuitions.

In this section I will argue that (i) through (iv) are all false.

V.2

The two-part philosophical thesis that the natural sciences (and in particular fundamental physics, chemistry, and biology) adequately and truly explain everything in terms of fundamental physical properties, and that all knowledge claims are adequately justified only to the extent that that they are warranted by empirical evidence and by

natural scientific methods alone, is *Scientific Naturalism*, which is most crisply and gnominically expressed by Sellars's well-known slogan:

In the dimension of describing and explaining the world, science is the measure of all things, of what is that it is, and of what is not that it is not.⁵⁷

X-Phi is clearly a sub-species of Scientific Naturalism—or even more specifically, X-Phi is clearly Scientific Naturalism *as* applied either to rational intuitions in general or to philosophical rational intuitions in particular, and *with* a radical critical attitude towards them that is inherited directly from classical Humean Empiricism.

X-Phi is also committed to epistemic psychologism in the Quinean mode. Now by *Mathematical Psychologism*, I mean the thesis that mathematical computation and mathematical knowledge are adequately explained and justified by empirical scientific psychology, e.g., contemporary cognitive neuroscience. Mathematical Psychologism is directly entailed by Scientific Naturalism and also by X-Phi. The leading contemporary proponent of Mathematical Psychologism is Penelope Maddy,⁵⁸ and although (as far as I know) she is not *officially* a member of the X-Phi movement, she is certainly a fellow traveller.

Now consider the following item reported in Newsweek in February 2010:

Native Chinese speakers use a different region of the brain to do simple arithmetic ($3 + 4$) or decide which number is larger than native English speakers do, even though both use Arabic numerals. The Chinese use the circuits that process visual and spatial information and plan movements (the latter may be related to the use of the abacus). But English speakers use language circuits. It is as if the West conceives numbers as just words, but the East imbues them with symbolic, spatial freight. ... “One would think that neural processes involving basic mathematical computations are universal,” says [Tufts psychologist Nalinin] Ambady, but they “seem to be culture-specific.”⁵⁹

What should we conclude from this? Here is what *I* would want to conclude:

Well-formed and sound mathematical computations in simple arithmetic, as performed by rational human animals, although universally and necessarily true and also objectively knowable a priori by fully reliable basic authoritative mathematics intuition, are nevertheless multiply instantiated in, and are therefore not identical to, neural computational processes, which in some cases are culturally specific.

But here is what a proponent of Mathematical Psychologism⁶⁰ would argue:

The two kinds of psychological processes (roughly, Western mathematical cognition and Eastern mathematical cognition) are non-identical. In which case there would not be a single mental kind multiply realized (after all, the processing differs in important ways). Content properties of the neural vehicles can be shared (i.e., the neural structures can share content-constituting—say, causal—relations to objectively existing mathematical properties in the world); so, the naturalist can still have her mathematical realism. But, to the extent that these content properties are relational complexes individuated by their relata (some of which are the varying neural vehicles), the relational complexes as wholes are of distinct kinds in the two cases. Thus, beyond the mathematical properties themselves, there remains only one shared portion across cultures: the content-determining relations various neural structures bear to mathematical properties; and these relations are reducible—to patterns of causal relations, in the first instance. Problem solved.

In immediate reply to the Mathematical Psychologist, I would want to claim that Ockham's Razor—which says that the entities postulated by explanations and theories ought not be multiplied without necessity— for a change favors the non-reductionist side of this debate, and also that it seems significantly more explanatorily economical to postulate one non-reducible mathematical human cognitive process-type (i.e., the process-type of consciously and self-consciously calculating that $3+4=7$) with two distinct culturally specific neurobiological instances, than to postulate two distinct mathematical human cognitive process-types, each of which is then physically reducible to a culturally specific brain process-type. That is not only pleasingly philosophically ironic, but also a point in favor of Mathematical Anti-Psychologism: Given these interesting empirical data, Mathematical Anti-Psychologism is a *simpler* theory than Mathematical Psychologism. In short, Ockham's Razor cuts *two ways*: sometimes towards the reductionist, and sometimes towards the non-reductionist.

Needless to say, Scientific Naturalists generally and Mathematical Psychologists in particular will not accept my thesis that sometimes non-reductionists have a better all-things-considered claim on the use of Ockham's Razor than reductionists. In any case, quite apart from the somewhat controversial issue of how

correctly to apply Ockham's Razor in philosophical explanations, I also think that there is a much deeper problem here that Mathematical Psychologism needs to face up to, and by implication that both Scientific Naturalism in general and X-Phi in particular need to face up to, in view of the fact that all tokens of human cognitive process-types in simple arithmetic *are also constructive finitist proofs in Primitive Recursive Arithmetic*, or PRA,⁶¹ which in turn is a necessary proper part of elementary or Peano arithmetic.

Now elementary arithmetic, or Peano Arithmetic, is defined by the following five axioms:

- (1) 0 is a number.
- (2) The successor of any number is a number.
- (3) No two numbers have the same successor.
- (4) 0 is not the successor of any number.
- (5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.,

together with the primitive recursive functions. Primitive recursive functions are the basic calculations or basic operations over the natural numbers—the successor function, addition, multiplication, exponentiation, etc. More precisely then, PRA is the fundamental fragment of elementary or Peano arithmetic that contains the quantifier-free theory of the natural numbers and the primitive recursive functions. Or otherwise put, PRA is *simple arithmetic*. PRA or simple arithmetic, in turn, is consistent, complete, sound, and decidable, and has all the primitive “logical perfections”—sharply *unlike* elementary or Peano arithmetic, which, as Gödel's Incompleteness Theorems show, is (i) consistent if and only if it is incomplete, and (ii) such that its ground of truth must outside of the system of Peano arithmetic itself.⁶²

Granting PRA, the much deeper problem for Mathematical Psychologism, Scientific Naturalism, and X-Phi alike is this. Consider the following fully reliable basic

authoritative philosophical intuition, which I will call **The Reliability of Rational**

Intuitions in Simple Arithmetic:

At least some of the truths of Primitive Recursive Arithmetic, or PRA, are authentically knowable a priori by fully reliable basic authoritative rational intuitions on the basis of Hilbert-style “objects of finitary reasoning,” i.e., on the basis of our cognitive construction and manipulation of sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata.

The Reliability of Rational Intuitions in Simple Arithmetic, in turn, captures a specifically Kantian intuitionist interpretation of William Tait’s deeply important philosophical insight about finitism, which I have already quoted as the sixth epigraph of this essay:

[A]lthough we cannot speak of the absolute security of finitism, there is a sense in which we can speak of its *indubitability*. That is, any nontrivial reasoning about number will presuppose finitist methods, and there can be no preferred or even equally preferable method from which to launch a critique of finitism. In other words, it is simply pointless to doubt it.⁶³

Now in giving natural scientific explanations and justifications any kind, we actually presuppose and use mathematics, and in particular elementary or Peano arithmetic, especially including PRA or simple arithmetic. As a direct consequence of this circularity, it follows that EITHER

(1) Mathematics, and in particular elementary or Peano arithmetic, especially including PRA or simple arithmetic, is just inexplicable and unjustifiable., OR

(2) We presuppose at least one fully reliable basic authoritative philosophical intuition, namely **The Reliability of Rational Intuitions in Simple Arithmetic**, which entails that mathematics and in particular elementary or Peano arithmetic, especially including PRA or simple arithmetic, is inexplicable and unjustifiable *by means of natural science alone*, and nevertheless *can* be adequately explained and justified, but *only* by appealing to properties that are not (merely) fundamental physical properties, to evidence that is not (merely) empirical, and to methods of inquiry that extend beyond those of the natural sciences, even though they also include those of the natural sciences: hence Scientific Naturalism, Mathematical Psychologism, RSARI, RSAPIO, and X-Phi are all false.

I call this *The Mathematico-Centric Predicament* because it is relevantly similar to another important circularity problem in the philosophy of the formal sciences, first

clearly articulated by the Harvard logician Harry Sheffer (discoverer of the Sheffer stroke-function), and now known as the problem of *The Logocentric Predicament*:

The attempt to formulate the foundations of logic is rendered arduous by a ... “logocentric” predicament. In order to give an account of logic, we must presuppose and employ logic.⁶⁴

Here is my slightly more explicit reformulation of Sheffer’s deep worry:

In order to explain or justify logic, logic must be presupposed and used. As a direct consequence of this circularity, it seems to follow that logic is inexplicable and unjustifiable.⁶⁵

The Logocentric Predicament forces philosophers of logic to face up to the task of explaining and justifying logic. Correspondingly, The Mathematico-Centric Predicament forces defenders of Scientific Naturalism in general, Mathematical Psychologism more specifically, and X-Phi in particular to face up to the fact that they cannot explain and justify mathematics and in particular elementary arithmetic, including PRA or simple arithmetic, without *also* presupposing at least one fully reliable basic authoritative philosophical intuition, i.e., **The Reliability of Rational Intuitions in Simple Arithmetic**, thereby showing the falsity of Scientific Naturalism, Mathematical Psychologism, RSAIO, RSAPIO, and X-Phi alike.

It seems to me obvious that defenders of Scientific Naturalism, Mathematical Psychologism, and/or X-Phi will *not* want to hold that elementary or Peano arithmetic, especially including PRA or simple arithmetic, is inexplicable and unjustified. How could they plausibly claim that $3+4=7$ or any other part of PRA or simple arithmetic, is inexplicable or unjustified, in view of the fact that they are already actually presupposing and using PRA or simple arithmetic in their attempts to explain and justify mathematics by means of the natural sciences?

I have just indicated the relevant similarity between The Mathematico-Centric Predicament and The Logocentric Predicament. But there is also a certain *dissimilarity* between them that is important, and needs to be made explicit. The Logocentric Predicament starts from the premise that in order to explain or justify logic, logic *must* be presupposed and used. But I am *not* making a parallel claim about mathematics and in particular elementary or Peano arithmetic, especially including PRA or simple arithmetic. In principle, you could at least *try* to explain or justify mathematics without using even PRA or simple arithmetic. You could at least *try* to explain or justify mathematics by using pure logic alone, without any appeal whatsoever to the primitive recursive functions. In particular, that would mean trying to explain or justify mathematics without any appeal whatsoever to *counting* or *enumeration*, including *equinumerosity*. You could not even appeal rationally to calculations by means of an abacus, your fingers, or Hilbert-style stroke diagrams. Even the most radical Logicists have never tried to do that. But it is not impossible to try. It is just *pragmatically self-contradictory* and *rationally self-stultifying*.

The Mathematico-Centric Predicament should be carefully distinguished from the well-known Quine-Putnam *Indispensability Argument* for the existence of numbers and other mathematical entities.⁶⁶ This argument says that mathematics is *indispensable* for the natural sciences, and that therefore numbers and other mathematical entities *exist*. I am *not* arguing that mathematics and in particular elementary or Peano arithmetic, especially including PRA or simple arithmetic, is *indispensable* for the natural sciences, and that therefore mathematics *must* be presupposed and used, and I am *not* thereby arguing for the existence of numbers and other mathematical entities. The

Indispensability Argument may or may not be sound, and in this essay I am taking no stand on that. What I am arguing, by contrast, is that mathematics and in particular elementary or Peano arithmetic, especially including PRA or simple arithmetic, *is just in fact presupposed and used in the actual current practice of the natural sciences*. No one could deny this. But since mathematics and in particular elementary or Peano arithmetic, especially including PRA or simple arithmetic, is just *in fact* presupposed and used in the actual current practice of the natural sciences, then either this actual presupposing and using is inexplicable and unjustified, or else it presupposes at least some fully reliable basic authoritative philosophical intuitions—i.e., **The Reliability of Rational Intuitions in Simple Arithmetic**—and thus it is explicable and justified only by something beyond the natural sciences themselves, so that Scientific Naturalism, Mathematical Psychologism, RSARI, RSAPIO, and X-Phi are all self-refutingly false. That is The Mathematico-Centric Predicament.

Moreover, it is also just a fact that primitive recursive functions are presupposed and used in the actual current practice of computability theory, via The Church-Turing Thesis, which says that effective decidability is the same as general recursiveness, and that all general recursive functions are Turing-computable.⁶⁷ That doctrine, in turn, is actually and highly successfully applied in the real-world construction of laptop computers, the Internet, cell or mobile phones, and other mobile Internet-related technology, all of which, I am sure, are used on a daily basis by all Scientific Naturalists, Mathematical Psychologists, and Experimental Philosophers. So it is very hard to see how defenders of RSARI or RSAPIO could ever provide an “error-theory” for PRA or simple arithmetic without pragmatic self-contradiction and rational self-stultification—

i.e., without committing *cognitive suicide*, and without doing something that is *rationaly impermissible*.

In other words, I think that The Mathematico-Centric Predicament decisively shows that Scientific Naturalism, Mathematical Psychologism (as a sub-species of Scientific Naturalism), RSARI, RSAPIO, and X-Phi (as a sub-species of Scientific Naturalism and Mathematical Psychologism alike) are all *false*. Correspondingly, I think that this five-part negative result collectively provides a sufficient reason for holding that Preservationism about intuitions, or PARI, is *true*.

VI. Kantian Structuralism

Number ... is a representation that summarizes the successive addition of one homogenous unit to another. Number is therefore nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition.

--I. Kant (*CPR* A142-143/B182)

Time provides a universal source of models for the numbers.... What would give time a special role in our concept of *number* which it does not have in general is not its necessity, since time is in some way necessary for all concepts, nor an explicit reference to time in numerical statements, which does not exist, but its sufficiency, because the temporal order provides a representative of the number which is present to our consciousness if any is present at all.

--C. Parsons⁶⁸

VI.1

The key to achieving a positive or anti-skeptical solution to Benacerraf's

Dilemma, or BD, I think, is precisely how one interprets step (4) in my reconstruction, which says:

(4) Given (1) and (3), our standard, uniform semantics of truth in natural language, as applied to true mathematical statements, commits us to a truth-making ontology of abstract mathematical objects and also to the non-empirical knowability of these statements.

It is *very* natural, and all-too-easy, to interpret the notion of "a truth-making ontology of abstract mathematical objects" in terms of classical Platonism. Classical Platonism about mathematics says that mathematical objects, which are the truth-makers of mathematical statements, have a mind-independent, substantial existence in a separate non-spatiotemporal realm, that these objects have intrinsic non-relational properties, and that the natures of these objects are strictly determined by their intrinsic non-relational properties, i.e., by their "real essences." In short, classical Platonism interprets mathematical objects as what Kant would have called *things-in-themselves*.⁶⁹ This classical Platonist interpretation of the truth-making ontology of abstract mathematical objects postulated in step (4), I think, is precisely *the snake in the Garden of Eden*, by

which I mean that I think that this interpretation is precisely the false and vitiating assumption which leads inevitably to to BD and to Skepticism, and I hereby reject it.

Granting that rejection as a starting point, then my positive or anti-skeptical intuition-based solution to BD, as I previewed it in section **II**—has two parts:

- (1) Kantian Structuralism, and
- (2) Kantian Intuitionism.

In the rest of this section, I want to develop and defend Kantian Structuralism. Then I will go on to develop and defend Kantian Intuitionism in section **VII**. In section **VIII**, I will critically compare and contrast Kantian Structuralism and Kantian Intuitionism with Parsons's theory. In sections **IX** and **X**, I will extend Benacerraf's Dilemma, Kantian Structuralism, and Kantian Intuitionism to logic. Then finally, in section **XI**, I will extend Kantian Structuralism and Kantian Intuitionism to morality too. As I mentioned in sub-section **I.3**, my argument for the existence and full reliability of basic authoritative philosophical intuitions will emerge naturally from the modal epistemology of mathematics, logic, and morality.

VI.2

Mathematical Structuralism, as an explanatory metaphysical thesis in the philosophy of mathematics—defended, e.g., by Benacerraf himself,⁷⁰ by Michael Resnick,⁷¹ by Stewart Shapiro,⁷² and most recently by Parsons⁷³—says that mathematical entities (e.g., numbers or sets) are not ontologically autonomous or substantially independent objects, but instead are, essentially, *positions* or *roles* in a mathematical structure, where a mathematical structure is a complete set of formal relations and operations that defines a mathematical system. What counts as an individual object of the system is thereby uniquely determined by the system as a whole—that is, any such

individual object is identical to whatever possesses a specific set of intrinsic structural system-dependent properties. So every individual object of the system is essentially a role in the relevant mathematical system, and thus metaphysically dependent, and indeed *strongly supervenient*, on the whole system.

The significant philosophical payoffs of Mathematical Structuralism are twofold.

First, Structuralism gets between the rock of Platonism and the hard place of Nominalism, because according to Structuralism mathematical objects are metaphysically absorbed into mathematical structures, hence they lack independent existence (contra Platonism), and yet it is also not true that there are no mathematical objects (contra Nominalism), since the objects continue to exist in a theoretically transformed way *as* positions or roles in the structure.

Second, because according to Mathematical Structuralism the mathematical objects, as embedded in the relevant mathematical structure, continue to have whatever metaphysical status the relevant embedding structure has, then there is no longer any serious metaphysical “identity problem” about precisely *which* objects should be identified with the natural numbers, since we look to the embedding structures and not to the objects themselves for any relevant metaphysical identity conditions.

In a way that is highly analogous to Functionalism in the philosophy of mind,⁷⁴ there are at least two distinct ways we can interpret Mathematical Structuralism. On the one hand, we can identify mathematical objects with *the roles* determined by the mathematical system as a whole. Or on the other hand, we can identify mathematical objects with *the role players* of the mathematical roles determined by the system as a whole. Which interpretation of Mathematical Structuralism should we accept?

In the analogous case of Functionalism in the philosophy of mind, I think that there is good reason to take the Role-Player interpretation seriously because we think that it is intuitively plausible to identify a mind with whatever it is that actually does all the causally efficacious things that cognitive systems are empirically known to do, and not merely to identify it with the set of causally relevant abstract patterns or rules that actual cognitive systems follow or instantiate. If a mind were merely identical with a set of causal-functional *roles*, then it would be open to the classical inverted qualia argument, Searle's Chinese Room argument, and Block's Chinese Nation argument (a.k.a. "the absent qualia argument"),⁷⁵ not to mention the deeper worry that causal relevance does not entail causal efficacy,⁷⁶ which yields the unhappy result that even *the representational mind* would be epiphenomenal if the Roles interpretation were true.

Correspondingly, and now to use an everyday non-philosophical analogy, it seems clearly and distinctly right to say that an ice hockey player is a person who actually and in a causally efficacious way does all the things that hockey players are supposed to do, according to the rules of ice hockey—and obviously, a real hockey player is *not merely* the same as a set of causally relevant abstract rules that hockey players follow or instantiate. So if we want minds to be *real causal players*, as it were, in physical nature, not to mention being *really capable of consciousness or subjective experience* in addition to mental representation or intentionality, then I think that we should defend a *dual* Roles interpretation *and* Role-Player interpretation of Functionalism, as opposed to a Roles interpretation alone or a Role-Player interpretation alone.⁷⁷ We should say that for *some* rational purposes, the mind should be identified with functional roles, and also that for *other* rational purposes, the mind should be identified with the role-players of the roles.

By analogy, then, and for essentially the same basic reasons, I will adopt a *dual* Roles interpretation *and* Role-Player interpretation of Mathematical Structuralism, as opposed to a Roles interpretation alone or a Role-Player interpretation alone. To be sure, we want the natural numbers to be identified for many rational purposes with their abstract roles in the mathematical structure of elementary or Peano arithmetic, especially including the finitist sub-structure of PRA or simple arithmetic. But for other rational purposes we also want the unique, intended model (i.e., the one and only real truth-maker) of Peano arithmetic, especially including the finitist sub-structure of PRA or simple arithmetic, *to be consciously knowable according to a reasonable epistemology*, which is the direct analogue of an adequate response to the problem of qualitative conscious experience for the Roles interpretation of Functionalism.⁷⁸ And we also want natural numbers and true statements about natural numbers *to be applicable to the actual spacetime world*, which is the direct analogue of an adequate response to the problem of Epiphenomenalism for the Roles interpretation of Functionalism.⁷⁹

So as I see it, Mathematical Structuralism should hold that mathematical objects are essentially the same, for some rational purposes, as the roles in a given mathematical structure, and *also* essentially the same, for some other rational purposes, as the role players of the specific mathematical roles in a given mathematical structure, and *not reducible* either to those roles themselves or to the role-players themselves. The roles tell us precisely what will count as the unique intended model of that abstract mathematical structure, but they neither exhaust the total nature of the mathematical objects nor do they eliminate the objects altogether. The mathematical objects are strongly superveniently determined by the abstract structure as regards the precise roles they play, but they are

also something over and above the abstract structure as regards their role-player status. Different objects can play the same mathematical roles; the same objects can play different mathematical roles; and as a consequence, there is no intelligible worry whether the natural number 12 is the same as or different from the real number 12. This metaphysical dependency relation between abstract mathematical structure and mathematical object in Mathematical Structuralism thereby provides a precise analogue of *natural or nomological* strong supervenience, as opposed to either downwards identity or *logical* strong supervenience (i.e., reduction), in the philosophy of mind.

An important and secondary *meta*-philosophical pay-off of this way of thinking about Mathematical Structuralism is the theoretically fruitful recognition that *the philosophy of mind* and *the philosophy of mathematics* are not only *formally analogous* to one another in certain ways, but also *necessarily connected* to one another in certain ways.⁸⁰

But the primary and *first-order* philosophical pay-off of this way of thinking about Mathematical Structuralism is its application to BD. BD clearly and distinctly shows us that we do *not* want numbers to be the kind of abstract entities that are also unknowable things-in-themselves and inapplicable to the actual spacetime world, lest we render a priori mathematical truth and knowledge impossible. Or otherwise and more positively put, BD clearly shows us that the abstractness of the numbers *must* somehow correlate directly with what is consciously knowable according to a reasonable epistemology. This is possible, I think, if (and indeed also only if) the abstractness of the numbers is *not* the abstractness of independent objects in a causally inert non-spatiotemporal realm, but instead *must be* nothing but the abstractness of the roles in a

non-empirical or a priori humanly consciously-accessible *cognitive* mathematical structure. More precisely, on this philosophical picture, the natural numbers are abstract because they are essentially roles in a *weakly or counterfactually transcendentally ideal* structure.

Before going on, then, I need to say more about Transcendental Idealism.

According to Kant, a mental representation is *transcendental* when it is either part of, or derived from, our non-empirical (hence a priori) innately-specified spontaneous cognitive capacities (*CPR* A11/B25) (*Prolog* 4: 373n.). This allows me to spell out Transcendental

Idealism as a two-part thesis: *Transcendental Idealism = (1) Representational Transcendentalism + (2) Cognitive Idealism.*

(1) **Representational Transcendentalism** = Necessarily, all the forms or structures of rational human cognition are generated a priori by the empirically-triggered, yet stimulus-underdetermined, activities of our innately-specified spontaneous cognitive capacities (= cognitive faculties, cognitive powers).

(2) **Cognitive Idealism** = Necessarily, all the proper objects of rational human cognition are nothing but appearances or phenomena (i.e., mind-dependent, sensory, spatiotemporal, directly perceivable objects) and never things-in-themselves or noumena (i.e., mind-independent, non-sensible, non-spatiotemporal, real essences constituted by intrinsic non-relational properties) (*CPR* A369 and *Prolog* 4: 293-294, 375).

Now (1) + (2) = Kant's "Copernican revolution" in metaphysics:

Up to now it has been assumed that all our cognition must conform to the objects; but all attempts to find out something about them *a priori* through concepts that would extend our cognition have, on this presupposition, come to nothing. Hence let us once try whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognition, which would agree better with the requested possibility of an *a priori* cognition of them, which is to establish something about objects before they are given to us. This would be just like the first thoughts of Copernicus.... (*CPR* Bxvi),

which I will rationally reconstruct as *The Conformity Thesis*:

It is *not* the case that rational human minds passively conform to the objects they cognize, as in Classical Rationalism and Classical Empiricism. On the contrary, necessarily, all the proper objects of rational human cognition conform to—i.e., they have the *same* form or structure as, or are *isomorphic* to—the forms or structures that are non-empirically generated by our innately-specified spontaneous cognitive capacities. So necessarily the form or structure of the manifestly real natural world we cognize is *mind-dependent*.

In this way, all versions of Transcendental Idealism hold that the manifest natural world we directly perceive must *in some sense* conform to the non-empirical structures of our innate cognitive capacities. Many Kantians are also committed to *Strong* Transcendental Idealism, or STI for short, which says:

- (1) Things-in-themselves (a.k.a. “noumena,” or Really Real things, i.e., things as they could exist in a “lonely” way, altogether independently of rational human minds or anything else, by virtue of their intrinsic non-relational properties) really exist and cause our perceptions, although rational human cognizers only ever perceive mere appearances or subjective phenomena.
- (2) Rational human cognizers actually impose the non-empirical structures of their innate cognitive capacities onto the manifest natural world they cognize, i.e., necessarily, all the immanent forms or structures of the proper objects of human cognition are literally *type-identical* to the forms or structures that are non-empirically generated by our innately-specified spontaneous cognitive capacities., and
- (3) Necessarily, if all rational human cognizers went out of existence, then so would the manifest natural world they cognize.

But some Kantians—or at least *one* contemporary Kantian, i.e., R.H.—think(s) that STI is objectively false and are committed instead only to the objective truth of *Weak* or

Counterfactual Transcendental Idealism, or WCTI for short, which says:

- (i) Things-in-themselves are logically possible, but at the same time it is necessarily unknowable and unprovable whether things-in-themselves exist or not, hence for the purposes of metaphysics, epistemology, and ethics, they can be ignored (= *methodological eliminativism about things-in-themselves*).
- (ii) Necessarily, all the proper objects of rational human cognition have the *same* forms or structures as—i.e., they are *isomorphic* to—the forms or structures that are non-empirically generated by our innately-specified spontaneous cognitive capacities, but at the same time those worldly forms or structures are *not* literally type-identical to those mental forms or structures (= *the necessary-conformity-without-literal-identity thesis*).
- (iii) It is a necessary condition of the existence of the manifest natural world that if some rational human animals *were* to exist in that world, then they *would* directly cognize that world via essentially non-conceptual (i.e., intuitional) representational content, at least to some extent (= *the counterfactual cognizability thesis*)., and
- (iv) The manifest natural world has at some earlier times existed without rational human animals to cognize it directly, and could exist even if no rational human animals existed to cognize it directly, even though some rational human animals now actually exist in that

world—e.g., I (R.H.) now actually exist in the manifest natural world—who do in fact cognize it directly, at least to some extent (= *the existential thesis*).

Here is a slightly more precise formulation of WCTI's crucial thesis (iii), the counterfactual cognizability thesis:

$$\text{Syn Ap} \Box (\forall x) (\exists y) [\text{MNW}_x \rightarrow \{(\text{RHA}_y \ \& \ \text{MNW}_y)\} \Box \rightarrow \text{DC}_{yx}]$$

Definitions:

Syn Ap \Box = synthetically a priori necessarily

$P \Box \rightarrow Q$ = If P were the case, then Q would be the case.

MNW_x = x belongs to the manifest (apparent) natural world

MNW_y = y belongs to the manifest (apparent) natural world

RHA_y = y is a rational human animal

DC_{yx} = y directly cognizes (i.e., essentially non-conceptually cognizes) x, at least to some extent

Natural Language Translation:

Synthetically a priori necessarily, anything that belongs to the manifest natural world is such that if some rational human animals *were* to exist in that world, then they *would* directly cognize that thing, at least to some extent.

2 Crucial Implications:

- (1) The counterfactual cognizability thesis holds even if no rational human animals actually exist.
- (2) If anything is such that rational human animals are unable to cognize it directly, at least to some extent—e.g., things-in-themselves—then that thing does *not* belong to the manifest natural world.

Having stated WCTI as carefully as I can, there are at least two significant philosophical questions that can still be raised about it.

The **first** question is the *historical* philosophical question of whether Kant's own Transcendental Idealism should be understood as STI or instead as WCTI. My own view on this question, for what it is worth, is that Kant himself simply *oscillated between* STI on the one hand and WCTI on the other hand. Some Kant-texts support one reading, and other Kant-texts support the other reading. The Transcendental Aesthetic and the Analytic of Concepts in the first *Critique* mostly support the STI reading. But Kant's remarks about "empirical realism," the Refutation of Idealism, and the Analytic of

Principles more generally (especially the Postulates of Empirical Thought), mostly support the WCTI reading.

The **second** question—and for me, the massively more important of the two questions—is the *objective* philosophical question of whether either STI or WCTI is in fact true, or whether both are in fact false. My own view on this question, again for what it is worth, is that STI is plausibly arguably *false*, whereas WCTI is plausibly arguably *true*. And here are my basic reasons for holding that STI is false, and that WCTI is true.

On the one hand, I think that it is clearly false that if all actual human minds including mine went out of existence, then the manifest natural world would necessarily go out of existence too. I think that it is clearly false that, e.g., the actual existence of Pike’s Peak (a 14,000 foot mountain near Colorado Springs, CO, USA, with a cog railway that runs right to the summit⁸¹) strictly depends on the actual existence of human minds including mine. Clearly, I think, Pike’s Peak *can* exist even if we, including myself, do not, and in fact I think that Pike’s Peak *actually existed millions of years before any conscious minds of any kind existed*, including of course the conscious minds of all rational human animals, including mine. In this way a great many things, including mountains like Pike’s Peak, exist *objectively*—as do shoes, ships, sealing wax, cabbages, kings, seas that do not boil, and pigs without wings. They are, all of them, *neither* subjective (strictly dependent on individual minds) *nor* relative (strictly dependent on cultures or societies). They are all *moderately mind-independent*. So STI is clearly false.

But on the other hand, I do also think that it is clearly true that necessarily, if the manifest natural world were not directly cognizable by conscious rational animals like us, at least to some extent, then the manifest natural world would not exist. The manifest

natural world, insofar as it now actually exists in its moderately mind-independent way, could not be such that *it is inherently impossible to cognize it directly*; and the manifest natural world, insofar as it now actually exists in its moderately mind-independent way, could not be such that *its actual existence renders our conscious rational animal actual existence impossible*. How could that be the case, given the actual fact that the manifest natural world actually exists now in its moderately mind-independent state, given the other actual fact that we ourselves do actually exist now as rational human animals in the manifest natural world, and given the further actual fact that we do now directly perceive, and thus essentially non-conceptually cognize, some parts of the actual manifest natural world, e.g., our own living animal bodies in actual space and actual time?⁸² Therefore, necessarily, the actual existence of the manifest natural world does not render our conscious rational human animal actual existence in that world impossible. On the contrary, the actual existence of the manifest natural world *renders our conscious rational human animal actual existence in that world necessarily possible*. Here, and now more explicitly, I am arguing in the following way, by using one empirical premise and two modal principles, in addition to the familiar classical logical principle of **Existential Generalization**:

Empirical premise: I, a rational human animal, actually exist in the manifest natural world.

Modal principle 1: Actually P \rightarrow Possibly P

Modal principle 2: Possibly P \rightarrow Necessarily Possibly P (i.e., the characteristic modal axiom of S5).

(1) I, R.H., a rational human animal, actually exist in the actual manifest natural world. (**Empirical premise**.)

(2) Some rational human animals actually exist in the actual manifest natural world. (From (1), by **Existential Generalization**.)

(3) Therefore, given the actual existence of the manifest natural world, some rational human animals actually exist in that world. (From (2).)

(4) Whatever is actual is also possible. (Premise, from **Modal principle 1.**)

(5) Therefore, given the actual existence of the manifest natural world, it is possible that some rational human animals actually exist in that world. (From (3) and (4).)

(6) If anything is possible, then it is necessarily possible. (Premise, from **Modal principle 2.**)

(7) Therefore, given the actual existence of the manifest natural world, it is necessarily possible that some rational human animals actually exist in that world. (From (5) and (6).) **QED**

This argument is sound whether, on the one hand, the modalities are analytic, conceptual, or logical (a.k.a. weakly metaphysically a priori necessary or possible), or on the other hand, they are synthetic a priori, essentially non-conceptual, or non-logical (a.k.a. strongly metaphysically a priori necessary or possible). For these reasons, then, it is plausibly arguable that STI is false and that WCTI is true. So, to say that the natural number structure provided by elementary or Peano arithmetic, especially including the finitist sub-structure of PRA or simple arithmetic, is weakly or counterfactually transcendentally ideal, is just to say that synthetic a priori necessarily, to the extent that this mathematical structure *is* immanent in the manifest natural world, then *were* rational human cognizers to exist in that world, they *would* directly (i.e., essentially non-conceptually) cognize that structure, at least to some extent.

In other words, then, I am proposing a specifically Kantian and Weak or Counterfactual Transcendental Idealist version of what Parsons calls “non-eliminative structuralism.”⁸³ Even more specifically, however, I think that the natural numbers are essentially the same, for *some* rational purposes, as roles in the abstract mathematical structure provided by elementary or Peano arithmetic, especially including the finitist sub-structure of PRA or simple arithmetic, when this is interpreted as certain kind of non-empirical or a priori humanly consciously-accessible *cognitive* structure, and also that the

numbers are essentially the same, for *other* rational purposes, as the role players of the natural number roles in the manifest spacetime world, i.e., the natural numbers are just the set of manifest directly perceivable material objects in actual spacetime, insofar as they fall under, and are immanently structured by the elementary or Peano arithmetic, and also especially by the primitive recursive or finitist arithmetic,⁸⁴ of the natural numbers. I will come back to this thesis again shortly.

Even if we have decided to adopt a dual Roles and Role-Players interpretation of structuralism, there are also several further basic distinctions between different kinds of Mathematical Structuralism that need to be made more explicit. The two main divisions are these:

- (a) *Reductive* Structuralism vs. (b) *Non-Reductive* Structuralism,
- (c) *In Rebus* Structuralism vs. (d) *Ante Rem* Structuralism.

Reductive Structuralism, as I am construing it, says that the objects of the mathematical system are either strictly identical with various elements and relations of the system or *logically* strongly supervenient on the whole system and thus nothing over and above the whole system. By contrast, Non-Reductive Structuralism says that the objects of the system are *strongly supervenient on the whole system but still something over and above the whole system*, hence neither strictly identical to various elements and relations of the system nor logically strongly supervenient on the whole system. In other words, the Reductive vs. Non-Reductive distinction applies to the *objects* of mathematical structural systems. Correspondingly, the Role-Players interpretation, on its own, entails Non-Reductive Structuralism, and the Roles interpretation, on its own, is consistent with both Non-Reductive Structuralism and Reductive Structuralism.

In Rebus Structuralism, as I am construing it, says that both the existence and specific character of the mathematical system are necessarily dependent on and determined by material things in the natural world, and that the systemic structures are not only literally proper parts of those material things but also ontologically non-detachable and epistemically non-abstractible from them. By contrast, *Ante Rem* Structuralism says that the existence and specific character of the system are neither necessarily dependent on nor determined by the existence of material things in the natural world, and that the systematic structures are both ontologically detachable and also epistemically abstractible from those material things, even if they are also literally proper parts of them. In other words, the *In Rebus* vs. *Ante Rem* distinction applies not to the *objects* of mathematical structural systems, but instead to the *structural systems* themselves. For example, *In Rebus* Structuralism would be defended by a mathematical Structuralist who is both a Reductive or Scientific Naturalist and also an Empiricist/Nominalist, like Hartry Field⁸⁵ or Penelope Maddy,⁸⁶ whereas *Ante Rem* Structuralism would be defended by a mathematical Structuralist who is both a Platonist and also a Rationalist/Realist, like Shapiro.

Significantly, and perhaps because of the example set by Field, Shapiro identifies Reductive Structuralism with *In Rebus* Structuralism, and Parsons identifies both Reductive Structuralism and *In Rebus* Structuralism alike with what he calls “eliminative structuralism.”⁸⁷ But strictly speaking, at least in principle, one could consistently defend both *In Rebus* Structuralism and also Non-Reductive (a.k.a. “non-eliminative”) Structuralism. Consider, e.g., a specifically *Wittgensteinian* Mathematical Structuralism,⁸⁸ in which numbers are identified with the entities that play the roles

specified by living mathematical linguistic practices but not identified with those practice-specified roles, and in which those living mathematical linguistic practices themselves, conceived as rule-systems, are the enframing mathematical structural systems in which mathematical objects are embedded as the role-players of the roles in the structures. This Wittgensteinian Structuralism would be both *in rebus* and non-reductive. I myself am not going to defend Wittgensteinian Structuralism. But the very possibility of it does have a relevant bearing on the Kantian Intuitionist theory of mathematical a priori knowledge that I will defend in section **VII**, because I do think that mathematical *knowledge* is partially determined by living mathematical linguistic practices, even if mathematical *truth* is not so determined.

The brand of Structuralism I am proposing, *Kantian* Structuralism, is a non-reductive and *ante rem* version of Mathematical Structuralism, that also presupposes Weak or Counterfactual Transcendental Idealism, i.e., WCTI. More specifically, it is based on

- (i) the abstract formal structures of space and time as we essentially non-conceptually, or directly referentially, cognize them in sense perception, together with
- (ii) formal concepts, including the ramified abstract formal structures of classical logic and conservative extensions of it, as we understand them in thinking,

insofar as rational human animals are capable of essentially non-conceptually, or directly referentially, cognizing, i.e., rationally intuiting (in the specific sense of rational “intuition-of”), those perceptually-embedded spatiotemporal structures, and also capable of understanding those conceptually-embedded logical structures. Otherwise put, Kantian Structuralism takes the necessity and apriority of mathematical truths at face value and then metaphysically explains those semantic features in terms of weakly or counterfactually transcendently ideal spatiotemporal structures of human sense

perception, the innately-specified capacity for essentially non-conceptual or directly referential cognition that picks out those structures, the innately-specified capacity for conceptual understanding, the conceptual structures picked out by the proper operations of that capacity for conceptual understanding or thinking, and their corresponding weakly or counterfactually transcendently ideal logical structures.

By sharp contrast to Kantian Structuralism, however, Field's and Maddy's Structuralism is both reductive and *in rebus* because it says that numbers are nothing over and above their being positions in modal or physical structures, and also that mathematical truth is reducible to fundamental physical facts about the physical world. And by another sharp contrast to Kantian Structuralism, Shapiro's Structuralism is both reductive and *ante rem* because it says that numbers are nothing over and above their being positions in non-modal structures, and also that mathematical truth is reducible to non-physical facts about non-spatiotemporal classically platonic structures. The comparisons and contrasts between Kantian Structuralism and *Parsons's* version of Mathematical Structuralism are more domestic and subtle, however, and I will work them out in detail in section **VIII**.

But more precisely, and with respect to the elementary arithmetic of the natural numbers, i.e., Peano arithmetic, in particular, and especially including PRA, Kantian Structuralism says the following:

- (1) The natural numbers are essentially positions or roles in the mathematical natural number structure provided by elementary or Peano arithmetic in its full generality and infinitude, even beyond its finitist sub-structure of PRA or simple arithmetic, and also including the conservative extensions of Peano arithmetic such as Cantorian or transfinite arithmetic.⁸⁹,
- (2) The mathematical natural number structure provided by Peano arithmetic (and PRA, and Cantorian arithmetic) is abstract only in the sense that it is weakly or counterfactually transcendently ideal, which is to say that this structure is identical to the formal

structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human comprehension of Peano arithmetic (and PRA, and Cantorian arithmetic), that we cognize through conceptual understanding or thinking, and
 (3) In our actual world, the unique, intended model of the abstract natural number structure provided by Peano arithmetic (and PRA, and Cantorian arithmetic) is just the *immanent structure* that is fully embedded in the set of manifest, directly or essentially non-conceptually perceivable spatiotemporal material objects in nature, insofar as they are the role players of the Peano-arithmetic-(and-PRA-and-Cantorian-arithmetic)-specified natural number roles in the abstract formal structure of time as we directly or essentially non-conceptually cognize it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human comprehension of Peano arithmetic (and PRA, and Cantorian arithmetic), that we cognize through conceptual understanding or thinking.

In this way, Kantian Structuralism adequately explains why something that is *abstract, ideal, and necessary* like Peano arithmetic in its full generality and infinitude, even beyond its finitist sub-structure of PRA, and also including the conservative extensions of Peano arithmetic such as Cantorian arithmetic, can really and truly apply to the hurly-burly *concrete, manifestly real, natural, and contingent* world of rational human animals and other natural things and processes, and thereby really and truly apply to all the manifest, essentially non-conceptually or directly referentially perceivable material spatiotemporal objects in our actual manifest natural world. For according to Kantian Structuralism, *since* the formal structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception is an immanent structure of all manifest directly perceivable material spatiotemporal objects in nature, and *since* this essentially non-conceptually or directly referentially cognizable immanent structure, when it is taken together with the weakly or counterfactually transcendently ideal abstract formal structure of any classical logical system rich enough to capture Peano arithmetic (and PRA, and Cantorian arithmetic), that we cognize through conceptual understanding or thinking, just *is* the unique, intended model of Peano arithmetic (and

PRA, and Cantorian arithmetic), *then* it follows with synthetic a priori necessity that Peano arithmetic and PRA, and Cantorian arithmetic) applies to all manifest essentially non-conceptually or directly referentially perceivable material spatiotemporal objects in nature. The abstractness, ideality, and necessity of Peano arithmetic (and PRA, and Cantorian arithmetic) is captured by the *number roles* in the composite structure of time and Peano arithmetic (and PRA, and Cantorian arithmetic), insofar as it can be conceptualized and understood by rational human animals. Correspondingly, the concreteness, reality, and contingency of the things and people in the manifest natural world to which arithmetic applies is captured by the *number role players* in the composite structure of humanly cognizable time and humanly cognizable Peano arithmetic (and PRA, and Cantorian arithmetic). Therefore rationally and humanly directly or essentially non-conceptually cognizable time-structure is the weakly or counterfactually transcendently ideal *metaphysical glue* that ineluctably binds Peano arithmetic (and PRA, and Cantorian arithmetic) to our manifest natural world; or to re-use Parsons's apt phrase, rationally and humanly directly or essentially non-conceptually cognizable time-structure is precisely what

get[s] us across the divide between the fuzzy *Lebenswelt* with its everyday objects and the sharp, precise realm of the mathematical, in terms of which mathematical conceptions of the physical world are developed.

Otherwise put, Kantian Structuralism clearly and distinctly solves the classical *application problem* for the philosophy of arithmetic.⁹⁰

VI.3

So, finally, I am now in a position to solve BD by using Kantian Structuralism. I will begin by supposing that the two preliminary assumptions of BD are true, and that

they express fully reliable basic authoritative philosophical intuitions. That obviously satisfies steps (1) and (2) of BD.

It also obviously raises an important issue about the epistemic status of fully reliable basic authoritative philosophical intuitions. What about the skeptical claims of those philosophers who in fact *reject* either of the two preliminary assumptions of BD? Since if I am correct, all fully reliable basic authoritative rational intuitions are completely convincing, intrinsically compelling, or self-evident, then EITHER

(1) Some purportedly fully reliable basic authoritative philosophical intuitions are *not* completely convincing, intrinsically compelling, or self-evident, and I am wrong about the nature of authoritative rational intuitions, OR

(2) These philosophers have so far failed to understand the meanings of these preliminary assumptions, OR

(3) These philosophers have so far failed to be sufficiently rationally reflective about the implications of the meanings of these preliminary assumptions, and have thereby also so far failed successfully to undertake the intentional performance of rendering their cognition of these assumptions “clear, distinct, and indubitable” hence their rational intuitions to the effect that these assumptions are false are merely *prima facie* intuitions and fairly unreliable.

My claim is that, in all likelihood, (3) is true. The conditions under which possible cases of (2), or a failure to understand the relevant meanings, could occur, include: agnosias or other cognitive disabilities, being drugged or drunk, cognitive immaturity, inattention, insanity, linguistic performance errors of an adventitious nature (i.e., brief slips of the eye or ear, or of the innate grammatical abilities for parsing verbal syntax or accessing one’s lexicon/repertoire of concepts, etc.), seizures, sleepiness, and so-on. But obviously *those* conditions are quite unlikely to hold for *these* philosophers in *this* particular connection: indeed, we can even reasonably assume that they *fully* understand the meanings of these preliminary assumptions. By sharp contrast, however, the conditions under which possible cases of (3), or insufficient rational reflectiveness about the relevant implications

of the relevant meanings, could occur, are radically more sophisticated and subtle, and include all the characteristic stages of the dialectic of philosophical and scientific inquiry, short of the final, rationally conclusive stage. Such preliminary stages can involve: commission of any of the classical informal or formal logical fallacies, confusion, dogmatism, equivocation, ignorance of relevant facts, intellectual arrogance, intellectual laziness, unacknowledged false assumptions or presuppositions, uncharitableness of interpretation, either unclarity or indistinctness of cognition more more generally, and perhaps the most important and insidious error-causing condition of all, “being in the grip of a bad picture (*schlechtes Bild*)” in the later Wittgenstein’s pregnant sense of that phrase:

112. A simile that has been absorbed into the forms of our language produces a false appearance, and this disquiets us. “But *this* isn’t how it is!”—we say. “Yet *this* is how it has to be!”

113. “But *this* is how it is—“ I say to myself ove and over and over again. I feel as though, if only I could fix my gaze absolutely sharply on this fact, get it in focus, I must geasp the essence of the matter.

114.... One thinks that one is tracing the outline of the thing’s nature over and over again, and one is merely tracing round the frame throuhg which we look at it.

115. A *picture* held us captive. And we could not get outside it, for it lay in our language and language seemed to repeat it to us inexorably.⁹¹

The very idea of a *bad* philosophical picture entails a fundamental meta-philosophical distinction between

(i) *confusion-inducing* or bad philosophical pictures, and

(ii) *clarity-inducing* or good philosophical pictures,

and points up their correspondingly seminal roles in philosophical reasoning. For the present purposes, it suffices to say that obviously I do think that the Tarskian and minimal Empiricist reasons I cited in section III for accepting the two preliminary

assumptions of BD *are* rationally conclusive, and that, in view of those reasons, both of these assumptions inherently express clarity-inducing or good philosophical pictures.

Now I will further suppose that Kantian Structuralism is true, and that it adequately explains the apriority and necessity of mathematical truth. This satisfies step (3) of BD.

This in turn allows me to re-interpret the realistic ontology of abstract objects described in step (4) of BD as the weakly or counterfactually transcendently ideal abstract formal structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception, when taken together with the weakly or counterfactually transcendently ideal abstract formal structure of any classical logical system rich enough to capture Peano arithmetic (and PRA, and Cantorian arithmetic), insofar as it can be comprehended by rational human animals via conceptual understanding or thinking. This dual abstract structure is itself of course causally isolated and inert, which satisfies step (6) of BD.

But this dual abstract structure is also *intrinsically temporal*, and in our actual world it uniquely determines the unique intended model of the natural number structure, which is the essentially non-conceptually or directly referentially cognizable manifest natural world of spatiotemporal objects in nature *just insofar as they are the role players of the Peano-arithmetic-(and-PRA-and-Cantorian-arithmetic)-specified natural number roles in the abstract structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception*. So the dual abstract structure consisting of the consciously-representable abstract formal structure of time together with Peano arithmetic (and PRA, and Cantorian arithmetic) is *causally relevant*, even though it is *not*

causally efficacious. Therefore in our actual world the unique intended model (i.e., the one and only real truth-maker) of the natural number structure is identical to the manifest natural world of causally efficacious essentially non-conceptually or directly referentially cognizable material spatiotemporal objects just insofar as they exist in certain configurations, which obviously solves the application problem for Peano arithmetic (and PRA, and Cantorian arithmetic); and mathematical knowledge is thereby possible on the assumption that a “reasonable epistemology” of cognizing true (mathematical) statements is modelled on a theory of sense perception which includes

causally efficacious, contact-involving or efficient, directly referential, non-inferential, and spatiotemporal relations between human linguistic knowers and the known objects themselves,

and thereby satisfies premise (5) of BD.

Hence if Kantian Structuralism is true, then all of (1) to (6) are true, under plausible interpretations of them, but the unacceptably skeptical conclusion of BD—step (7)—is clearly avoided, and mathematical knowledge is really possible after all. I will spell all this out more carefully in section **VIII**, when I explicitly compare and contrast Kantian Structuralism and Kantian Intuitionism with Parsons’s account.

It should be particularly emphasized here that I am construing the fully reliable basic authoritative philosophical intuition lying behind Benacerraf’s premise (2)—i.e., his assumption of a “reasonable epistemology”—to be essentially equivalent to the Kantian thesis I cited in section **IV** when I was unpacking the very idea of apriority, to the effect *that necessarily, human cognition begins in causally-triggered sensory experience*:

Although all our cognition commences **with** experience, yet it does not on that account all arise **from** experience... It is therefore a question requiring closer investigation, and one not to be dismissed at first glance, whether there is any such cognition independent of all experience and even of all impressions of the senses. One calls such **cognitions a priori**, and distinguishes them from **empirical** ones, which have their sources *a posteriori*, namely in experience. (*CPR* B1-2)

At the same time, as Kant explicitly points out here, it does *not* follow from the necessary fact that all human cognition begins in causally-triggered sense experience—i.e., it does *not* follow from Benacerraf’s “reasonable epistemology”—that all human cognition is *derived* from sensory experience, which I am construing as meaning that cognition is *strongly supervenient on* actual or possible empirical facts (i.e., sensory experiential and/or contingent facts), whether merely naturally or nomologically supervenient (i.e., non-reductively strongly supervenient) or logically supervenient (i.e., reductively strongly supervenient). So Kant’s point is that Minimal Empiricism must also be combined with the denial of classical Humean or Maximal Empiricism. Hence explicitly adopting a theory of sense perception that necessarily includes a causal component, and thereby anchors all human cognition in direct causally-triggered sense perception of the natural world, does *not* explanatorily or ontologically reduce all human cognition to causal factors.

It is also fully relevant here to note that Benacerraf himself does not postulate the causal dimension of his “reasonable epistemology” within any kind of reductive theoretical framework, although many or perhaps even most readers of “Mathematical Truth” have taken it that way. But in fact and on the contrary, Benacerraf is perfectly in line *with Kant* on this point. To postulate a necessary causal dimension in human knowledge is *not* thereby to assert *The Causal Theory of Knowledge*.

Considered for a moment apart from its ability to help us achieve a positive solution to BD, and also apart from its ability to solve the classical application problem for arithmetic, what other reasons could we have for defending Kantian Structuralism? I think that there are at least five other very good reasons.

First, Kantian Structuralism offers a clean-and-simple solution to another important problem pointed up by Benacerraf, which is that many different models satisfy the abstract structure of any logical system rich enough to express elementary or Peano arithmetic, hence the second-order logic of Peano arithmetic underdetermines the identity conditions of the natural numbers.⁹² Otherwise put, Benacerraf's *other* problem is that there seems to be in principle no way of determining or identifying just *which* of the many distinct models that satisfy the logic of Peano arithmetic is *really* the natural numbers. This is what Parsons calls the "multiple reduction" problem,⁹³ and what others, following Frege, have called the "Caesar" problem, or the "Identification" problem. According to Kantian Structuralism, however, the abstract formal structure of the asymmetric successively synthesized series of moments (or simple events) in time, insofar as we directly or essentially non-conceptually cognize it in sense perception, is the unique, intended model of Peano arithmetic (and PRA, and Cantorian arithmetic). On this picture, a "standard" model of Peano arithmetic (and PRA, and Cantorian arithmetic), is any possible world in which either time as we directly or essentially non-conceptually cognize it in sense perception exists, or else something isomorphic to the time-structure exists.⁹⁴

But then the proper part of the model that satisfies a particular natural number-role in the abstract system of Peano arithmetic (and PRA, and Cantorian arithmetic), *just is* anything in our actual manifest natural world that occurs in time as we essentially non-conceptually or directly referentially cognize it in sense perception, *insofar as* it intrinsically instantiates the thermodynamically asymmetric successive serial structure of time as we essentially non-conceptually or directly referentially cognize it in sense

perception, and thereby plays at least some of the Peano-arithmetic-(and-PRA-and-Cantorian-arithmetic)-specified natural number roles. The natural numbers themselves exist in non-actual possible worlds as *the Peano-arithmetic-(and-PRA-and-Cantorian-arithmetic)-specified and temporally-specified natural number roles*, and in our actual manifest natural world as the unique intended model of Peano arithmetic (and PRA, and Cantorian arithmetic), namely *the totality of manifest natural Peano-arithmetic-(and-PRA-and-Cantorian-arithmetic)-specified and temporally-specified natural number role-players*. Now the actual inhabitants of time as we essentially non-conceptually or directly referentially cognize it in sense perception are essentially non-conceptually or directly referentially perceivable, manifest material spatiotemporal objects that contain spatiotemporal immanent structural properties. So in our actual world, the unique intended model of the natural number structure is identical to the totality of essentially non-conceptually or directly referentially perceivable, manifest material spatiotemporal objects just insofar as they are the role players of the Peano-arithmetic-(and-PRA-and-Cantorian-arithmetic) specified natural number roles in the abstract formal structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception.

(I hereby apologize, Dear Reader, for the stylistic gallumphing-ness of all these “insofar as”-es, and other repeated phrases. But there does not seem to be any other way of stating this doctrine precisely.)

Second, if Kantian Structuralism can offer a unified solution to BD *and* Benacerraf’s other problem, then that seems to be another strong point in its favor. For as Benacerraf himself has argued, BD and Benacerraf’s other problem are essentially

interdependent. So an adequate solution to BD must *also* solve Benacerraf's other problem.⁹⁵

Third, Kantian Structuralism crisply explains why classical Logicism failed, and why it seems so clear that the arithmetic of the natural numbers is not reducible to second-order logic plus the Peano axioms alone. According to Kantian Structuralism, the elementary or Peano arithmetic of the natural numbers can be determined only by the ramified logical formal structure of Peano arithmetic (and PRA, and Cantorian arithmetic), insofar as it can be conceptually understood or thought by rational human animals, together with any formal structure that is isomorphic to the structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception. To be sure, contemporary *Neo-Logicists* have shown that adding Hume's Principle (which says that the number of Fs = the number of Gs if and only if there are as many Fs as Gs) to second-order logic plus the Peano axioms logically entails the elementary arithmetic of the natural numbers.⁹⁶ But it seems to be intelligibly arguable that Hume's Principle is *not* an analytic, conceptual, or logical truth, precisely because it *presupposes* the formal structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception, and also whatever is isomorphic to the formal structure of time as we essentially non-conceptually or directly referentially cognize it in sense perception. If so, then ironically enough the actual success of *Neo-Logicism* is metaphysically best explained by *Kantian Structuralism*, and not by postulating the analyticity, conceptual truth, or logical truth of *Hume's Principle*, as the *Neo-Logicists* have done. But then in that case, *Neo-Logicism* is most adequately and correctly formulated as the thesis that Peano arithmetic is *transcendentally explained in terms of*

second-order logic, Hume's Principle (which is synthetically, essentially non-conceptually, or non-logically a priori necessary), and Kantian Structuralism, and not adequately or correctly formulated as the thesis that Peano Arithmetic is analytically, conceptually, or logically derivable from and explanatorily reducible to second-order logic and Hume's Principle.

Fourth, if that is true, then Kantian Structuralism would also crisply explain why, contrary to both classical Logicism and Neo-Logicism, mathematical truths clearly seem *not* to be analytically, conceptually, or logically necessary truths, but instead to be *synthetically, essentially non-conceptually, or non-logically a priori necessary truths.*

Now Gödel's incompleteness theorems—according to which

(i) There must be logically unprovable true sentences in any formal system rich enough to contain the axioms of Peano arithmetic., and

(ii) All such systems are consistent (i.e., non-contradictory) if and only if they are incomplete (i.e., not all the truths of the system are theorems of the system) and have their ground of truth outside the system itself.

—already strongly suggest to the Kantian Structuralist that the nature of mathematical truth outruns logical provability precisely because mathematical truths are synthetically, essentially non-conceptually, or non-logically necessary a priori, not analytically, conceptually, or logically necessary.

But another and even more decisive reason for thinking that mathematical truths are not true in every logically possible world, hence are not analytic, is the clear and distinct conceivability and hence logical possibility, of either

(1) possible worlds with *nothing whatsoever* in them—which would of course entail the non-existence of numbers in those worlds, and thus the non-truth of many sentences of Peano arithmetic (and PRA, and Cantorian arithmetic) in those worlds,⁹⁷ or

(2) possible worlds with *non-standard arithmetics* of the natural numbers in them, e.g., a world in which the standard primitive recursive function of addition or “plus” is replaced

by Kripke's non-standard primitive recursive function of "quaddition" or "quus"—which would of course directly entail the non-truth of many sentences of Peano arithmetic (and PRA, and Cantorian arithmetic) in those worlds.⁹⁸

If mathematical truths are necessarily true but not analytically necessary, then according to Kantian Structuralism the explanation for this striking fact is that the truth and meaningfulness of mathematical propositions presuppose the abstract formal structure of time insofar as we consciously represent it in sense perception, which is *not* itself a *purely logical or conceptual fact* that attaches to every logically possible world. On the contrary, the presence either of the abstract formal structure of time insofar as we consciously represent it in direct sense perception, or of some other abstract structure isomorphic to the abstract formal structure of time insofar as we consciously represent it in direct sense perception, in a given possible world, is a "*strongly metaphysically necessary a priori*" fact that attaches to only a *restricted class* of logically possible worlds, i.e., to all and only the logically possible worlds in which the very same spacetime structure, causal-dynamic structure, and mathematical structure as that of our actual world, also exist. This is also the strongly metaphysically necessary a priori class of all and only the possible worlds in which rational human animal *consciousness* and *intentionality* are really possible, and thus both WCTI and Liberal Naturalism are vindicated by the very idea of the synthetic a priori, when it is embedded within the theoretical framework of Kantian Structuralism.⁹⁹

On this view, possible worlds without denumerable objects in them are all time-structureless worlds, and all time-structureless worlds are possible worlds without denumerable objects in them. So if Kantian Structuralism is true, then the metaphysical explanation for *Modal Dualism*—which is the classical Kantian thesis that there are two essentially different kinds of necessary truth, namely

(1) *analytic a priori necessary truth*, i.e., truth about the kind of necessity which flows from the nature of logic and concepts, which thereby includes logical truth and conceptual truth, and

(2) *synthetic a priori necessary truth*, i.e., truth about the kind of necessity which flows from the nature of the immanent structures of things in the world, which thereby includes mathematical truth¹⁰⁰

—comes along for free.

If Kantian Structuralism is true, then it fully explains how the elementary arithmetic of the natural numbers, i.e., Peano arithmetic, is true. What about the rest of mathematics? The general answer provided by Kantian Structuralism is that all of the rest of mathematics, *including* its most abstruse and ontologically rich parts—e.g., iterative set theory and Cantorian arithmetic—can be built up as conservative extensions from Peano arithmetic (and PRA), and the abstract formal structure of time insofar as we consciously represent it in sense perception, *together with* all the formal concepts, classical logical constructions, and specific patterns of logical inference required by those other parts of mathematics, that are encoded in standard mathematical linguistic practices, insofar as mathematical language can be understood by rational human animals. I will have more to say about this crucial point in section **VIII**. It suffices to say, for now, that rational intuitions of mathematical truths of the conservatively extended mathematical theories built up in this way will then be only fairly reliable or *constructed* rational intuitions, not fully reliable or authoritative rational intuitions, whether basic or non-basic.

Fifth, this line of thinking indicates how Kantian Structuralism might also be able to offer a new solution to the classical *Problem of the Continuum*. Very simply put, The Problem of the Continuum is this: What is the correct characterization of the quantitative structure of the spacetime world we consciously experience, i.e., the intuitively-given

continuum? According to the famous Continuum Hypothesis—or CH for short—proposed by Cantor, the quantitative structure of the continuum has either the infinite denumerable cardinality of the natural numbers (= aleph null, i.e., \aleph_0) or the infinite non-denumerable cardinality of the real numbers (= 2 to the power of aleph null, i.e., 2^{\aleph_0}) and there is no number applicable to the continuum with a cardinality that falls strictly between that of the naturals and that of the reals. What Kantian Structuralism proposes about the continuum is that:

- (i) The continuum definitely has the infinite denumerable cardinality of the natural numbers.,
- (ii) The continuum definitely has the infinite non-denumerable cardinality of the real numbers., and
- (iii) The continuum definitely has no other cardinality strictly between those two.

Kantian Structuralism is *able* to say this *precisely because*, according to Kantian Structuralism, the real number structure is logico-mathematically a priori *constructible* from the set of all consciously experienceable points and stretches in spacetime, together with the set of *all possible degrees of any consciously experienceable sensory quality, for each consciously experienceable point or stretch in spacetime*.

What I mean is that it is an a priori fact about the nature of human experience that any set of points or stretches of experienceable spacetime can instantiate any degree of some or another sense-experienceable quality. Building on that a priori fact, the Kantian Structuralist thesis is that for each distinct point or stretch in sense-experienceable spacetime, of which there is a denumerably infinite number, we can also find a denumerably infinite number of different degrees of some or another sense-experienceable quality. Then we can think of the latter cardinal number as an *exponent* of

the former cardinal number in an operation that yields the former's *power set*,¹⁰¹ i.e., the set of all its subsets. The cardinality of the result of that power set operation is the same as the first transfinite number, \aleph_1 , which in turn has the same cardinality as the real numbers, i.e., 2^{\aleph_0} . Putting the same point in specifically Kantian terminology, Kantian Structuralism says that the basic structure of the continuum is the non-empirical *extensive quantity* structure as described in the Axioms of Intuition insofar as it is also exponentiated, according to the power set operation, by the non-empirical *intensive quantity* structure as described in the Anticipations of Perception. In this sense, the basic structure of the continuum is the Kantian *synthesis* of the extensive quantity structure and the intensive quantity structure.

Therefore Kantian Structuralism says that The CH is synthetic a priori true—i.e., that The CH is determinately true in every humanly experienceable world, and a truth-value gap in every other logically possible world that lacks the spatiotemporal structure of human experience. The fundamental mathematical issue raised by The CH is whether there is any number structure with a cardinality strictly between the denumerable infinite cardinality of the natural numbers and the non-denumerable infinite cardinality of the real numbers. Kantian Structuralism says that synthetically a priori necessarily there is *no such intervening number structure*, precisely because *human experience is just so structured as to rule this out*, and precisely because—given WCTI—*necessarily the world is correspondingly just so structured that if human cognizers were to exist, then they would cognize that world directly both a priori and a posteriori, at least to some extent*, including coming to know The CH as a synthetic a priori truth.

But this is not Pre-Established Harmony. Leopold Kronecker famously or notoriously said that God made the integers and everything else was done by humans.¹⁰² Kantian Structuralism is even more radically anthropocentric than this, and explicitly excludes anything *noumenal or theological* from the metaphysical foundations of mathematics. According to Kantian Structuralism, *the formal constitution of rational human animal nature made the natural numbers, and logico-conceptual construction by rational human animals*, together with their innate capacity for logical and linguistic cognition, *did all the rest*. So in that sense, mathematics is *all about us*. But this Kantian Structuralist account does *not* entail any sort of Anti-Realism. On the contrary, assuming the truth of WCTI, then necessarily the manifestly real natural world inherently possesses the self-same mathematical structures that rational human animals are inherently capable of consciously detecting in that world. As a matter of logical necessity, the manifestly real natural world did not *have* to be that way. It just *is* that way. It is a *brute essential fact* about nature. But on the working assumption that the manifestly real natural world, as it just so happens, really *is* that way, and also that it really is *necessarily* that way, precisely because it flows from its essence, then the fundamental formal coordination between rational human animal minds and the manifestly real natural world holds with synthetic a priori necessity. So Kantian Structuralism is just about as objectively realistic as it is possible to be. Or again: *objectivity has a human face, with robustly normative rationality written all over it*.

Suppose, now, as a working hypothesis, that Kantian Structuralism is true. We still need to explain more precisely *how* mathematical a priori knowledge is possible. And that is where Kantian Intuitionism comes in.

VII. Kantian Intuitionism

The epistemologically pregnant sense of self-evidence (*Evidenz*) ... gives to an intention, e.g., the intention of judgment, the absolute fullness of content, the fulness of the object itself. The object is not merely meant, but in the strictest sense *given*, and given as it is meant, and made one with our meaning-reference.... It is said of every percept that it grasps its object directly, or grasps this object *itself*. But this direct grasping has a different sense and character according as we are concerned with a percept in the narrower or wider sense, or according as the directly grasped object is *sensible* or *categorical*. Or otherwise put, according as it is a *real* or *ideal* object.

--E. Husserl¹⁰³

Self-evidence (*die Einleuchten*), of which Russell has said so much, can only be discarded in logic by language itself preventing every logical mistake. That logic *a priori* consists in the fact that we *cannot* think illogically.

--L. Wittgenstein¹⁰⁴

VII.1

As I formulated it above, Kantian Intuitionism holds that *a priori* knowledge in mathematics, by means of mathematical intuition, is the joint product of two rational human animal abilities operating in tandem:

(1) a rational human animal's *veridical sensible-form-in-Kantian-a-priori-intuition-or-mental-model-or-mental-diagram-or-mental-picture-or-structural-imagery-or-schema-constructing-and-manipulating* abilities, which are innately specified in her mind and also inherently present, as necessary ingredients, in all rational human sense perception, and which entail her self-conscious and self-reflective cognition of phenomenologically self-evident formal structures of rational human sense perception, together with

(2) a rational human animal's *logic-and-language-constructing-and-manipulating* abilities, which are innately specified in her mind and also inherently present, as necessary ingredients, in all rational human empirical conceptualizing and perceptual judgment, and which entail her self-conscious and self-reflective cognition of phenomenologically self-evident formal conceptual contents and specific patterns of logical inference in classical or non-classical logics.

As I also formulated it above, the basic idea behind Kantian Intuitionism is that authoritative or self-evident mathematical intuition can be construed in such a way as to preserve both the abstractness and causal inertness of **the truth-makers** of mathematical statements and also the causal relevance of **the intentional targets** of mathematical intuition, as well as the causal efficacy of **the evidential verifiers** of mathematical

beliefs. There I emphasized the point that truth-makers, intentional targets, and verifiers can be different sorts of things, even if they are essentially connected. What I gave as an example there is what I explicitly want to argue now, namely

- (i) The truth-maker is a *mathematical immanent structure in the manifestly real natural world*,
- (ii) The intentional target is a *veridical sensible form in Kantian a priori intuition, mental model, mental diagram, mental picture, structural image, or schema*, of at least part of that very structure, and
- (iii) The evidential verifier is a *manifestly real natural worldly fact, picked out by direct sense perception*, which *implements* the immanent structure and thereby *satisfies* the abstract mathematical structure, and also strictly *conforms* to the veridical sensible form in Kantian pure intuition, mental model, mental diagram, mental picture, structural image, or schema.

VII.2

The precise nature of the connection between (i), the truth-maker, and (ii), the veridical sensible form in Kantian a priori intuition, mental model, mental diagram, mental picture, structural image, or schema—the Hilbert-style objects of finitary reasoning—is particularly crucial to my overall account. This, in turn, is due to my conception of *the nature of knowledge*, which I take to flow naturally from the widely-known and almost universally-accepted “Gettier counterexamples” to the classical analysis of knowledge, according to which knowledge (K) is the same as justified true belief (JTB), i.e., $K = JTB$.¹⁰⁵

Here is what I mean by that. The simplest kind of Gettier counterexample goes like this. I look at my watch, and it says that it is 11:00 am. And my watch has been working fine for months. So I have a sufficient reason for asserting that it is 11:00 am. And, as it happens, it really is 11:00 am. But, unbeknownst to me, my watch has been broken since 11:00 pm, when, by a malfunction of the mechanism, it started reading

11:00 am and also stopped working right at that very moment. So even though I have a sufficient reason for asserting that it is 11:00 am, and it is true that it is 11:00 am, and I believe that it is 11:00 am, I do not know that it is 11:00 am. So knowledge is not justified true belief, i.e., $K \neq \text{JTB}$.

How should we understand this result? My own take on the Gettier counterexamples is that they trade on a special internal feature of the concept and fact of epistemic justification: Epistemic justification is structurally *two-dimensional*, in the sense that by its very nature it is either (1) *Low-Bar*, or (2) *High-Bar*.

(1) *Re Low-Bar*. The “Low-Bar” dimension of justification allows for justification to be detached from truth, and means: *Whatever provides a sufficient reason for the believer to assert her belief-claim, even if that belief turns out false*, in which case that belief obviously is not knowledge. But most importantly for the Gettier counterexamples, low-bar justification is also consistent with cases (like my broken watch case) in which the believer’s claim is actually true, yet that actual truth is not *inherently or intrinsically* connected to the believer’s reason for asserting her belief-claim. Otherwise put, the truth of the claim in these cases is only *accidentally or extrinsically* connected to the believer’s sufficient reason for asserting her belief-claim.

Now this clearly and distinctly points up the fact that knowledge requires an inherent or intrinsic connection between the truth of a believer’s belief-claim and the believer’s sufficient reason for asserting her belief-claim, because in the cases in which there is only an accidental or extrinsic connection, the believer’s belief-claim could just as easily have been false with no change whatsoever in the believer’s sufficient reason for

asserting her belief-claim. So knowledge is *not* the same as Low-Bar justified true belief, i.e., $K \neq L\text{-BJTB}$.

(2) ***Re High-Bar.*** By sharp contrast, the “High-Bar” dimension of justification requires that justification be inherently or intrinsically connected to truth, and means: *Whatever provides a sufficient reason for the believer to assert her belief-claim and is inherently or intrinsically connected to the truth of that belief-claim.* An example of this would be the variant case in which my watch says it is 11:00 am, and my watch is still working fine, and it is actually 11:00 am, and I believe that it is 11 am. So I know that it is 11:00 am. In turn, then, High-Bar justified true belief (H-BJTB) *is* the same as knowledge, i.e., $K = H\text{-BJTB}$.

In this way, what the Gettier counterexamples show us are two distinct synthetic a priori philosophical truths about knowledge, each of which is known by means of a distinct fully reliable basic authoritative philosophical intuition:

- (I) Knowledge is *not* the same as Low-Bar justified true belief ($K \neq L\text{-BJTB}$), and
- (II) Knowledge *is* the same as High-Bar justified true belief ($K = H\text{-BJTB}$).

Therefore, any theory of knowledge which adequately establishes an inherent or intrinsic connection between the sufficient reason for a believer’s assertion of her belief-claim and the truth of her belief, thereby constitutes an adequate philosophical explanation of knowledge.

And that is not all. Provided that the inherent or intrinsic connection is a *non-logically, strongly metaphysically, or synthetically a priori necessary* one, and not a *logically, weakly metaphysically, or analytically a priori necessary* connection, then there is no risk that the explanation will be circular, since circularity in arguments or theories is always a strictly logical, weakly metaphysical, or analytic connection of some

sort. Furthermore, this conception of a philosophical explanation of knowledge—that it adequately establishes an inherent or intrinsic connection between the the sufficient reason for a believer’s assertion of her belief-claim and the truth of her belief—perhaps surprisingly, is largely compatible with Timothy Williamson’s highly plausible “knowledge first” approach to epistemology in *Knowledge and its Limits*.¹⁰⁶ This large measure of compatibility flows directly from the fact that, according to my conception of the theory of knowledge, which I call *Categorical Epistemology*,¹⁰⁷

- (i) Knowledge itself is the primitive, non-analyzable, non-reducible, immanently structured, and categorically normative highest or supreme good and ideal standard of rational human cognition with which epistemology is fundamentally concerned.,
- (ii) High-Bar justification, truth, and belief are the metaphysically non-detachable, essentially-related elements of knowledge., and
- (iii) A priori knowledge via fully reliable authoritative rational intuitions is the *perfection* of our capacities for rational human cognition.

Or in other words, Categorical Epistemology is a *perfectionist* Kantian morality of rational human cognition. No doubt, Williamson would sharply disagree with me about the robust rational normativity of knowledge—not to mention sharply disagreeing with my contemporary Kantianism. But at the same time, we do both hold that

- (a) Knowledge is a primitive, non-analyzable, non-reducible phenomenon with which all serious explanatory epistemology must begin., and
- (ii) Knowledge is inherently *mentalist* and *factive*.

So there is significant common ground shared between us.

It should also be fully noted here that the fundamental distinction between High-Bar justification and Low-Bar justification in Categorical Epistemology is itself only a specification of a more general and necessary structure of human rationality, which I call *Two-Dimensional Rational Normativity*. Two-Dimensional Rational Normativity is the

fact that the conditions on normative evaluations of rationality fall into two importantly different kinds:

(1) Low-Bar Rational Normativity: The necessary and sufficient conditions for *minimal or nonideal rationality*, which include the possession of online, uncompromised versions of all the cognitive, practical, and moral capacities constitutive of intentional agency., and

(2) High-Bar Rational Normativity: The necessary and sufficient conditions for *maximal or ideal rationality*, which include all the necessary and sufficient conditions for Low-Bar Normativity as individually necessary but *not* jointly sufficient conditions, and also include the *perfection*, or *correct and full self-realization*, of all the cognitive and practical capacities constitutive of intentional agency, as individually necessary *and* jointly sufficient conditions.

Non-satisfaction of the conditions for *Low-Bar Rational Normativity* entails non-rationality and non-agency. But this is certainly *not* the case for non-satisfaction of the conditions of *High-Bar Rational Normativity*.

This point, in turn, makes it possible to see very clearly the fundamental flaw in *One-Dimensional* theories of rational normativity, no matter how plausible and sophisticated these theories might otherwise be.¹⁰⁸ According to a One-Dimensional theory, any failure to meet the ideal standards of rational normativity entails *non-rationality*, *non-agency*, and *non-responsibility*. Or in other words, if you are not ideally or *perfectly rational*, then you are a *rationally defective* or *irrational animal*, and *off the hook*. For example, if you fail to know perfectly (i.e., to have High-Bar justified true belief), then you are not in any sense a rational or responsible cognitive agent. And if you fail to act in a practically or morally perfect way (i.e., to have a Good Will in Kant's sense), then you are not in any sense a rational or responsible practical or moral agent. Disastrously, these results of One-Dimensionalism play directly into the hands of radical cognitive, practical, and moral skeptics, since as a matter of fact no actual rational human animal ever manages to meet *all* or even *most* of the High-Bar standards of rational

normativity, but instead is doing extremely well indeed if she *ever* manages to meet *some* of them—e.g., successfully performing some fully reliable basic authoritative rational intuitions in mathematics, logic, morality, and/or philosophy. How convenient for the radical skeptic, then, that most or all of us, most or all of the time, turn out to be *irrational animals*. Perhaps even more disastrously, these results also play directly into the hands of “human, all too human” intentional agents looking for a fast track out of their everyday cognitive, practical, and moral difficulties in a desperately nonideal world. How convenient for them that falling short of rational perfection should entail the suspension of responsibility: If Rationality is Dead, then Everything is Permitted, and they can take *the nihilist’s way out*, like the pathetically wicked character Smerdyakov in

The Brothers Karamazov:

“Take that money away with you, sir,” Smerdyakov said with a sigh.

“Of course, I’ll take it! But why are you giving it to me if you committed a murder to get it?” Ivan asked, looking at him with intense surprise.

“I don’t want it at all,” Smerdyakov said in a shaking voice, with a wave of the hand. “I did have an idea of starting a new life in Moscow, but that was just a dream, sir, and mostly because ‘everything is permitted’. This you did teach me, sir, for you talked to me a lot about such things: for if there’s no everlasting God, there’s no such thing as virtue, and there’s no need of it at all. Yes, sir, you were right about that. That’s the way I reasoned.”¹⁰⁹

For these reasons, it is clear that One-Dimensional theories of rational normativity are false.

On The Two-Dimensional theory, however, things are very different. Satisfaction of the conditions for Low-Bar Rational Normativity is a necessary and sufficient condition of the cognitive, practical, and moral *responsibility* of intentional agents, but it does not guarantee that any of the further conditions of High-Bar Rational Normativity are actually satisfied. In other words, it is fully possible for an intentional agent to be minimally and nonideally rational, but *in a bad or wrong way*, to any degree of badness or wrongness, all the way down to the lowest limiting case of *cognitive, practical, or*

moral monstrosity within its kind, for all of which the intentional agent is also fully cognitively, practically, or morally responsible, and thus correspondingly blameworthy to any of those degrees, down to the limiting case. At the same time, it is also fully possible for an intentional agent to be minimally and nonideally rational *in a good or right way*, to any degree of goodness or rightness, all the way up to the highest limiting case of *cognitive, practical, or moral perfection within its kind*—e.g., successfully performing some fully reliable basic authoritative rational intuitions in mathematics, logic, morality, and/or philosophy—for all of which, again, the intentional agent is also fully cognitively, practically, or morally responsible, and thereby correspondingly praiseworthy to any of those degrees, up to the limiting case.

VII.3

Now back to my theory of intuitional objective a priori knowledge, and my leading thought that the precise nature of the connection between (i) the truth-maker, and (ii) the veridical sensible form in Kantian a priori intuition, mental model, mental diagram, mental picture, structural image, or schema, is particularly crucial to my overall account. As I have stressed, all rational intuitions, even the fully reliable authoritative ones, are fallible, i.e., it is *not analytically, conceptually, or logically necessary* that they be true. But fallibilism is not skepticism. Hence, as it happens, some authoritative intuitions *just are* objectively necessarily true and fully justified, i.e., *High-Bar justified*, and insofar as this is so, these following further two condition both hold:

1. LOCKED-ONTO: The veridical sensible form in Kantian pure intuition, mental model, diagram, structural image, or schema, is *locked onto the truth-maker*, i.e., there is an intrinsic isomorphism between the representational form of the the veridical sensible form in Kantian a priori intuition, etc., to the worldly form of the truth-maker, such that they are *structurally identical*, i.e., there is a “bijective map” running homomorphically from the form of the veridical sensible form in Kantian a priori intuition, etc., to the form

of the truth-maker, and also homomorphically from the form of the truth-maker to the form of the veridical sensible form in Kantian a priori intuition, etc., and

2. STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC.: For every authoritative intuition *AI*—

(2.1) *Either* its characteristic veridical sensible form in Kantian a priori intuition, etc., is locked onto a truth-maker, in which case *AI* is a case of authentic a priori knowledge, i.e., sufficiently justified belief in a necessary truth, *or else* its characteristic veridical sensible form in Kantian a priori intuition, etc., is *not* locked onto a truth-maker, in which case *AI* is only a case of failed a priori belief, i.e., insufficiently justified because *untrue* a priori belief,

(2.2) There is no common content shared between a veridical sensible form in Kantian a priori intuition, etc., and a non-veridical sensible form in Kantian a priori intuition, etc., and

(2.3) The only thing shared between veridical sensible forms in Kantian a priori intuition, etc., and non-veridical sensible forms in Kantian a priori intuition, etc., is the further extrinsic fact that under some cognitive conditions, some or another rational human cognitive subject of an *AI* actually fails to tell the difference between the two inherently distinct mental representations (veridical vs. non-veridical), although

(2.4) necessarily, at least in principle, under appropriate cognitive conditions, every such rational human cognitive subject *could* correctly discriminate between the two.

Fallibilism, as I have said, or at least have clearly implied, is the thesis that *no act or state of belief, even a fully reliable authoritatively rationally intuitive one, analytically, conceptually, or logically entails its own truth*. Hence every act or state of belief, even a self-evident one, *can* be false, as a matter of logical possibility. But if LOCKED-ONTO is satisfied, then the relation between the representational form of the veridical sensible form in Kantian a priori intuition, etc., in a fully reliable authoritative intuition and the worldly form of the truth-maker of that belief, is *inherent or intrinsic*. The characteristic properties of that relation are therefore *robustly necessary* properties, i.e., synthetic a priori necessary properties. Hence although my being in that mental act or state of fully reliable authoritative rational intuition does not *analytically, conceptually, or logically (a.k.a. weakly metaphysically) necessitate* the necessary truth or High-Bar justification of

that rational intuition, nevertheless it does *synthetically a priori, essentially non-conceptually, or non-logically (a.k.a. strongly metaphysically) necessitate* the necessary truth and high-bar justification of that rational intuition. Hence it is *authentic a priori* knowledge that is also perfectly consistent with *real* fallibilism.

In this way, my Categorical Epistemology of rational intuitions has a significant advantage over other recent or contemporary proponents of fallibilism about a priori knowledge, who have been unable to combine the real fact of fallibility with robust necessitation in the a priori knowledge-relation, because, as Modal Monists, they lack the fact and also the very idea synthetic a priori, essentially non-conceptual, or non-logical (strong metaphysical) necessity. This is true, e.g., of Bealer's "strong modal tie to the truth" between idealized intuitions at the end of the relevant historical processes of communal inquiry, and their truth-makers. For Bealer, at the idealized end of communal inquiry, the real human fallibility of rational intuition mysteriously turns into an unreal superhuman *infallibility*.¹¹⁰

The philosophical provenance of Kantian Intuitionism has four primary sources:

- (1) Kant's theory of "a priori intuition" and "productive imagination" in the *Critique of Pure Reason*,
- (2) Husserl's specifically *phenomenological* approach to the epistemology of necessary truth in *Logical Investigations*,
- (3) Wittgenstein's specifically *linguistic* approach to the epistemology of necessary truth in the *Tractatus*, and
- (4) Parsons's theory of Mathematical Structuralism and mathematical intuition in *Mathematical Thought and its Objects*, which in turn is significantly influenced by Kant's intuitionism, Brouwer's intuitionism, and Hilbert's finitism.¹¹¹

In view of those sources, I will argue for Kantian Intuitionism in two stages.

First, I will state what I take to be the deep epistemological ideas lying behind Husserl's doctrine of "categorial intuition" and also behind Wittgenstein's doctrine that "language itself prevent[s] every logical mistake" by virtue of the fact that "we *cannot* think illogically."

Then **second**, in section **VIII**, I will briefly sketch and criticize Parsons's theory, and compare and contrast it with Kantian Structuralism and Kantian Intuitionism.

Husserl and Wittgenstein. For our purposes here, Husserl's deep epistemological idea is that the abstract formal structures characteristic of logic or mathematics are immediately represented in our essentially non-conceptual, pre-reflective or first-order conscious awareness of the logico-syntactic and sortal-semantic structures of the meaningful sentences we use to frame true logical or mathematical judgments, and that the truth of those judgments is directly verified in direct perceptual experience of the manifestly real and intrinsically spatiotemporal natural world. This direct verification, in turn, is *phenomenological self-evidence*. My underlying proposal is that at least some phenomenologically self-evident mental acts or states, which Husserl calls "categorial intuitions," satisfy both LOCKED-ONTO and STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC., and that this threefold fact is also inherently characteristic of a certain kind of competent rational human language use.

To present the notions of phenomenological self-evidence and categorial intuition properly, I want to sketch the basic concepts of Husserl's early phenomenology, and also trace them back to some more fundamental Kantian ideas.

Phenomenology, as Husserl understood it in 1900 in the first edition of the *Logical Investigations*, is an elaboration of “descriptive psychology” in Brentano’s sense, as he worked it out in *Psychology from an Empirical Standpoint*, part I. More precisely, Husserlian phenomenology in 1900 is the first-person, introspective, non-reductive philosophical psychology of consciousness and intentionality, as opposed to the natural science of empirical psychology (*LI V*, §7). As a specifically *philosophical* psychology, its basic claims, if true, are non-logically or synthetically necessarily true and a priori.

As Husserl points out in Investigation V, consciousness (*Bewußtsein*) is a subject’s capacity for “lived experience” or *Erlebnis*, i.e., phenomenal awareness, together with her capacity for *intentionality*. By general philosophical agreement, intentionality is *the “aboutness” of the mind, the “of-ness” of the mind, or the directedness of mind to its objects*. Here the notion of an “object” is very broadly construed so as to include existing or non-existing individuals, properties, relations, facts, temporal events, spatial locations, other minds, and one’s own mind, as possible targets of intentionality; and the acts or states of intentionality can include all sorts of cognitive activities and psychological attitudes, e.g., perception, memory, abstract thinking, love, hate, or fear.

The contemporary concept of intentionality, it is usually held, originally derives from Scholastic philosophy, and emerged during the Phenomenological tradition that begins with Brentano’s *Psychology from an Empirical Standpoint*, and continues on through Meinong, Husserl, early Heidegger, Sartre, and Merleau-Ponty. Intentionality is also a basic theme in the early Analytic tradition, starting with Russell’s theory of acquaintance, singular reference, and singular thought, and continuing on through

Wittgenstein both early and late, John Searle, Gareth Evans, John McDowell, Christopher Peacocke, John Campbell, and many others.

Now on my view, the theory of intentionality in the Phenomenological tradition to which Husserl centrally belongs in fact originally derives from *Kant's theory of human cognition*, and not from Scholastic philosophy, which is at most a remote influence on Brentano's concept of intentionality.¹¹² Kant's *Critique of Pure Reason* is, in one sense, a treatise on epistemology and metaphysics. But Kant's way of doing epistemology and metaphysics is sharply different not only from Classical Rationalism and Classical Empiricism, but also from contemporary Analytic epistemology and contemporary Analytic metaphysics. This is because Kant's grounds epistemology and metaphysics on the theory of human *cognition (Erkenntnis)*, or *conscious objective mental representation (CPR A320/B376-377)*. This is explicitly stated in the famous letter to Marcus Herz in 1772:

[I] was then making plans for a work that might perhaps have the title "The Limits of Sense and Reason." I planned to have it consist of two parts, a theoretical and a practical. The first part would have two sections, (1) general phenomenology and (2) metaphysics, but only with regard to its nature and method.... As I thought through the theoretical part, considering its whole scope and the reciprocal relations of its parts, I noticed that I still lacked something essential, something that in my long metaphysical studies I, as well as others, had failed to pay attention to and that, in fact constitutes the key to the whole secret of hitherto still obscure metaphysics. I asked myself: What is the ground of the reference of that in us which we call "representation" to the object? (PC 10: 129-130)

A theory of human cognition, or conscious objective mental representation, in Kant's sense focuses on the nature of the various *conscious acts/states, mental contents, and objects* of conscious objective mental representation, and tries to explain how human cognition in precisely this sense is *possible*. Now a theory of content is also a theory of *meaning*. So Kant's theory of human cognition, or conscious objective mental representation, is also a *cognitive semantics*.¹¹³

According to Kant, the central fact about the human mind is its capacity to represent (*vorstellen*) (*CPR* A320/B376-377), which is to say that

- (i) the human mind has something *X* “to put before” (*stellen ... vor*) it, and
- (ii) that which puts *X* before the human mind is a mental “representation” (*Vorstellung*).

Our mental representational capacity cannot be further explained—it is simply a primitive fact about us:

What representation (*Vorstellung*) is cannot really be explained. It is one of the simple concepts that we necessarily have. Every human being knows immediately what representation is. Cognitions (*Erkenntnisse*) and representations are of the same sort.... Every representation is something in us, which, however, is related to something else, which is the object. Certain things represent something, but we represent things. (*BL* 24: 40)

Mental representations, in turn, can be either conscious or nonconscious (*CPR* A78/B103).¹¹⁴ The primary cognitive role of consciousness (*Bewußtsein*) is to contribute subjective integrity, or a well-focused and uniquely egocentric organization, to a mental representation (*CPR* B139). A conscious mental representation is thus an “idea” in the broadest possible sense. *Subjective* conscious mental representations are internal or immanent to consciousness and lack fully determinate form or structure. *Objective* conscious mental representations, by contrast, are determinate ways of referring the mind to any sort of objects (i.e., some topic or target of the mind—what the representation is *about* or *of* or *directed to*), including the self considered as an object, as in self-consciousness or apperception. Objects of conscious mental representation also include existent or non-existent objects, and actual or possible objects. In short, conscious objective mental representation in Kant’s sense is essentially what the Phenomenologists later call *intentionality*.

For Kant, every objective conscious mental representation has both

- (i) a “form” (*Form*), and

(ii) a “matter” (*Materie*) or “content” (*Inhalt*) (*CPR* A6/B9) (*JL* 9:33).

The form of an objective conscious mental representation is its *intrinsic structure*. For example, sensory perceptions have intrinsic spatial and temporal form or structure, and judgments have intrinsic logical form or structure. *Materie* is qualitative sensory content.

Inhalt by contrast is *mental content*: this is also what Kant calls the “sense” or *Sinn* of an objective conscious mental representation, and its “meaning” or *Bedeutung* (*CPR* A239-240/B298-299) as well. The mental content, sense, or meaning of an objective conscious mental representation is the *information* that the mind has about its objects. Since the same object can be represented in different ways, there is a many-to-one relation between mental contents (senses, meanings) and their corresponding objects. This doctrine was later recapitulated and reworked by Frege, in an explicitly linguistic context, as the distinction between “sense” (*Sinn*) and “reference” (*Bedeutung*).¹¹⁵

Unfortunately, Kant also sometimes uses the term “form” to refer to purely psychological components of our use or grasp of an objective conscious mental representation (*BL* 24: 40). Form in this sense is somewhat similar to what Descartes called the “formal reality” of an idea, and the mental content (sense, meaning) of an objective conscious mental representation in Kant’s sense is somewhat similar to what Descartes called the “objective reality” of an idea. More precisely, for Kant the form of an objective conscious mental representation is what for lack of a better name I will call its *representational character*, by analogy with the “phenomenal character” of phenomenal consciousness. Representational character includes

(i) the difference between clarity and unclarity, and between distinctness and indistinctness,

(ii) different subjective attitudes of all sorts, or what Locke called “postures of the mind,” including but not restricted to propositional attitudes, and

(iii) our direct conscious awareness of, and ability to distinguish between and generalize over, types of mental acts or mental operations of all different sorts (e.g., analysis, synthesis, memory, imagination, thought, judgment, etc.), which Kant calls “reflection” (*Überlegung*) (CPR A260/B316) and which is somewhat similar to Locke’s “ideas of reflection.”

Conscious mental representations can be either subjective or objective, but in either case they are necessarily accompanied by “sensations” (*Empfindungen*). The “matter” or phenomenal content of sensations—or what we would now call “phenomenal characters”—are qualitative intrinsic properties of all conscious representations. More precisely, however, sensation is “the effect of an object on the capacity for representation, insofar as we are affected by it” (CPR A19-20/B34), or in other words, a sensation together with its content is nothing but the subject’s direct response to endogenously- or exogenously-caused changes in its own state. Endogenously-caused sensations are “subjective sensations” (CPJ 5:206) or feelings, and exogenously-caused sensations are “objective sensations,” such as the sensations that accompany the perception of external objects (CPJ 5: 206).

Objective conscious mental representations, as I have mentioned already, are also known as *cognitions* (*Erkenntnisse*) (CPR A320/B376-377), and this Kantian usage is essentially equivalent with the use of the term ‘cognition’ in contemporary cognitive psychology. Just to make things confusing however, in the B edition of the first *Critique* (see, e.g., at CPR Bxxvi, n.) Kant also sometimes uses the term ‘cognition’ in a somewhat narrower sense to mean an objective conscious cognition of an actual or possible object of sense perception, an actual or possible empirical object, or empirical

state-of-affairs: namely, *an empirically meaningful or objectively valid judgment*.¹¹⁶

This notion of a cognition then directly contrasts with the notion of mere *thought*, which is a conscious conceptual mental representation of any sort of object whatsoever, whether or not it is an object of actual or possible sense perception.

So according to Kant, and in relation to this narrow sense of ‘cognition’, there are two categorically or essentially different kinds of *intentional objects*:

- (1) *cognizable objects*, or “thick” objects, and
- (2) *merely thinkable objects*, or “thin” objects.

As to the merely thinkable or thin objects, Kant explicitly points out that

Once I have pure concepts of the understanding, I can also think up objects that are perhaps impossible, or that perhaps possible in themselves but cannot be given in any experience since in the connection of their concepts something may be omitted that yet necessarily belongs to the condition of a possible experience (the concept of a spirit), or perhaps pure concepts of the understanding will be extended further than experience can grasp (the concept of God). (CPR A96, underlining added)

It is crucial to understand what Kant means by saying that “I can also think up objects that are perhaps impossible.” This does *not* mean that I can think up objects that are *logically or analytically impossible*, since he explicitly says that logical and analytic consistency is a necessary condition of all thinkability and of all thinkable objects:

I can **think** whatever I like, as long as I do not contradict myself, i.e., as long as my concept is a possible thought, even if I cannot give any assurance whether or not there is a corresponding object somewhere within the sum total of all possibilities. (CPR Bxxvi n., underlining added)

Therefore what Kant must mean when he says that “I can also think up objects that are perhaps impossible” is that it is possible to think *metaphysically impossible* or *synthetic a priori impossible* objects, i.e., objects that are logically and analytically self-consistent, and thereby *merely thinkable*, and thereby conceivable, yet nevertheless also inherently *uncognizable*, because they cannot be given via any actual or possible sensible intuition, and thus are *humanly unintuitable*:

The transcendental use of a concept in any sort of principle consists in its being related to things **in general** and **in themselves**; its empirical use, however in its being related merely to **appearances**; i.e., objects of a possible **experience**. But that it is only the latter that can ever take place is evident from the following. For every concept there is requisite, first, the logical form of a concept (of thinking) in general, and then, second, the possibility of giving it an object to which it is to be referred. Without this latter it has no sense (*Sinn*), and is entirely empty of content (*Inhalt*), even though it may contain the logical function for making a concept out of whatever sort of *data* there are. (CPR: A238-239/B298)

Back now to Husserl. As Husserl points out in Investigation V, “consciousness” (*Bewusstsein*) is *subjective experience*, where the notion of “experience” includes both

- (i) *Erlebnis*, i.e., “lived experience” or *phenomenal awareness*, and
- (ii) *Erfahrung* in Kant’s sense, i.e., “objective experience” or *intentionality* that is directed towards either cognizable objects (thick objects) or merely thinkable objects (thin objects).

In turn, for Husserl every conscious intentional mental state *M* has four individually necessary and jointly individuating features:

- (1) *M* is a mental *act* (*psychischer Akt*) with its own “immanent content” or “act-matter” and its own specific character (i.e., phenomenal character) (*LI V*, §§11, 14, 20),
- (2) *M*’s mental act falls under a specific intentional *act-type* or “act-quality,” e.g., perceiving, imagining, remembering, asserting, doubting, etc. (*LI V*, §20),
- (3) *M*’s mental act has an intentional *target*, which at the very least has ontic status or “being” (*Sein*) and perhaps also actual existence or “reality” (*Wirklichkeit*), although this target *need not necessarily* have reality—hence intentional targets can include fictional objects, impossible objects, abstract objects, ideal objects, etc. (*LI V*, §§11, 17, 20), and
- (4) *M*’s mental act has an intentional *meaning content* or “semantic essence” (*bedeutungsmässige Wesen*), which presents its object in a certain specific way, where this meaning content is either *propositional* or *referential* (*LI V*, §§21, 31-36).

It is crucial to note that this general phenomenological analysis holds *both* for the intentionality of judgment and belief, which presupposes pure formal logic and necessarily requires the existence of natural language and the intentional subject’s linguistic competence, *and also* for the intentionality of perception and other modes of sensory cognition such as imagination and memory, which do not presuppose pure formal logic or necessarily require the existence of natural language or linguistic competence.

In Investigation VI, Husserl argues that truth (*Wahrheit*) is the structural and semantic intrinsic conformity of a judgment to the very fact that satisfies its propositional content, and that authentic knowing (*Erkennen*) or “self-evidence” (*Evidenz*)—whether authentic a priori knowledge or authentic a posteriori knowledge—is the sufficiently justified conscious intentional recognition of necessary or contingent truth (*LI VI*, §§6-12, 20, 28, 36-39). Self-evidence has its own characteristic phenomenology. The basic structure of this phenomenology of self-evidence is the goal-directed advance from “empty” intentions to “filled” intentions, whereby

(1) Empty intentions are logico-linguistically structured propositional contents insofar as they are *conceptually understood* by an intentional subject to specify the very facts that *could or would* satisfy those contents and thereby *make* those propositions true., and

(2) Filled intentions are logico-linguistically structured propositional contents insofar as the very facts that could or would satisfy them are also *non-conceptually intuited* by an intentional subject as *actually satisfying* those contents and thereby *making* those propositions true.

In other words, and now formulated in an explicitly Kantian way, for early Husserl the phenomenological profile of authentic knowledge or self-evidence is a systematic advance from conceptual “understanding” (*Verstand*) to essentially non-conceptual “intuition” (*Anschauung*), and this holds whether the authentic knowledge is a priori or a posteriori, and whether the truth-making fact that is intuitively experienced in intentional fulfillment as satisfying the relevant propositional content is a non-empirical or ideal (necessary or possible) abstract fact, or an empirical or real (contingent) concrete fact. In the case of non-empirical or ideal facts, then the essentially non-conceptual intuition by which the fact is self-evidently known is a *categorial* intuition. (*LI VI*, §§40-58). Categorial intuitions are intentional states containing phenomenal characters that intrinsically and specifically pick out the formal and structural elements of the very facts

that are known via intentional fulfillment, either by means of formal elements of perceptual consciousness, or by means of formal elements of logico-linguistic consciousness. In other words, categorial intuitions are phenomenologically self-evident acts or states of belief that satisfy both LOCKED-ONTO and STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC., and are therefore High-Bar justified true beliefs. So they are inherently or intrinsically connected to the truth-makers of those beliefs, and they thereby constitute authentic a priori knowledge.

For my purposes here, two paradigmatic examples of categorial intuition would be—

(i) the way in which aggregates of directly perceived objects (say, seven martinis) are essentially non-conceptually and pre-reflectively or first-order consciously “subitized” into finite groups (say, groups of 3 or 4), e.g.,

Y Y Y Y Y Y Y

and

(ii) the way in which an evidentially verifying state-of-affairs as described by a statement or judgment appears to have the very same grammatical form as the sentence used to describe it., e.g.,

The seven martinis are sitting on the table. Y Y Y Y Y Y Y

Correspondingly, when rational human animals use simple arithmetic sentences like ‘3+4=7’ in making necessarily true statements like “3+4=7,” then we are thereby essentially non-conceptually and pre-reflectively or first-order consciously aware of an intrinsically-structured structured *temporal* flow of mental images associated with our visual or auditory cognition of those inscriptions or utterances. Indeed, recent empirical research on memory strongly indicates that the essentially non-conceptual, pre-reflective

or first-order conscious phenomenal look and sound of language is processed separately from the propositional cognition of linguistic meaning.¹¹⁷ For example, I can vividly recognize and remember the look or sound of certain German sentences and words—e.g.,

Die Welt is alles, was der Fall ist

or

Wovon man nicht sprechen kann, darüber muss man schweigen

(as, perhaps, screeched by the brilliant Finnish absurdist composer and singer M.A. Numminen¹¹⁸)—without recognizing or remembering what they mean.

Thus the mathematical propositions or statements that we express by means of the self-conscious, reflective, intentional conceptual acts of cognizing the linguistic meanings of arithmetic sentences, are also directly combined with an essentially non-conceptual, pre-reflective or first-order conscious grasp of the formal structure of experiential or lived time that essentially conforms to what L.E.J. Brouwer calls the “first act of intuitionism,” which is

completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twofold thus born is divested of all quality, it passes into the empty form of the common substratum of all twofolds. And it is this common substratum, this empty form, which is the basic intuition of mathematics.¹¹⁹

And then in turn, whenever we directly perceive a configuration of manifestly real material objects in the natural world that partially confirms the necessarily true arithmetic propositions or statements that we express—say, we see the three martinis on the kitchen table sitting alongside the four other martinis, yielding the look of seven martinis sitting on the kitchen table, e.g.,



—then the non-conceptual, pre-reflective or first-order conscious direct sense perceptions of those manifestly real material objects, supplemented by the self-conscious, self-reflective epistemic perceptions based on those direct perceptions, when taken together with their perceptual, imaginational, and memory-based synthesis in time as we explicitly or implicitly count them up, collectively immediately deliver to us a phenomenological formal structure that is also intrinsically isomorphic to the addition operation over the natural numbers 3 and 4 in the system of elementary or Peano arithmetic, especially including PRA or simple arithmetic, and thus *also* based essentially on a conscious directly referential sense perception of Hilbert’s “objects of finitary reasoning.” That essentially non-conceptual, pre-reflective or first-order conscious visual experience is a *categorical intuition* in Husserl’s sense, and it necessarily impresses itself upon us as *mathematically completely convincing, intrinsically compelling, or self-evident*, where this necessarily includes the satisfaction of LOCKED-ONTO and also the satisfaction of STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC. ¹²⁰.” Or in other words, Husserl’s phenomenological notion of a categorical intuition fuses *Brouwer’s temporal-representation-based intuitionist epistemology of mathematics* with *Hilbert’s spatial-representation-based finitist epistemology of mathematics*. But as regards the *logico-semantic foundations* of mathematics, we need not suppose that either Intuitionism or Finitism is true. Indeed we can even suppose that they are both false, and that Kantian Structuralism instead is true.

In this way, as a rational human animal and conscious intentional subject, in categorially intuiting that $3+4=7$, you rationally cannot help believing the propositional

content associated with precisely that essentially non-conceptual, pre-reflective or first-order conscious visual experience, precisely because it completely convincing, intrinsically compelling, or self-evident. But, furthermore, it is also fully reliable objective a priori knowledge of necessary truth, precisely because that sensible form in Kantian a priori intuition, etc., is *locked onto* its truth-maker, and because *strong disjunctivism about sensible forms in Kantian a priori intuition, etc.*, is also true of it, and these two facts jointly yield High-Bar justified true belief. This updated Husserlian doctrine, in its Kantian Structuralist and Kantian Intuitionist context, and with its Browerian and Hilbertian epistemological background, I think, provides a robustly realistic phenomenological interpretation of the classical Cartesian idea of *clear, distinct, and indubitable rational intuition* that is also perfectly consistent with fallibilism.

Correspondingly, as I see it, the Tractarian Wittgenstein's equally deep epistemological idea is that to have logical or mathematical a priori knowledge is just

(i) To be a conscious rational human animal who possesses an innately specified cognitive capacity for essentially non-conceptually and pre-reflectively or first-order consciously constructing, understanding, and using natural languages:

Human beings possess the capacity of constructing languages, in which every sense can be expressed, without having an idea of how and what each word means—just as one speaks without knowing how the single sounds are produced. Ordinary language is a part of the human organism and is not less complicated than it.,¹²¹

and

(ii) Actually to apply the meaningful logical and mathematical sentences or statements of those natural languages—e.g., “ $3+4=7$ ”—according to the implicit normative rules of logic and of those natural languages, to a world of directly perceivable manifestly real material objects in the natural world, whose configurations inherently satisfy those sentences or statements.

So if, plausibly, we take early Wittgenstein's remarks about cognizing language to be anticipations of a broadly *Chomskyan* theory of language,¹²² then essentially non-conceptually, non-self-consciously, pre-reflectively or first-order consciously, and thus

“tacitly” knowing the logical and mathematical *parts* of a natural language is just a sub-species of essentially non-conceptually, non-self-consciously, pre-reflectively or first-order consciously, and thus “tacitly” knowing *a natural language* more generally.

This is objective priori knowledge in the mode of knowing exactly but also only essentially non-conceptually and pre-reflectively or first-order consciously *how* to construct and manipulate or use the language according to categorically normative rules of human rationality,¹²³ but not objective a priori knowledge in the mode of self-consciously and self-reflectively knowing exactly *what* one is doing or *that* one is doing it, whenever one actually does it. Or in other words, Wittgenstein is adumbrating the notion of a conceptually-apt, but also essentially non-conceptual and pre-reflective, first-order conscious *robustly normative a priori mathematical and logical linguistic competence*.

VIII. Parsons, Kantian Structuralism, and Kantian Intuitionism

The question is how it is possible for a priori intuition to be “of” objects that are not given a priori. Kant’s own solution to the puzzle ... appeals to the idea that a priori intuition contains only the form of our sensibility. This evidently removes the causal dependence of intuition on the object. It is a nice question what is left of the characterization of intuition that gives rise to the puzzle. Kant’s solution seems to allow the *phenomenological* presence of an object to be preserved, but it is a further question whether what one has is a representation of a physical object, not individually identified and not really present, or a representation of a mathematical object. The former is not ruled out by the a priori character of pure intuition, as the “presence” might be that characteristic of *imagination* rather than sense. In fact, a number of passages in Kant indicate that just that is his position. Kant’s puzzle may have force for us, but we are not likely to accept the position that pure intuition contains only the form of sensibility, a central part of Kant’s transcendental idealism, at least not as Kant understood it.

--C. Parsons¹²⁴

VIII.1

Now I want to look at the basic points of the Parsons’s theory of Mathematical Structuralism and mathematical intuition in *Mathematical Thought and its Objects*, especially chapters 2-3, 5, and 9, and then formulate six constructive worries about it. My working hypothesis is that *although* Parsons’s theory has been explicitly and significantly influenced by Kant (and also by Brouwer and Hilbert), and *although* Parson’s theory is highly philosophically interesting and suggestive for my purposes, *nevertheless* the underlying problem with it is that it is *insufficiently Kantian*. The worries are “constructive” in the sense that I will use them in order to elaborate and defend Kantian Structuralism and Kantian Intuitionism *conjointly* somewhat beyond what I have already done *severally* in sections **VI** and **VII**.

VIII.2

(Parsons 1) According to Parsons, intuition in the specifically philosophical sense is of two different basic kinds:

(i) *intuition-that P* (propositional intuition, a.k.a., “conceptual intuition” or “propositional intuition”), and

(ii) *intuition-of X* (object-directed intuition, a.k.a. “non-conceptual intuition” or “perceptual intuition”).

This distinction, in turn, maps quite closely onto the classical Russellian distinction between:

(i) knowledge-by-description, and

(ii) knowledge-by-acquaintance.¹²⁵

It is relevant to note here that Russell’s knowledge-by-description vs. knowledge-by-acquaintance distinction is clearly an updated version of *Kant’s* distinction between conceptualization and intuition (*Anschauung*). Notice also, however, that Parsons’s intuition-of (i.e., knowledge-by-acquaintance) is at least *minimally non-conceptual* in the sense that representational states are not determined by conceptual or propositional capacities alone, does not presuppose the possession of concepts, and does not presuppose the application of concepts. Intuition-of can also be directed to *propositions* taken as objects, as in “By the way, $3+4=7$. I love *that* proposition.”

(Parsons 2) According to Parsons, *rationality* is any mental capacity or process essentially related to the provision of reasons, justification, logical inference, and logical principles, including consistency and systematization. *Ideal* rationality, in turn, is rationality that *fully and successfully* conforms to and satisfies all the basic norms and principles of reason. *Nonideal* rationality, by contrast, is rationality that *tries to* conform to and satisfy all the basic norms and principles, even if it does not always manage to do so fully or successfully. The crucial point here is that nonideal rationality is *still* rational and *not* either irrational or arational.

(Parsons 3) According to Parsons, rational intuition or intuition-that is *non-infallible* (defeasible, fallible) yet also *intrinsically compelling* (completely convincing,

self-evident)—and this is said to be relevantly similar to Quine’s notion of the “obviousness” of basic logical truths.¹²⁶ It is important to notice in this connection that the distinction between intrinsic compellingness and infallibility teases apart two different senses of *indubitability*:

- (i) the indubitability of *evidence* (especially a priori evidence), and
- (ii) the indubitability of *truth* (especially necessary truth).

Obviously these are logically independent notions, although just as obviously, they are also mutually consistent.

(Parsons 4) According to Parsons, rational intuition or intuition-that is *non-inferential*, i.e., not needing to be derived by inference or from premises. In this sense, rational intuition is logically and justificationaly *self-contained*, although nothing inherently rules out an auxiliary inferential justification of it, whether deductive, inductive, abductive, or transcendental. The combination of the intrinsic compellingness and the non-inferentiality of rational intuition-that is basically the same as *authoritative rational intuition* in the sense spelled out by me above, by Husserl via his phenomenological notion of *Evidenz*, and by Wittgenstein via his Tractarian linguistic transformation of Russell’s notion of “self-evidence” or *die Einleuchten*. The grounds of truth (especially necessary truth) must be explained independently.

(Parsons 5) Parsons explicitly raises the question: “What accounts for the intrinsic compellingness and non-inferentiality of rational intuition or intuition-that, and in particular, what accounts for the the intrinsic compellingness and non-inferentiality of *mathematical* intuition-that?” For example, what accounts for the the intrinsic compellingness and non-inferentiality of the rational intuition-that $3+4=7$ or any other truth of PRA? Kant’s two-part answer, also explicitly adopted by Parsons, is

(i) that *mathematical intuition-of* accounts for the intrinsic compellingness and non-inferentiality of rational intuition-that, and

(b) that mathematical intuition-of is in some way or another linked fundamentally to human *sense perception*.

(Parsons 6) According to Parsons, much of mathematics is too abstract and complicated to be suitable for mathematical intuition-of, e.g., the more complex parts of number theory, analysis, set theory, or geometry.

(Parsons 7) According to Parsons, because of Benacerraf's Dilemma, there is no good reason to think that numbers themselves, taken as *abstract objects in the classical Platonic sense*, can be the proper objects of mathematical intuition. Mathematical intuition has to be sense-perception-like.

(Parsons 8) According to Parsons, what is the nature of numbers and other mathematical objects? He rejects both Platonism and Nominalism, and asserts Mathematical Structuralism as I spelled it out in section VI. Parsons is explicitly a Non-Eliminative Structuralist, but remains officially neutral on the question of *ante rem* vs. *in rebus* Structuralism.

(Parsons 9) According to Parsons, as a Non-Eliminative Structuralist, mathematical intuition is directed specifically to mathematical objects that are something over and above their merely being positions or roles in structures. Moreover, Parsons holds that if any part of mathematics is capable of being intuited, then it must belong to *elementary arithmetic*, i.e., Peano arithmetic.

Now, Parsons asks himself, what class of objects satisfies both of the following criteria:

(i) They inherently belong to the relevant elementary arithmetic/Peano structure as positions/roles in the structure (Mathematical Structuralism), and

(2) They are also something over and above the structure, i.e., they do not explanatorily and ontologically “disappear” into the structure, as in Eliminative Structuralism (Non-Eliminative Structuralism)?

Parsons thinks that *Brouwer’s intuitionist epistemology*¹²⁷ and *Hilbert’s finitist epistemology*¹²⁸ each provide crucial clues. From Brouwer, he takes the idea that the intuitable part of mathematics is *constructible* in repeatable acts of human sensory intuition aided by the imagination. And from Hilbert, he takes the idea that the domain of construction is the domain of *tokens* of simple linguistic *types*, e.g., visually perceivable strokes such as our old friends—

||||||

According to Parsons, linguistic types are *quasi-abstract* in the sense that they are fully repeatable (multiply instantiable, multiply realizable) like platonic universals, yet they repeat (instantiate, realize) only *in* space and time.

(Parsons 10) Granting **(Parsons 9)**, then Parsons’s basic idea about mathematical intuition is that any calculation in elementary or Peano Arithmetic can be represented intuitively in terms of calculations using strokes, e.g.,

3+4=7

is intuitively representable in sense perception, e.g., via our other old friends

||| + |||| = ||||||

More generally, any natural number can be represented in terms of simple stroke calculations. We see this by using our capacity for non-conceptual *sense-perception* together with our capacity for *imagination*—both in the form of *memory* and also in the form of the ability to create what Kant calls “schemata.” The relevant stroke construction, as perceived or imagined (via memory or Kantian schemata) is itself a *model* in the mathematical sense of any corresponding mathematical proposition or structure that

describes elementary arithmetic or the natural numbers. Otherwise put, according to Parsons's Non-Eliminative Structuralism and Mathematical Intuitionism, at least some mathematical objects are *perceivable and imaginable role players of the natural number roles*, i.e., all the actual and possible stroke-constructions, and *these are the objects of mathematical intuition*.

VIII.3

So that is Parsons's doctrine in a nutshell. For *me*, however, these stroke constructions count as *evidential verifiers* of mathematical beliefs, not *truth-makers* of mathematical statements. If Kantian Structuralism is correct, then the truth-makers are *the mathematical structures themselves, insofar as they are implemented in the manifestly real natural world of the spatiotemporal material objects of human experience*, as directly represented by formal essentially non-conceptual content, a.k.a. "Kantian a priori intuition." With that leading caveat in front of us, I now turn to six worries about Parsons's account.

First, I have a worry about Parsons's minimal Non-Conceptualism about sense perception. Many contemporary philosophers of cognition (e.g., McDowell) are defenders of *Conceptualism*, and as I have argued elsewhere,¹²⁹ there are some quite strong Conceptualist arguments against minimal or "state" Non-Conceptualism that Parsons has not addressed. In particular, *the content* of a minimally non-conceptual state could still be conceptual, even if the state itself is not determined by conceptual capacities and does not entail concept-possession or concept-application. In other words, for all that Parsons has said, what I have called *Highly Refined Conceptualism* could still be correct.

Second, because Parsons is a Mathematical Structuralist, he still has to account for our knowledge of mathematical *structures*. A natural Kantian-Browerian-Hilbertian suggestion here is that mathematical structures are grasped by our innately-specified capacity for non-empirically generating formal essentially non-conceptual contents in sense perception, memory, or imagination, via sensible forms in Kantian pure intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata, together with our innately-specified capacity for *conceptualization*, together with our innately-specified capacity for *logical cognition*. But Parsons never explicitly says this.

Third, because Parsons remains officially neutral about the difference between *ante rem* Structuralism vs. *in rebus* Structuralism, then if it turns out that he is ultimately an *ante rem* structuralist, he would still have a significant commitment to Platonism, and would therefore correspondingly still have a significant problem with BD. Indeed, and I think revealingly, Parsons specifically *avoids* facing up to BD in *Mathematical Thought and its Objects*.

Fourth, one basic worry about allowing in stroke-constructions *as* mathematical objects themselves is that they do not seem to be *precise* in the way that classical mathematical objects are. One possibility here is that the *productive* imagination in the Kantian sense (see, e.g., *CPR* B151-152) might be used as a precisifying representational capacity—e.g., you see the martini, and then you turn away, and after some productive imaginal processing in episodic memory, you have generated a martini-iconic or martini-like sensible form in Kantian a priori intuition, etc.—but, again, Parsons never actually says this.

Fifth, in order to represent all the natural numbers using stroke constructions, the imagination must be an *infinitary* cognitive capacity, at least in the sense that the cognizing subject can *always imagine adding one more stroke to an existing stroke sequence*. But that is a significant cognitive power which appears to be *spontaneous* and also *a priori* in Kant's sense. And Parsons never explicitly asserts this.

Sixth, even if infinitary stroke constructions are allowed, nevertheless the method of stroke construction does not verify *all* of even *elementary* arithmetic, i.e., Peano Arithmetic. More specifically, Peano's *axiom (5)* is not verified by stroke constructions, and requires the ability to grasp *quantifications* over all the numbers. So it seems clear that at most *quantifier-free* arithmetic, i.e., Primitive Recursive Arithmetic or PRA, i.e., simple arithmetic, could be verified by mathematical intuition in Parsons's sense. This puts serious epistemic limits on our mathematical intuition. Perhaps that would not be a genuine problem, if Parsons's view were simply the combined Kantian-Brouwerian-Hilbertian epistemological doctrine that *nothing will count as mathematical knowledge of any kind unless it presupposes our ability to know at least some of the finitist sub-structures of PRA or simple arithmetic via authoritative rational intuition, via the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, etc.*, but yet again he does not actually say that.

Now it seems to me that I can respond to these six worries about Parsons's account just by helping myself to some (I think, independently defensible) Kantian ideas and also to some ideas of my own, and that this conjunction yields the defensible two-part theory of Kantian Structuralism in conjunction with Kantian Intuitionism.

Re problem 1: I think that we should accept a *maximal* or *content* non-conceptualism, namely what I call *Kantian essentialist content Non-Conceptualism*, or *Kantian Non-Conceptualism* for short, which, again as I have argued elsewhere,¹³⁰ says that

- (i) Non-conceptual content is categorically or essentially different in structure and psychological function from conceptual content, and that
- (ii) Non-conceptual mental states or acts are defined by their inclusion of essentially non-conceptual content,

which specifically also includes

- (iii) a Kantian theory of *formal* essentially non-conceptual content, or a priori intuition, according to which we veridically represent the formal structures of space and time via subjective a priori forms of our empirical sensibility in inner sense and outer sense.

Re problem 2: I think that we should accept the combined Kantian-Brouwerian-Hilbertian epistemological doctrine that mathematical structures are grasped by our innately-specified spontaneous capacity for non-empirically representing formal essentially non-conceptual contents in sense perception, memory, or imagination, via sensible forms in Kantian a priori intuition, etc., *plus* our innately-specified spontaneous capacity for conceptualization, *plus* our innately-specified spontaneous capacity for logical cognition.

Re problem 3: I think that we should accept the specifically Kantian idea that mathematical structures are all *weakly or counterfactually transcendentally ideal*, that is, *necessarily conforming to* the pure a priori mental representations of those structures. The Non-Eliminative Structuralism that we need must be *non-platonistic*, and the version of Transcendental Idealism that we should accept is Weak or Counterfactual Transcendental Idealism or WCTI, *not* Strong Transcendental Idealism or STI.

Re problem 4: I think that we should accept the specifically Kantian idea that the imagination can be used as a *precisifying* representational capacity—e.g., you see the martini, then you turn away, and then, via its veridical representation in minimal episodic memory, you generate an *empirical schema* of a martini. This effectively mediates between actual perception and Kantian formal essentially non-conceptual content, i.e., Kantian a priori intuition.

Re problem 5: Following on directly from that, I think that we should also accept the specifically Kantian theory of the *productive* imagination, as an innately-specified, spontaneous and also *a priori* capacity for constructing and manipulating sensible forms in Kantian a priori intuition, etc.

Re problem 6: Finally, I think that we should accept the following Kantian-Brouwerian-Hilbertian epistemic principle, or **The KBH** for short, as a fully reliable non-basic authoritative philosophical intuition about the nature of mathematical knowledge:

The KBH: Nothing will count as mathematical knowledge of any kind unless it presupposes our innately-specified rational human ability to know at least some of the finitist sub-structures of PRA or simple arithmetic in authoritative rational intuition, via the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata.

In other words, all mathematical knowledge of any kind, no matter how abstruse, presupposes that all rational human animals have at the very least an innately-specified *cognitive competence* for knowing at least some true statements of PRA or simple arithmetic via sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata. It is hard to see how anyone could seriously deny **The KBH**, as Tait so crisply points out:

[A]lthough we cannot speak of the absolute security of finitism, there is a sense in which we can speak of its *indubitability*. That is, any nontrivial reasoning about number will presuppose finitist

methods, and there can be no preferred or even equally preferable method from which to launch a critique of finitism. In other words, it is simply pointless to doubt it.¹³¹

But in any case, even at the risk of philosophical overdetermination, here is an explicit reductio argument for **The KBH**. Suppose, e.g., that we conceive of someone—let us call her *The ZF Superstar*—who by hypothesis has full knowledge of the basic principles of Zermelo-Fraenkel set theory. Now add to it the further postulate that *The ZF Superstar* has *no cognitive competence whatsoever* for PRA or simple arithmetic. But that is that is clearly and distinctly absurd. So **The KBH** is true.

Let me now elaborate that reductio argument a little further, in order to bring out some other important points that lurk nearby. By a “cognitive competence for PRA or simple arithmetic,” I mean *an essentially non-conceptual, pre-reflectively conscious ability* for knowing PRA or simple arithmetic via the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, etc., as opposed to *an occurrent conceptual, reflective, and self-conscious grasp of that very intentional performance* that occurrently yields an authoritative rational intuition of PRA or simple arithmetic, and thereby also occurrently yields authentic objective a priori knowledge of it. For example, an ordinary young child who can already speak his own natural language somewhat can come to know that $3+4=7$ by counting on an abacus, his fingers, or a Hilbert-style stroke diagram, but obviously he will fail to have an occurrent conceptual, reflective, and self-conscious grasp of the sentence or statement “ $3+4=7$.” The ordinary young somewhat linguistic child thereby possesses a skill, or know-how, for generating and manipulating *a constructive procedure* by means of which it is possible to have an occurrent conceptual, reflective, and self-conscious grasp of the sentence or statement “ $3+4=7$,” yet without actually having an occurrent conceptual, reflective, and self-

conscious grasp of that sentence or statement. By deploying that skill, or know-how, he does *not* authentically know objectively a priori that $3+4=7$, where authentic knowledge is High-Bar justified true belief. But at the same time, but he *does constructively prove* that $3+4=7$, and thus she has *warranted* true belief, or Low-Bar justified true belief, but not authentic knowledge, that $3+4=7$. Otherwise put, the ordinary young somewhat linguistic child's successful counting procedure, for all intents and purposes, is *just another Gettier example* which shows, yet again, that Low-Bar justified belief is not authentic knowledge, i.e., $K \neq L\text{-BJTB}$.

It does not seem at all impossible, then, that The ZF Superstar might lack an occurrent conceptual, reflective, self-conscious grasp of PRA or simple arithmetic. After all, the great Indian mathematician Ramanujan was able to have Low-Bar justified true belief about certain highly abstruse parts of prime number theory, without also having any occurrent conceptual, reflective, self-conscious grasp of elementary proof theory,¹³² i.e., without having High-Bar justified true belief, or authentic objective a priori knowledge, about those parts of prime number theory.

But that possibility is *not* what I am specifically postulating for the purposes of my thought-experiment. What I am specifically postulating is that The ZF Superstar lacks even *an essentially non-conceptual, pre-reflectively conscious ability*, or cognitive competence, for knowing PRA or simple arithmetic via sensible forms in Kantian a priori intuition, etc.,. So she does not even have Low-Bar justified true belief about PRA or simple arithmetic. In particular, The ZF Superstar cannot count up to 10, or 5, or 2, or even to 1 by using an abacus, her fingers, or a stroke diagram. And she has not the slightest skillful or reflective grasp of what *zero* is. Shee cannot add, subtract, multiply,

or divide. And so-on. In other words, The ZF Superstar cannot effectively enumerate the membership of even the smallest sets, or tell the difference between an empty set and a non-empty set, much less effectively perform any of the primitive recursive functions over the members of any sets. How then could she ever know *any* higher set theory?

The answer, of course, to echo Tweedledum and Tweedledee, is: *Nohow*. The very idea of a fully-knowledgeable mathematician of any highly sub-specialized area in mathematical theory who also lacks even a cognitive competence for knowing PRA or simple arithmetic via the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, etc., is absurd and unintelligible. In other words, The ZF Superstar, minus a cognitive competence for knowing PRA or simple arithmetic via authoritative rational intuition, is *not* the Ramanujan of set theory. The ZF Superstar, any other purported mathematical Superstar, or indeed any other ordinary *rational human animal*, minus a cognitive competence for knowing PRA or simple arithmetic via sensible forms in Kantian a priori intuition, etc., is simply a *non-mathematical animal*, i.e., in effect a mathematical *dunce*, no matter how rational she might be in the *other* parts of her human animal life. In short, my thought experiment shows the absurdity and unintelligibility of the thought that one could know any mathematics *whatsoever* without at least a cognitive competence for PRA or simple arithmetic via the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, etc.

We are now in a position to revisit, in a constructive way, my positive intuition-based solution to BD in section **VII**. Let us start with step 6 in the original formulation of BD:

(6) But on the other hand, given (4), and since all abstract objects are causally isolated and inert, it then follows that all abstract mathematical objects are causally isolated and inert.

Now let us modify (6), and then complete BD in the following way, according to Kantian

Intuitionism:

(6*) The original step (6) assumes that causally inert abstract mathematical objects, the truth-makers of mathematical statements, are *things-in-themselves* or *noumenally real things*, i.e., mind-independent, non-spatiotemporal, non-sensory entities constituted by “real essences,” i.e., intrinsic non-relational properties. But that assumption is highly questionable, given the Kantian view that things-in-themselves are unknowable by cognizers like us, and therefore we should reject it.

(7*) On the contrary, we should assume instead that mathematical objects, the truth-makers of mathematical statements, are just transcendently ideal a priori immanent structures of manifestly real spatiotemporal material objects in nature (phenomena), knowable by means of formal essentially non-conceptual contents in sense perception, memory, or imagination, and more specifically by means of the cognitive construction and manipulation of veridical sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata, according to the thesis of Weak or Counterfactual Transcendental Idealism or WCTI, and also satisfying the High-Bar normative epistemic principles of LOCKED-ONTO and STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC.

(8*) Now since manifestly real spatiotemporal material objects in nature are causally efficacious, then the formal essentially non-conceptual veridical sensible forms in Kantian a priori intuition, etc., of their transcendently ideal a priori immanent structures must be at least causally relevant.

(9*) Therefore, the causally inert abstract mathematical structures that are necessarily implemented in the manifestly real spatiotemporal material natural world, which are *the truth-makers* of mathematical statements, inherently correspond to veridical sensible forms in Kantian a priori intuition, etc., of those immanent structures, which are *the intentional targets* of mathematical intuition, and in turn inherently correspond to directly perceivable manifestly real spatiotemporal material objects in nature, which are the causally efficacious *evidential verifiers* of mathematical beliefs or judgments in elementary or Peano arithmetic, especially including PRA or simple arithmetic.

(10*) Therefore, authentic objective a priori knowledge of at least some necessary mathematical truths, via mathematical rational intuition, is really possible.

This completion constitutes a positive, intuition-based solution for BD. Here are two further elaborative comments on this solution.

First, it needs to be re-emphasized that according to Kantian Structuralism and Kantian Intuitionism, the infinitary mathematical structures of elementary or Peano arithmetic, especially including the finitist sub-structures of PRA or simple arithmetic,

are only *weakly or counterfactually transcendentally ideal*, that is, necessarily *conformable* to our formal essentially non-conceptual non-empirical/a priori mental representations of space and time, precisely to the extent that these spatiotemporal representations are *taken together with* our possession of innately-specified formal a priori meta-logical concepts and our innately-specified cognitive-linguistic capacity for constructing all classical or non-classical logical systems.¹³³ Thus our formal essentially non-conceptual non-empirical or a priori representations of space and time do not in any way *exhaust* elementary or Peano arithmetic, especially including PRA, much less the rest of mathematics: nevertheless those representations are *presupposed by* Peano arithmetic, especially including PRA, and also by the rest of mathematics.

Second, I am interpreting the “causal contact” criterion in WCTI-based and Manifest Realist terms as either a direct sense perception of Hilbert-style stroke-constructions, *or any minimal-episodic-memory-based and imaginatively precisifiable sensible form in Kantian a priori intuition, etc., whatsoever, provided it has a veridical manifestly real spatiotemporal material natural structural basis*. Thus direct sense perception of the manifestly real material natural world gets us the evidential verifiers of mathematical beliefs or judgments, and veridical minimal episodic memory together with the productive imagination smoothly mediates between actual direct sense perception and formal essentially non-conceptual content, or Kantian a priori intuition.

According to Kantian Intuitionism, then, our knowing mathematical truths a priori by means of mathematical beliefs or judgments involves the very same sorts of essentially non-conceptual, pre-reflective, first-order conscious but also conceptually-driven cognitive activities as knowing factual truths by means of ordinary linguistic

perceptual judgments, in accordance with what Kant calls *Empirical Realism*, and what others have called “Direct Perceptual Realism,” or what I call *Radically Naïve Realism*.¹³⁴ In this way, our innately-specified conceptual capacity for constructing, understanding, and using the logical and mathematical parts of natural language, together with our innately-specified essentially non-conceptual capacity for direct sense perception and pre-reflective consciousness, via the cognitive construction and manipulation of veridical sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata, when conjointly triggered appropriately by the world of directly perceivable manifestly real material spatiotemporal objects in nature, and when correctly conjointly implemented by us, just *is* basic authoritative rationally intuitive mathematical objective a priori knowledge in the classical sense. That is, and more briefly: You authentically know some necessary mathematical truths objectively a priori in the classical sense in basic authoritative rational intuition when you are *both* essentially non-conceptually and pre-reflectively or first-order consciously and *also* conceptually self-consciously and self-reflectively thinking or talking about mathematics correctly, and *furthermore* the underlying mathematical structures of the manifestly real natural world uniquely satisfy the mathematical statements generated in your language of thought or in your outer speech.

As I mentioned in section **V**, elementary or Peano arithmetic is defined by the following five axioms:

- (1) 0 is a number.
- (2) The successor of any number is a number.
- (3) No two numbers have the same successor.
- (4) 0 is not the successor of any number.
- (5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers,

together with the primitive recursive functions (basic calculations or basic operations) over the natural numbers—the successor function, addition, multiplication, exponentiation, etc. But axiom (5) is not verifiable in an *essentially non-conceptual* way, and on the contrary requires the inherently *conceptual* and self-conscious or reflective ability to grasp denumerably infinitary quantifications over all the numbers. Nevertheless, given our basic or non-basic authoritative rational intuitive knowledge of all the true propositions or statements covered by the first *four* axioms, in the finitist sub-structure captured by PRA or simple arithmetic, via veridical sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata, then there is no need whatsoever for a further reply to epistemic skepticism, since High-Bar justified true belief has thereby been achieved. Therefore PRA or simple arithmetic, for Kantian, Brouwerian, Hilbertian, Husserlian, Wittgensteinian, and Parsonsian epistemological reasons, paradigmatically exemplifies basic or non-basic authoritative rational intuition, and also paradigmatically exemplifies authentic objective a priori knowledge, precisely because all the rational human abilities required to grasp it are grounded in an essentially non-conceptual and pre-reflectively or first-order conscious cognitive competence for knowing PRA or simple arithmetic via the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, etc., and also insofar as we can also have an occurrent conceptual, reflective, self-conscious grasp of PRA.

Although I cannot argue for this here and now, it seems to me there must *also* be precisely analogous paradigmatic exemplifications of basic or non-basic authoritative rational intuition and authentic objective a priori knowledge in Elementary Geometry,

Elementary Set Theory, and Elementary Logic—e.g., *minimal Euclidean* geometry (roughly, classical Euclidean geometry minus The Parallel Postulate), *basic* set theory (roughly, classical set theory minus The Naïve Comprehension Axiom),¹³⁵ and *first-order monadic* logic (roughly, classical first-order predicate logic minus polyadic predication and multiple quantification).¹³⁶ If so, then minimal Euclidean geometry, basic set theory, and first-order monadic logic, along with PRA or simple arithmetic, are *the essential starting points* of any adequate general theory of rational intuition and objective a priori knowledge.

It is crucial to note that the scope of a priori knowledge in mathematics and logic, not to mention a priori knowledge in morality and philosophy, far exceeds the scope of basic or non-basic authoritative rational intuition. For example, a priori knowledge in non-Euclidean geometry and topology, Zermelo-Fraenkel set theory, and classical first-order polyadic logic, not even to mention the more recondite kinds of mathematics and logic—is either constructedly rationally intuitive or else inferential, conceptual, and at best fairly reliable, and of course also still defeasible or fallible. But, given **The KBH**, all *non-authoritative* and at best *fairly* reliable mathematical and logical a priori knowledge nevertheless presupposes the *fully* reliable and *authoritative* rationally intuitable parts of mathematics and logic, and constantly explicitly or implicitly draws upon them as it carefully advances from the less easily challenged, virtually uncontested, and more epistemically secure domains, towards the more challengeable, more contested, and less epistemically secure domains. This epistemic advance is beautifully symbolically mirrored in the situation of Adam and Eve as they leave Paradise at the end of *Paradise*

Lost, with a hard-won awareness of what is and what is not really possible for rational animals like us, in our “human, all too human” condition:

They looking back, all the eastern side beheld
Of Paradise, so late their happy seat,
Waved over by that flaming brand, the gate
With dreadful faces thronged a fiery arms.
Some natural tears they dropped, but wiped them soon;
The world was all before them, where to choose
Their place of rest, and Providence their guide.
They hand in hand with wandering steps and slow,
Through Eden took their solitary way.¹³⁷

We can now see, I think, that Kantian Intuitionism is logically consistent, coherent, theoretically elegant, and also fully vindicated by a philosophical inference-to-the-best-explanation, as well as by a *transcendental* explanation (see section **X** for details). This can be shown in four steps. **First**, we take the innately-specified psychological capacities included in ordinary human sense perception and ordinary human linguistic cognition seriously, especially including episodic memory and imagination. **Second**, we take contemporary mathematical science and natural science seriously. **Third**, we accept Kantian Structuralism and Weak or Counterfactual Transcendental Idealism (WCTI). **Fourth** and finally, if Kantian Structuralism and WCTI are both true, then Kantian Intuitionism is also true, precisely because our actual world of directly perceivable manifestly real material spatiotemporal objects intrinsically carries with it and necessarily implements the abstract formal denumerable infinitary structures of the system of elementary or Peano arithmetic, especially including the finitist sub-structures of Primitive Recursive Arithmetic or PRA, and also its conservative non-denumerably infinitary extensions such as Cantorian or transfinite arithmetic, and thus essentially non-conceptually or directly referentially perceptually presents the system of natural numbers, i.e., the intended model of Peano arithmetic, to any rational human

animal who is also cognitively competent in the mathematical parts of her own natural language. Therefore, Kantian Intuitionism is the best overall explanation of mathematical knowledge.

IX. The Extended Benacerraf Dilemma

These considerations bring us up to the problem: In what sense is logic something sublime? For there seemed to pertain to logic a peculiar depth—a universal significance. Logic lay, it seemed, at the bottom of all the sciences.—For logical investigation explores the nature of all things. It seeks to see to the bottom of things and is not meant to concern itself whether what actually happens is that or that.—It takes its rise, not from an interest in the fact of nature, nor from a need to grasp causal connexions: but from an urge to understand the basic, or essence, of everything empirical.

--L. Wittgenstein¹³⁸

It is easy enough to extend BD to logic, and thereby raise the fundamental philosophical problem so evocatively identified by the later Wittgenstein: “In what sense is logic something sublime?” One need only substitute ‘logical’ for every occurrence of ‘mathematical’ in the original Benacerraf Dilemma, as follows, with the relevant substitutions boldfaced:

- (1) Natural language requires a standard, uniform semantics of truth. Hence: Truth is uniform and Tarskian. (Preliminary assumption I.)
- (2) A reasonable epistemology of cognizing true (**logical**) statements should be modelled on human sense perception. Hence: All human knowledge begins in causally-triggered sense experience. (Preliminary assumption II.)
- (3) **Logical** knowledge in a classical sense (i.e., as a priori knowledge of objectively necessary truth) exists as a basic feature of standard **logical** linguistic practices, so **logical** truth in a classical sense (i.e., as objectively necessary truth) also exists as a basic feature of those standard practices.
- (4) Given (1) and (3), our standard, uniform semantics of truth in natural language, as applied to **logical** truths, commits us to a truth-making ontology of abstract **logical** objects and also to the non-empirical knowability of true **logical** statements.
- (5) On the one hand, given (2), that fact that a reasonable epistemology of cognizing true (**logical**) statements should be modelled on human sense perception entails that knowledge involves causally efficacious, contact-involving or efficient, directly referential, non-inferential, and spatiotemporal relations between human linguistic knowers and the known objects themselves.
- (6) But on the other hand, given (4), and since all abstract objects are causally isolated and inert, it then follows that all abstract **logical** objects are causally isolated and inert.
- (7) So if we accept all of (1) – (6), then **logical** knowledge in the classical sense is both possible and impossible, which is absurd.

For convenience, I will call this sublimity-of-logic problem *The Extended Benacerraf Dilemma*. While it is easy enough to generate The Extended Benacerraf Dilemma, sadly, it is not so very easy to solve it. In the next section I will argue, **first**, that necessarily logic is transcendently ideal, and **second**, that Kantian Structuralism and Kantian Intuitionism can be smoothly extended from mathematics to logic and thereby solve The Extended Benacerraf Dilemma. If I am correct, then this solution to The Extended BD shows us that logic *really is* sublime in a precisely characterizable way, and that logic is sublime in this way *just insofar as* it is transcendently ideal, but *not* otherwise.

X. Why Logic Must Be Transcendental

[The logic of the general use of the understanding] contains the absolutely necessary rules of thinking, without which no use of the understanding takes place, and it therefore concerns these rules without regard to the difference of the objects to which it may be directed.... Now general logic is either pure or applied logic. In the former we abstract from all empirical conditions under which our understanding is exercised.... A **general** but **pure** logic therefore has to do with strictly *a priori* principles, and is a **canon of the understanding** and reason, but only in regard to what is formal in their use, be the content what it may.... A **general logic**, however, is called applied if it is directed to the rules of the use of the understanding under the subjective empirical conditions that psychology teaches us.... In general logic the part that is to constitute the pure doctrine of reason must therefore be entirely separated from that which constitutes applied (though still general) logic. The former alone is properly science.... In this therefore logicians must always have two rules in view. 1) As general logic it abstracts from all contents of the cognition of the understanding and of the difference of its objects, and has to do with nothing but the mere form of thinking. 2) As pure logic it has no empirical principles, and thus draws nothing from psychology.... It is a proven doctrine, and everything in it must be completely *a priori*.

--I. Kant (*CPR* A52-54/B76-78)

Logic is not a theory but a reflexion of the world. Logic is transcendental.

--L. Wittgenstein¹³⁹

X.1

Both Kant and early Wittgenstein held the perhaps surprising thesis *that logic is transcendental*. I will call this *The L-is-T Thesis*. The L-is-T Thesis says:

Logic is objectively necessarily true, a priori, knowable by means of basic or non-basic authoritative rational intuition, and also *transcendentally explains* (a.k.a. "is the condition of the possibility of") all rational human cognition and thought.

Here, in turn, is the relevant notion of a *Transcendental Explanation*, via the preliminary notion of a *Transcendental Argument*.

An *argument* is a set of sentences or statements Γ (and possibly $\Gamma =$ the null set of sentences or statements), i.e., the premises, such that a sentence or statement S (which may or may not be a member of Γ), i.e., the conclusion, is held to follow validly or soundly from Γ . Then an argument is a *Transcendental Argument* if and only if

(i) Some version of Transcendental Idealism, whether Strong Transcendental Idealism (STI) or Weak or Counterfactual Transcendental Idealism (WCTI), is assumed to be true., and

(ii) That argument advances from a sentence or statement S , taken as a single premise, to an a priori necessary presupposition $APNP$ of S —i.e., “a condition of the possibility” of S —taken as a single conclusion, as follows:

- (1) S
- (2) S presupposes $APNP$.
- (3) Therefore, $APNP$.

For example, let S = “There are 7 martinis sitting on the kitchen table” and let $APNP$ = “ $3+4=7$ and **The Reliability of Rational Intuitions in Simple Arithmetic**”:

At least some of the truths of Primitive Recursive Arithmetic, or PRA, are authentically knowable a priori by fully reliable basic authoritative rational intuitions on the basis of Hilbert-style “objects of finitary reasoning,” i.e., on the basis of our cognitive construction and manipulation of sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata.”

- (1) There are 7 martinis sitting on the kitchen table., e.g.,

Y Y Y Y Y Y Y

- (2) The sentence or statement that there are 7 martinis sitting on the kitchen table presupposes the a priori necessary truth that $3+4=7$ and **The Reliability of Rational Intuitions in Simple Arithmetic**. For if it were *not* the case that $3+4=7$ holds as a paradigmatic instance of PRA that is authentically known by fully reliable basic authoritative rational intuition, that is, if it were *not* the case that the primitive recursive functions over the natural numbers, like addition, are known to hold by fully reliable basic authoritative rational intuitions, then it would be *neither* true that there are 7 martinis sitting on the kitchen table *nor* false that there are 7 martinis sitting on the kitchen table.

- (3) Therefore, $3+4=7$ and **The Reliability of Rational Intuitions in Simple Arithmetic**. (From (1) and (2).)

An $APNP$ can be either analytic a priori (indeed, trivially, every analytic truth is presupposed by every sentence or statement whatsoever) or synthetic a priori, but in either case it is known by basic authoritative philosophical rational intuition.

In turn, an *explanation* is a set of sentences or statements Γ (and Γ cannot be the null set of statements) and another sentence or statement S (which cannot be a member of Γ , on pain of circularity), such that some sort of necessitation relation is held to obtain between Γ and S , i.e.,

$$\square (\Gamma \rightarrow S]$$

Then an explanation is a Transcendental Explanation if and only if there is an a priori necessary presupposition *APNP* of a sentence or statement *S* such that *APNP*, when taken together with some or another set of true general and specific claims (*C1, C2, C3 ...Cn*) derived from natural science and/or Wide Reflective Equilibrium in the Rawlsian sense, is also related to *S* in the following way:

$$\text{Syn Ap } \square [\{APNP \& (C1, C2, C3 \dots Cn)\} \square \rightarrow S]$$

or in other words,

Synthetically a priori necessarily, if *APNP* and also some or another set of general and specific claims (*C1, C2, C3 ...Cn*) derived from natural science and/or Wide Reflective Equilibrium in the Rawlsian sense all *were to be* true, then *S would be* true.

Thus a sound Transcendental Explanation demonstrates an synthetic a priori subjunctive conditional relation between an *APNP*, which is known by basic authoritative philosophical rational intuition, and an *S*, which is known by any other reliable method of knowledge, via some body of fundamental knowledge claims provided by natural science and/or Wide Reflective Equilibrium in the Rawlsian sense. Otherwise put, a sound Transcendental Explanation demonstrates that *APNP* is “*the* condition of the possibility” of *S*.

For example, let *S* = “There are 7 martinis sitting on the kitchen table,” let *ANPP* = “3+4=7 and **The Reliability of Rational Intuitions in Simple Arithmetic**,” and let “(C1, C2, C3 ... Cn)” be a set of relevant general and specific claims taken from natural science and Wide Reflective Equilibrium in the Rawlsian sense, about martinis, tables, their causal-dynamic relations, and the nature of the sitting-on relation. Then the following is a sound Transcendental Explanation:

(1) There are 7 martinis sitting on the kitchen table., e.g.,



(2) Synthetically a priori necessarily, if “ $3+4=7$ and **The Reliability of Rational Intuitions in Simple Arithmetic**” and also a set of relevant general and specific claims about martinis, tables, their causal-dynamic relations, and the nature of the sitting-on relation, all *were to be true*, then it *would* be true that there are 7 martinis sitting on the kitchen table.

Now Kant held The L-is-T Thesis because he held that pure general logic is the strictly universal and a priori science of the laws of thought. Early Wittgenstein, by a significant contrast, held The L-is-T Thesis because he held that the classical second-order logic of Frege’s *Begriffsschrift*, and Russell and Whitehead’s *Principia Mathematica*, is built into the very nature of my language and also into the very nature of the world my language represents.

I fully agree with Kant and early Wittgenstein that The L-is-T Thesis is true. But two things about the The L-is-T Thesis are quite obscure in Kant’s and early Wittgenstein’s writings in philosophical logic:

- (1) Precisely which argument, or arguments, can adequately justify The L-is-T Thesis?., and
- (2) Precisely what are the basic implications of The L-is-T Thesis?

In the next sub-section I will present five arguments for The L-is-T Thesis, and also spell out their basic implications, which include Kantian Structuralism and Kantian Intuitionism about logic. Then in sub-section **X.3** I will show how The L-is-T Thesis solves The Extended Benacerraf Dilemma.

X.2

Argument 1: First-Order Monadic Logic and Pure General Logic are both Transcendental

The first argument is intended to show that both first-order monadic logic and pure general logic are, in addition to being objectively necessarily true, also a priori

necessary presuppositions (*APNPs*) of all rational human cognition and thought, hence “transcendental” in the sense specified in sub-section **X.1**.

It is both relevant and important to note that as early as C.I. Lewis’s seminal 1918 book *Survey of Symbolic Logic*, there was a fundamental distinction in the 20th century logical tradition between

(i) *formal* or *symbolic* logic, which is essentially a rigorous development of Kant’s notion of pure general logic, and

(ii) what Russell aptly called *mathematical* logic, which is *second-order* because it includes whatever logical or semantic machinery is needed to quantify over and talk about functions, predicates, and relations, and also other characteristically mathematical furniture like sets, numbers, and spaces.¹⁴⁰

The reason that this distinction is philosophically important is that for Kant, it is also possible to have a pure or completely a priori logic that is topic *specific*, or systematically sensitive to special ontological commitments, which is what he calls *transcendental* logic (*CPR* A62/B87). Strikingly, early Wittgenstein seems to have had, in effect, the very same idea about transcendental logic in the *Tractatus*, as we saw in this section’s second epigraph:

Logic is not a theory but a reflexion of the world. Logic is transcendental.¹⁴¹

In this way, *mathematical* logic in Russell’s sense would count as a transcendental logic for both Kant and the Tractarian Wittgenstein.

Transcendental logic in Kant’s sense, however, also inherently contains *necessarily true synthetic a priori statements*, which would not have been allowed by Wittgenstein in his Tractarian period. Nevertheless, from a Kantian standpoint, it seems that if early Wittgenstein had admitted necessarily true synthetic a priori statements into *his* transcendental logic, then this would have made it possible for him to provide a coherent account for the logico-semantic status of the infamous Two Colours Proposition,

or The TCP. Here is what early Wittgenstein says explicitly about The TCP in the

Tractatus:

For two colours ... to be at one place in the visual field, is impossible, logically impossible, for it is excluded by the logical structure of colour.¹⁴²

In this way, early Wittgenstein regards The TCP—i.e., “For two colours ... to be at one place in the visual field, is impossible”—as a logical truth of elementary logic. But this forces him into the dilemma of *either* giving up the logical independence of atomic propositions—e.g., the logical independence of the atomic propositions

(R) Point *P* in visual space is red all over., and

(G) Point *P* in visual space is green all over.

—*or else* devising some analysis of propositions like (R) and (G) which smoothly converts them and all their analogues into complex or molecular propositions, in order to be able to assert that that the obvious mutual exclusion relation between (R) and (G) is a purely logical relation. But for early Wittgenstein, facing up to this dilemma also means giving up his account of the nature of logic and logical analysis in the *Tractatus*, which is precisely what post-Tractarian Wittgenstein more or less explicitly does in 1929 in “Some Remarks on Logical Form,” by claiming that atomic propositions *can* be mutually logically contradictory,¹⁴³ and then by later observing to Waismann that this move in fact leads to absurdity:

Now suppose the statement “An object cannot be both red and green” were a synthetic judgment and the words ‘can not’ meant logical impossibility. Since a proposition is the negation of its negation, there must also exist the proposition, “An object can be red and green.” This proposition must also be synthetic. As a synthetic proposition it has sense, and this means that the state of things represented by it *can obtain*. If ‘can not’ means *logical* impossibility, we therefore reach the consequence that the impossible *is* possible.¹⁴⁴

From a Kantian standpoint, however, it seems to me obvious that the correct way out of this dilemma is to allow for two *essentially different* kinds of necessity, namely,

(1) analytic or logical a priori necessity, i.e., the necessity which flows from the nature of concepts (a.k.a. “conceptual necessity”), and

(2) synthetic or non-logical a priori necessity, i.e., the necessity which flows from the nature of things in the world (a.k.a. “essentially non-conceptual necessity”),

which is the same as to hold the thesis of *Modal Dualism*. Given Modal Dualism, and given the fact that impossibility is definable in terms of necessity and negation, one can coherently hold that (R) and (G) are logically independent propositions and yet also non-logically mutually exclusive propositions, by holding that the mutual exclusion relation between them is one of *synthetic or non-logical a priori impossibility*, not analytic or logical a priori impossibility.

In any case, as I have mentioned already, Kant holds that the truths of arithmetic and geometry are synthetic a priori, not analytic. One reason he does so is because he thinks that the representational content of mathematics rests on logic *plus* our a priori representations of the formal structures of asymmetrically-directional time (for the purposes of representing Primitive Recursive Arithmetic, or PRA, and its conservative extensions, including Peano Arithmetic¹⁴⁵) or orientable 3-D Euclidean space (for the purposes of representing Euclidean geometry and its conservative extensions, including classical Non-Euclidean geometry¹⁴⁶). But another, and ultimately equivalent, way of expressing the synthetic apriority of arithmetic and geometry is to point out that the logic which represents them must contain irreducibly *relational* predicates whose satisfaction conditions require the existence of at least one object in the actual world (e.g., in the case of identity) or otherwise the existence of at least two objects in the actual world, and in some cases (e.g., the case of the relational predicates needed to represent the standard Peano axioms for arithmetic) the existence in the actual world of at least a denumerably infinite number of objects. Thus all the logical truths of the first-order, inherently

polyadic, and multiply-quantified part of Frege's logic—i.e., classical first-order predicate logic with identity—in Kantian (or at least, contemporary Kantian) terms, are *synthetic a priori*, not analytic.

Frege's logic includes set theory, as well as an axiom that allows for the unrestricted formation of sets, and of course it leads directly to Russell's Paradox about the logically explosive (a.k.a. "impredicative") status of the set *K* of all sets that are not members of themselves, whose existence yields the unhappy paradoxical result that *K* is a member of itself if and only if it is not a member of itself. Russell's mathematical logic includes a principle—*The Vicious Circle Principle*—which stipulatively rules out the impredicativity that afflicts unconstrained iterative set theory.¹⁴⁷ But Russell's mathematical logic also includes something called *The Axiom of Infinity*, which posits the existence of at least a denumerably infinite number of objects in the domain of discourse, and which is arguably not a purely logical principle. Moreover, and in any case, Russell's mathematical logic still threatens to allow for paradoxical impredicativity with respect to *functions, predicates, and relations*, even if it stipulatively rules out impredicative sets, unless one makes a further empirical and clearly non-logical assumption Russell calls *The Axiom of Reducibility*.¹⁴⁸

In other words, the crucial issue here is whether the rational core of classical logic should be taken to be second-order logic in either the Fregean or Russellian sense, or instead is *elementary logic*: i.e., bivalent first-order polyadic predicate calculus with identity.¹⁴⁹

Tarski, e.g., both emphatically and explicitly supported the thesis that elementary logic, not second order logic, is the core classical logic:

The terms ‘logic’ and ‘logical’ are used [by most contemporary logicians] in a broad sense, which has become almost traditional in the last decades; logic is here assumed to comprehend the whole theory of classes and relations (i.e., the mathematical theory of sets). For many different reasons I am personally inclined to use the term ‘logic’ in a much narrower sense, so as to apply it only to what is sometimes called “elementary logic,” i.e., to the sentential calculus and the (restricted) predicate calculus.¹⁵⁰

But even *elementary logic* contains some arguably non-logical factors. For example, since

$$(1) a=a$$

is an instance of the law of identity and can be introduced into any line of a proof as a theorem of logic, and thus as depending on the empty set of premises, it follows immediately that

$$(1) (\exists x) x = x$$

which says *that something exists*, is *also* a theorem of logic, which seems highly implausible. Why couldn’t there be logically possible worlds that with no individual objects in them (i.e., the empty domain of discourse); and furthermore, why couldn’t there be logically possible worlds in which *nothing whatsoever* exists?¹⁵¹

Quine, significantly, holds that identity is indeed *part* of the rational core of classical logic, yet also *excludes* set theory from this core:

The upshot is, I feel, that identity theory has stronger affinities with its neighbors in logic than with its neighbors in mathematics. It belongs in logic.

We turn now from identity to set theory. Does it belong in logic? I shall conclude not.¹⁵²

By sharp contrast, for Kantians, both Frege’s logic and also Russell’s mathematical logic, and indeed *any* logic that is an inherently relational or polyadic logic and also includes identity, hence elementary logic, and also any logic that includes set theory, and any logic that is a second-order logic more generally, will all count as synthetic a priori *transcendental* logics, not pure general logics, precisely because they all include special ontological commitments that take them significantly beyond the scope of

pure general logic. To the same effect, in the specific case of set theory, Quine accurately and aptly points up the significant philosophical advantages of Kant's pure general logic over Frege's logic:

Altogether, the contrasts between elementary logic and set theory are so fundamental that one might well limit the word 'logic' to the former... and speak of set theory as mathematics in a sense exclusive of logic. To adopt this course is merely to deprive 'ε' of the status of a logical word. Frege's derivation of arithmetic would then cease to count as a derivation from logic; for he used set theory. At any rate we should be prepared to find that [Carnap's] linguistic doctrine of logical truths holds for elementary logic and fails for set theory, or vice versa. Kant's readiness to see logic as analytic and arithmetic as synthetic, in particular, is not superseded by Frege's work (as Frege supposed), if "logic" be taken as elementary logic. And for Kant logic certainly did not include set theory.¹⁵³

And basically the very same points could be made for the comparison and contrast between Kant's logic and Russell's mathematical logic, just by uniformly substituting 'Russell' for 'Frege' and 'second-order logic' for 'set theory' in that quotation from Quine.

This brings me to the heart of the matter. Kant thinks of *pure general logic* as the core classical logic because it is analytic, a priori, and strictly universal but also more fundamentally because it bears no burden of ontology, and holds equally for empty domains of discourse, and for worlds with nothing whatsoever in them, as well as for occupied domains, and worlds containing sets, functions, or relations. Now Kant's pure general logic, as it happens, is a *second-order intensional monadic logic*. It is second-order and intensional because it both includes and quantifies over finegrained, decomposable concepts, as well as possible-worlds extensions.¹⁵⁴ By another sharp contrast, Quine's and Tarski's elementary logic is an extensional logic, and not an intensional logic; moreover, elementary logic is also inherently polyadic or relational, and it includes identity. Nevertheless, where Kant's pure general logic and elementary logic *fully overlap* is precisely in *first-order monadic logic*, which is bivalent truth-

functional logic together with a restricted predicate logic employing quantification over individuals and into one-place predicates only.¹⁵⁵ In empty domains, or in completely empty possible worlds, *first-order monadic logic collapses to truth-functional logic*.

Therefore, if we zero in on first-order monadic logic and explicitly take into account how it collapses into truth-functional logic in empty domains and empty worlds, it follows that in first-order monadic logic we have before us an ultra-pasteurized version of Kant's pure general logic that is also the perfect candidate for being "sheer logic" in Quine's sense:

If sheer logic is not conclusive, what is? What higher tribunal could abrogate the logic of truth functions or of quantification?¹⁵⁶

In part, this is because of the following highly significant historical intersection of doctrines in the philosophy of logic:

- (1) Kant implicitly accepts first-order monadic logic as belonging to the rational core of classical logic.,
- (2) Frege implicitly accepts first-order monadic logic as belonging to the rational core of classical logic.,
- (3) Russell implicitly accepts first-order monadic logic as belonging to the rational core of classical logic.,
- (4) The Tractarian Wittgenstein implicitly accepts first-order monadic logic as belonging to the rational core of classical logic.,
- (5) Tarski implicitly accepts first-order monadic logic as belonging to the rational core of classical logic., and
- (6) Quine implicitly accepts first-order monadic logic as belonging to the rational core of classical logic.

Furthermore, as Quine implicitly showed us, first-order monadic logic is also *the paradigm of logical analyticity*. Therefore first-order monadic logic, as being logic in a way about which Kant, Frege, Russell, early Wittgenstein, Tarski and Quine could all fully agree, is *pure general, paradigmatically analytic, core classical, "sheer" logic*.

Indeed, when we realize that it was *precisely* the pure generality, paradigmatic analyticity, core classicality, and sheerness of first-order monadic logic that Kant implicitly had in mind when he wrote

That from the earliest times **logic** has traveled this secure course [of a science] can be seen from the fact that since the time of Aristotle it has not had to go a single step backwards.... What is further remarkable about logic is that until now it has also been unable to take a single step forward, and therefore seems to all appearances to be finished and complete. (*CPR* Bxviii-xix),

then we can clearly see that Kant's notorious remark was entirely apt, arguably self-evidently true, and precisely the reverse of outrageous.

Following out Kant's deep thoughts about the nature of pure general logic and (implicitly) first-order monadic logic, then, let us call the pure logical properties of truthful consistency, soundness, completeness, decidability, and logical truth or analyticity *The Logical Perfections*. As in standard treatments of contemporary logic, consistency is the property of the formal non-contradictoriness of statements, or alternatively the property of there being at least one interpretation in which all members of a given set of statements are true (a.k.a. the set of statements "has a model"). Soundness is the property such that all provable sentences or theorems in a logical system are logically true or tautologous. Completeness is the property such that all tautologies are theorems, or provable sentences. And decidability is the property such that there is a finite recursive procedure for determining tautologousness. By the perhaps slightly unfamiliar notion of the *truthful consistency* of given logical system Σ , moreover, I specifically mean that:

(i) Σ never includes arguments that lead from true premises to false conclusions (= truth-preservation)., and

(ii) Σ never includes contradictions as theorems of logic (= non-dialetheism—i.e., no "truth-value gluts" or "true contradictions" allowed).

We can think of truthful consistency as the *Highest or Supreme Good* of logic, and we can also think of this systemic feature together with all the other Logical Perfections as proper parts of the *Complete Good* of logic.

The Logical Perfections collectively specify the standards of High-Bar rational normativity for logic. But it is also true that each of The Logical Perfections is not independently essential to logic. *Dialetheic paraconsistent* logical systems are possible,¹⁵⁷ in which contradictions can occur as true sentences or statements or even as theorems of logic (= dialetheism), and such systems are thereby not truthfully consistent, provided that the system also contains an axiom that prevents every sentence or statement whatsoever from being entailed by any given contradiction (= paraconsistency), a logical phenomenon that is called “Explosion.” For example, arguably both The Liar Sentence (which asserts its own falsity)¹⁵⁸ and The Gödel Sentence (which provably asserts its own unprovability)¹⁵⁹ are true contradictions, and these true contradictions can arguably be allowed into logical systems as true sentences or even theorems, provided that Explosion is ruled out.

Correspondingly, some logical systems are not sound, e.g., dialetheic paraconsistent systems. Some logical systems are sound but not complete, e.g., elementary logic plus the standard Peano axioms for arithmetic. And some logical systems are undecidable, e.g., elementary logic. As Gödel showed, undecidability and indeed also logical unprovability both apply to some individual true statements in any formal system rich enough to contain elementary logic plus (enough of) the standard Peano axioms for arithmetic, and such systems are consistent if and only if they are incomplete and have their ground of truth outside the system. Decidability on its own,

however, can also apply to a formal system consisting entirely of what Kant would have regarded as irreducibly synthetic a priori truths, e.g., the truths of PRA.

More generally, it is only in the context of a logic of *analyticity* that decidability closes the tight systemic circle of all The Logical Perfections. Indeed, when we see that the tight circle of The Logical Perfections can actually be exemplified in at least two logics—i.e., either classical *truth-functional logic* or *first-order monadic logic*, both of which are truthfully consistent, sound, complete, decidable, and analytic—then we realize that each of these logics constitutes a *maximal, ideal, or High-Bar normative standard* of rational systematicity. This maximal, ideal, or High-Bar rational normative standard, as Kant points out, necessarily guides all rational and scientific inquiry in a *regulative* way. But this ideal must *not* also be regarded as *constitutive* in Kant’s sense. For the tragically mistaken thesis that the maximal, ideal, or High-Bar rational normative standard realized by classical truth-functional logic or first-order monadic logic applies to any *other* set of statements or body of knowledge will inevitably lead to fundamental metaphysical errors and insoluble logical paradoxes and puzzles, as The Transcendental Dialectic clearly shows in great detail (*CPR* A293-A704/B349-732).

In the Introduction to the *Jäsche Logic*, Kant himself uses the term “logical perfections” (*logische Vollkommenheiten*) in essentially the same way I have just used it (*JL* 9: 33-81). But Kant of course did not know about meta-logic. Now since Kant did not know about meta-logic, he also did not know that the first-order monadic logic that is embedded in his pure general logic is truthfully consistent, sound, complete, and decidable, although he did of course (at least implicitly) know that first-order monadic logic is analytic a priori, since (again, at least implicitly) he knew that second-order

intensional monadic logic is analytic a priori. Strikingly, and by contrast, classical first-order predicate logic with polyadic predicates and multiple quantification is truthfully consistent, sound, and complete, but *not* decidable, and (as we have seen) *not* analytic.

What are we to make of the fact that first-order monadic logic—or logic in a sense that Kant, Frege, Russell, Tarski, and Quine all implicitly but fully affirm as belonging to the rational core of classical logic—is provably truthfully consistent, sound, complete, decidable, *and also* analytic a priori? One possibility is that first-order monadic logic is the logic *which best captures our most unshakeable and thus authoritative “obviousness” intuitions¹⁶⁰ about logical analyticity in natural language*. Indeed, even Quine himself implicitly admits this, which can be easily enough seen by recalling his initial definition of analyticity, adding one minor qualifier to his famous remark about “sheer logic,” and then juxtaposing these two seminal Quinean texts:

[Analytic statements] fall into two classes. Those of the first class, which may be called *logically true*, are typified by:

- (1) No unmarried man is married.

The relevant feature of this example is that it not merely is true as it stands, but remains true under any and all reinterpretations of ‘man’ and ‘married’. If we suppose a prior inventory of *logical* particles, comprising ‘no’, ‘un-’, ‘not’, ‘if’, ‘then’, ‘and’, etc., then in general a logical truth is a statement which is true and remains true under all reinterpretations of its components other than the logical particles.¹⁶¹

If sheer logic is not conclusive, what is? What higher tribunal could abrogate the logic of truth functions or of [monadic] quantification?¹⁶²

Notice here that all analytic statements of the same form as “No unmarried man is married” involve first-order *monadic* quantification only. And not only the logic of truth functions but also the logic of first-order monadic quantification each counts as *conclusive, sheer* logic. But first-order monadic logic is the logic of truth functions plus the logic of first-order monadic quantification. So according to Quine, at least implicitly,

first-order monadic logic must be the logic which best captures our most unshakeable and thus authoritative “obviousness” intuitions about logical analyticity in natural language.

Now if first-order monadic logic is the logic which best captures our most unshakeable and thus authoritative “obviousness” intuitions about logical analyticity in natural language, then it is arguable that pure general logic, insofar as it inherently contains first-order monadic logic, along with fine-grained, decomposable intensions and possible-worlds extensions, is the *Universal Natural Logic* of human natural languages insofar as it best captures our most unshakeable and thus authoritative “obviousness” intuitions about *all kinds of analyticity* in natural language, just as Chomsky’s *Universal Grammar* best captures our most unshakeable and thus authoritative “obviousness” intuitions about *all kinds of grammaticality* in natural languages.¹⁶³

Here we need also to consider a distinct although, ultimately, closely related point. One of the great advances of 20th century logic was the discovery and development of non-classical logics. Non-classical logics are of two distinct kinds:

- (i) *extended logics*, which preserve all the tautologies, theorems, inference rules, syntactic rules, and semantic rules of classical logic, but add some new ones, and
- (ii) *deviant logics*, which reject some of the tautologies, theorems, inference rules, syntactic rules, or semantic rules of classical logic, and may also add some new ones.¹⁶⁴

Extended non-classical logics are *conservative*, while deviant non-classical logics are *radical*. For example, second-order logic and classical modal logic are extended logics, whereas Intuitionist logic (which rejects the universal principle of excluded middle, or PEM) and dialetheic paraconsistent logic (which as I mentioned above, rejects the universal principle of non-contradiction, or PNC, and accepts the existence of “truth-value gluts” or “true contradictions,” provided that it also contains an axiom that it rules

out the entailment of every statement whatsoever by any given contradiction, a.k.a. “Explosion”) are deviant logics.

Given the distinction between extended and deviant non-classical logics, and assuming the plausibility of my earlier claim that pure general logic, insofar as it inherently contains first-order monadic logic together with finegrained, decomposable intensions and possible-worlds extensions, is the logic which best captures our most unshakeable and thus authoritative “obviousness” intuitions about all kinds of analyticity in natural language, and is arguably the Universal Natural Logic of all natural languages, then I think that we can now also see that pure general logic plausibly arguably captures *the a priori essence of logic*, in the threefold sense that

- (i) Synthetically a priori necessarily, if *anything* counts as a logic, then pure general logic, insofar as it inherently contains first-order monadic logic, will count as a logic.,
- (ii) Synthetically a priori necessarily, if *anything* is either an extended or a deviant logic, then it is nothing but either a conservative extension or a deviant of pure general logic, insofar as it inherently contains first-order monadic logic., and
- (iii) Plausibly arguably, the conservative extension of first-order monadic logic to pure general logic captures the a priori essence of *logical analyticity*, since pure general logic is just second-order intensional monadic logic and plausibly arguably best captures our most unshakeable and thus authoritative “obviousness” intuitions about all kinds of analyticity in natural language.

Argument 2: The Absolute Unrevisability Argument

The second argument is intended to show that at least one logical principle which is fully presupposed by first-order monadic logic and pure general logic alike is itself absolutely unrevisable, and therefore, in addition to being objectively necessarily true, is also an a priori necessary presupposition (*APNP*) of all rational human cognition and thought, and thus is also “transcendental” in the sense specified in sub-section **X.1**. Here is that logical principle in its alethic version:

Not every sentence or statement in every language or logical system is both true and false (**The Minimal Logical Meta-Principle of Non-Contradiction**), i.e.,

$$\sim (\forall S) (S \ \& \ \sim S)$$

The Minimal Logical Meta-Principle of Non-Contradiction also has a deontic version formulated as a *logical categorical imperative*:

You categorically ought to accept as truths in any language or logical system only those sentences or statements which do not entail that it and all other sentences or statements in that language or logical system are both true and false.

This logical categorical imperative version of The Principle, in turn, guarantees what I will call *minimal truthful consistency*. Truthful consistency, as such, means that you must accept as truths in a language or logical system only those sentences or statements which do not entail that *any* argument in that language or system leads from true premises to false conclusions. By contrast, *minimal truthful consistency* means that you must accept as truths in any language or logical system only those sentences or statements which do not entail that *every* argument in that language or system leads from true premises to false conclusions. This latter notion of course is consistent with holding that *some* arguments in that language or system lead from true premises to false conclusions, and indeed is also consistent with holding that some arguments in the language or system lead from the *null* set of premises to *necessarily false* conclusions. If so, then some sentences or statements in that language or system are both true and false, hence are truth-value gluts or “true contradictions.” So minimal truthful consistency is consistent with dialethic paraconsistency.¹⁶⁵ In other words, then, The Minimal Logical Meta-Principle of Non-Contradiction essentially secures minimal truthful consistency, and rules out Explosion. It is not a strictly *truth*-preserving logical principle, and not even a strictly *consistency*-preserving logical principle, but it nevertheless strictly rules out

global inconsistency, i.e., logical anarchy or chaos, which is the ultimate result of Explosion: If every sentence or statement whatsoever follows from a contradiction, then the negation of every sentence or statement whatsoever also follows from a contradiction, and therefore every sentence or statement whatsoever is a truth-value glut or true contradiction.¹⁶⁶

In the 1980s, Hilary Putnam very plausibly argued that the negative version of this minimal logical meta-principle is the one absolutely indisputable a priori truth:

I shall consider the weakest possible version of the principle of [non-] contradiction, which I shall call the minimal principle of [non-] contradiction. This is simply the principle that *not every statement is both true and false...* [I]f, indeed, there are no circumstances in which it would be rational to give up our belief that *not every statement is both true and false*, then there is at least one *a priori truth*.¹⁶⁷

Although the 1980s Putnam apparently holds a sharply different theory of apriority from mine—indeed, arguably, he holds The Pragmatic or Quinean Conception of the A Priori, as opposed to my Kantian or strong-supervenience-failure, or strict underdetermination conception of the a priori (see section **IV**)—nevertheless our accounts do converge perfectly on the *transcendental* logico-semantic status of the statement that $\sim (\forall S) (S \ \& \ \sim S)$, i.e., on the transcendental logico-semantic status of The Minimal Logical Meta-Principle of Non-Contradiction.

Argument 3: The Logocentric Predicament Argument

The third argument is intended to show that, if the two first two arguments are sound and if I am correct that first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all “transcendental” in the sense specified in sub-section **X.1**, then this “transcendental fact” can be used to provide an adequate solution to the very hard philosophical problem of *The Logocentric*

Predicament. So we can then conclude that logic is transcendental *by an inference to the best philosophical explanation*.

The Logocentric Predicament is this: How can logic ever be justified or explained, if logic must be presupposed and used in order to justify or explain logic? As I mentioned in section V, this problem is essentially the same as the one that the Harvard logician Harry Sheffer—known best for his discovery of the Sheffer stroke function—called “the logocentric predicament” in a 1926 review of the second edition of *Principia Mathematica*:

The attempt to formulate the foundations of logic is rendered arduous by a ... “logocentric” predicament. In order to give an account of logic, we must presuppose and employ logic.¹⁶⁸

In 1895 Lewis Carroll had pointed up a closely related worry in “What the Tortoise Said to Achilles,” by arguing that the attempt to generate the total list of premises required to validly deduce the conclusion of an argument leads to a vicious regress.¹⁶⁹ Carroll’s argument was resuscitated in 1936 by Quine in “Truth by Convention,” where he pointed out that the attempt to define logical (or analytic) truth on the basis of syntactic meta-logical conventions alone is viciously circular in a Tortoise-like fashion, because pre-conventional logic is already required to generate the truths from the conventions.¹⁷⁰ And in 1976 Susan Haack raised what is in effect the same worry, but this time in the form of a worry about the very idea of a justification of logical deduction, by arguing as follows:

- (1) All justification is either non-deductive (e.g., inductive) or deductive.
- (2) On the one hand a non-deductive justification of deduction is too weak and on the other hand a deductive justification of deduction is circular.
- (3) Therefore, deduction cannot be justified.¹⁷¹

Philosophers of logic have attempted various solutions to The Logocentric Predicament, the Tortoise regress problem, and the problem of justifying deduction. I will

not canvass these attempts here, although I do cover them and critically analyze them in detail in another place.¹⁷² My intention here is just to suggest how we could use the notions of first-order monadic logic and pure general logic to solve The Logocentric Predicament and its associated problems. Suppose that pure general logic really does capture the a priori essence of logic *just because*, insofar as it contains first-order monadic logic, and also falls under The Minimal Logical Meta-Principle of Non-Contradiction, it thereby adequately captures all The Logical Perfections—truthful consistency, soundness, completeness, decidability, and above all, analyticity—and it is also The Universal Natural Logic. Then since all rational theorizing, explanation, and justification whatsoever presuppose logic, it follows that pure general logic must also be the a priori essence of all rational theorizing, explanation, and justification whatsoever.

More explicitly, this line of transcendental argument solves The Logocentric Predicament by showing us that pure general logic is the explanatory and justificatory *unique rationally obligatory theoretical primitive*. Pure general logic is *the one and only science necessarily by virtue of which and in terms of which* every judgment, belief, claim, inference, science, or more generally any theoretical activity or product that is in any way justifiable or explicable by reasons, *categorically ought to be* to be explained or justified. Pure general logic is then both adequately explained and justified when we learn that every explanation and justification *whatsoever*, including the explanation and justification *of every other logic*, both *has to* presuppose and use pure general logic, and has to presuppose and use it *alone*, and also *rightly does so*. Pure general logic—The Universal Natural Logic, the paradigm of logical analyticity—is that logic which, uniquely, we must and ought to presuppose and use in order to *construct* any other logic,

in order to *construct* any rational explanation whatsoever, in order to *construct* any rational justification whatsoever, and in order to *construct* any rational theory whatsoever. Hence the Kantian ethicist Onora O'Neill very aptly calls this line of argument "a constructivist vindication of formulas of logic."¹⁷³

The philosophical thesis of *Constructivism*, whether inside or outside of ethics, holds that human agents or the human mind play an active, basic role in determining and generating the content of all beliefs, truths, knowledge (especially including the knowledge of language), desires, volitions, act-intentions, and logical or moral principles. In this way, The Logocentric Predicament, the Tortoise regress problem, and the problem of justifying deduction are just ways of *showing us* pure general logic's primitive and unique a priori status in any cognitive, scientific, or more generally theoretical constructive activity or product, and in particular its absolutely unique a priori categorically normative status in *all* constructive theoretical explanation and justification whatsoever, including any attempt to construct theoretically an explanation or justification of logic itself. Pure general logic is *the one and only categorically normative a priori condition of the possibility* of all constructive theoretical explanation and justification whatsoever. Otherwise put, pure general logic *must be presupposed and used* in every constructive theoretical explanation and justification whatsoever. And *that is* why logic must be presupposed and used in any attempt to justify or explain logic. It is partially constitutive of our rational humanity. In this sense, pure general logic is not only *transcendental*,¹⁷⁴ but also our rational human logical *duty*.

Argument 4: The Non-Supervenience Argument

The fourth argument is intended to show, again, that first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all a priori in my specifically contemporary Kantian sense of that notion (see section III), but in a way that is interestingly distinct from that of **Argument 1**.

In section IV, I argued that even if the *existence* of necessary truths logically strongly supervened on everything, it would not follow that their *specific character* logically strongly supervenes too. For although all logically necessary truths in first-order monadic logic and pure general logic are necessarily equivalent, their *structural senses* are different in virtue of their inherently different logical forms. For example, “ $P \rightarrow P$ ” does not have the same structural sense as “ $P \vee \sim P$ ” because its logical form is inherently different. It is in virtue of *transformation* rules—e.g., De Morgan’s Equivalences—that we are able to move from one logical truth having a certain structural sense, to another logical truth having a distinct although necessarily equivalent structural sense. So their structural senses can vary independently of their being logically necessarily true, and this intensional fact is made manifest by the application of transformation rules. In turn, therefore, their structural senses do not logically strongly supervene on whatever it is that their existence logically supervenes on, under the supposition that their existence logically strongly supervenes on everything. And that is true in every logically possible world: logically necessary truths in first-order monadic logic and pure general logic with inherently different logical forms are all intensionally non-equivalent. So their specific character does not *logically* strongly supervene on *anything*, except of course on first-order monadic logic, pure general logic, and The

Minimal Logical Meta-Principle of Non-Contradiction themselves. Nor does their specific character *merely strongly* supervene on anything, except of course on first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction themselves. If their specific character does not either logically or merely strongly supervene on anything but first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction themselves, then since none of these is strongly supervenient on actual or possible sense experiences and/or contingent facts, *then they are all a priori*.

Argument 5: The Weak Transcendental Ideality Argument

Suppose that I am correct that first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all “transcendental” in the sense specified in sub-section **X.1**. The fifth and final argument is intended to *explain* why this is so, by showing that first-order logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all *weakly or counterfactually transcendently ideal*, or WC-ly TI, for short

So now I am going to argue explicitly that first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all WC-ly TI.

1. First-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are either (i) physical, (ii) platonic, (iii) sense-experiential, (iv) conventional or social, or (v) transcendently ideal, and there are no other relevantly distinct options. (Premise, justified by constructed philosophical rational intuition)
2. If either first-order monadic logic, pure general logic, or The Minimal Logical Meta-Principle of Non-Contradiction were physical, then they would be contingent. But first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all necessary. So first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are not physical. (Premise, justified by constructed philosophical rational intuition)
3. If either first-order monadic logic, pure general logic, or The Minimal Logical Meta-Principle of Non-Contradiction were platonic, then they would be unknowable by

Benacerraf's Dilemma considerations. But first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all authentically knowable a priori. So first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are not platonic. (Premise, justified by constructed philosophical rational intuition)

4. If either first-order monadic logic, pure general logic, or The Minimal Logical Meta-Principle of Non-Contradiction were sense-experiential, then they would be a posteriori. But first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all a priori. So both first-order monadic logic and pure general logic are not sense-experiential. (Premise, justified by constructed philosophical rational intuition)

5. If either first-order monadic logic, pure general logic, or The Minimal Logical Meta-Principle of Non-Contradiction were conventional or social, then they would be either physical, sense-experiential, logically strongly supervenient on physical facts or sense-experiential facts, or merely strongly supervenient on physical facts or sense-experiential facts. But neither first-order monadic logic, nor pure general logic, nor The Minimal Logical Meta-Principle of Non-Contradiction is either physical, sense-experiential, logically strongly supervenient on physical facts or sense-experiential facts, or merely strongly supervenient on physical facts or sense-experiential facts. So neither first-order monadic logic, nor pure general logic, nor The Minimal Logical Meta-Principle of Non-Contradiction is conventional or social. (Premise, justified by constructed philosophical rational intuition)

6. Therefore first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all transcendentally ideal. (From 1-5, and Disjunctive Syllogism)

7. If something is transcendentally ideal, then it is either strongly TI or else WC-ly TI and there are no other relevantly distinct options. (Premise, justified by constructed philosophical rational intuition)

8. Strong TI is false. (Premise, justified by constructed philosophical rational intuition)

9. Therefore first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all WC-ly TI. (From 7, 8, and Disjunctive Syllogism)

The argument I have just spelled out is clearly valid, since it is in the form of two simple disjunctive syllogisms. But at the same time, it is equally clear that its soundness rests on the seven premises, each justified by *constructed* philosophical rational intuition, whose rational support is therefore only *fairly* reliable, and does not flow from the highest kind of evidence, i.e., basic or non-basic authoritative rational intuition.

Nevertheless, I do think it can still be truly said that this argument makes a *fairly*

plausible case for the weak or counterfactual transcendental ideality of first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction.

If the five arguments I have just spelled out are all in fact sound, then The L-is-T Thesis is true for first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction. Now if first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all necessary, a priori, and do not logically supervene on anything but themselves, then none of them logically supervenes on anything physical, contingent, sense-experiential, or conventional or social. This in turn entails that not everything logically supervenes on the physical world, the contingent natural world, the sense-experiential natural world, or the social world. So Scientific Naturalism is false, Physicalism is false, and Maximal Empiricism is false, including Logical Empiricism. If first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all categorically normative for all rational human cognition and thought, then they are necessarily presupposed by, and also conditions of the possibility of, all rational human cognition and thought. Because first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all weakly or counterfactually transcendently ideal, it also follows that Platonism about logic is false. And finally, because first-order monadic logic, pure general logic, and The Minimal Logical Meta-Principle of Non-Contradiction are all transcendental in *all* senses of that notion as I specified it in sub-section **X.1**, it follows that human rationality, human cognition, human thought, first-order monadic logic, pure general logic, and The Minimal Logical Meta-

Principle of Non-Contradiction are all essentially bound up with one another, and stand or fall together. As Kant and early Wittgenstein so brilliantly saw, philosophical logic bottoms out in serious transcendental epistemology and serious transcendental metaphysics.

X.3

A Four-Part Transcendental Solution to The Extended Benacerraf Dilemma

From here on in, I will assume that The L-is-T Thesis is true and explicitly deploy it in order to work out a solution to The Extended Benacerraf Dilemma. Obviously, the heavy burden of proof for *any* adequate solution to The Extended Benacerraf Dilemma is the threefold task of

- (i) clarifying the nature of abstract logical objects,
- (ii) providing an account of the cognitive mechanism of logical intuition, and then
- (iii) showing how these are internally related to one another in logical a priori knowledge, i.e., High-Bar justified necessarily objectively true a priori belief.

In the rest of this sub-section, then, I will sketch a four-part *transcendental* theory of logical intuition that seems to do the job,¹⁷⁵ and also explicitly extends Kantian Structuralism and Kantian Intuitionism to logic.

Part One. The first part of the theory is Kantian Logical Structuralism. According to non-reductive Structuralism, as I have already pointed out in section VI, abstract objects of some specific kind are not construed as independently existing entities but instead are taken to be, *essentially*, distinct roles, positions, or offices in a *structure*, that is, an abstract formal relational system consisting of a coherent set of interlinked patterns or configurations.¹⁷⁶ So the thesis of my non-reductive Logical Structuralism is that each logical system is an abstract formal relational totality consisting of a coherent set of

logical patterns or configurations, and that logical objects are *nothing more than and also nothing less than* distinct roles, positions, or offices in some such system.

Both logical objects and their constitutive logical structures are abstract, so I need to say something more about the notion of abstractness I am using. On my view, something is abstract if and only if it is not uniquely located in spacetime. Concrete things, by contrast, are uniquely located in spacetime. This is a broad conception of abstractness that allows for both “platonic” abstractness and “non-platonic” abstractness. Something is platonically abstract if and only if it has extra-spatiotemporal existence: for example, *ante rem* universals, or the inhabitants of Frege’s “third realm.” Platonically abstract objects are intrinsically non-spatiotemporal, and have no connection whatsoever to the natural causal order. So they are causally irrelevant. By contrast, something is non-platonically abstract if and only if it has an infra-spatiotemporal and trans-spatiotemporal existence: for example, *in re* universals, rules, laws of nature, linguistic types, mental representation types (e.g., concepts), multiply realizable functional organizations, sets of concrete objects, etc.

The essential feature of non-platonic abstractness, aside from lack of unique location in spacetime, is actual or possible repetition, or multiple occurrence, multiple instantiation, multiple realization, etc., across spacetime. This is at least consistent with causal relevance. In this way, I can assert both non-reductive Logical Structuralism and the abstractness of logical structures while also not committing myself to the highly problematic thesis that logical objects and their constitutive logical structures are platonically abstract and therefore causally irrelevant. On the contrary, if I am correct, then logical objects and their constitutive structures are *non-platonically* abstract, and at

least possibly causally relevant, even if not causally efficacious, precisely because they are all *weakly or counterfactually transcendentally ideal*, and also cognitively constructed by rational human animals *in language*, whether in the language of thought or in a public language.¹⁷⁷ In this way, the non-platonic abstractness of logic is the abstractness of a *weakly or counterfactually transcendentally ideal linguistic structure*, a formal relational system consisting of a coherent set of interlinked patterns of linguistic types that necessarily conforms to the innately-specified capacities of the rational human mind.

Part Two. This brings me to the second part of the theory. Assuming that logical objects and their constitutive structures are non-platonically abstract because they are weakly or counterfactually transcendentally ideal, and also cognitively constructed by rational human animals in language, I am now also claiming that the primary cognitive mechanism of authoritative rational intuition in logic is *the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata*, and correspondingly, the phenomenal continuous isomorphism, spatial-structure-coincidence, or temporal-structure-coincidence that occurs in the specifically pattern-matching activities of rational human sense perception, minimal episodic memory, and/or the imagination. This, in turn, fully satisfies both LOCKED-ONTO and also STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC., and guarantees that authoritative rational intuitions in logic are High-Bar justified by virtue of being inherently or intrinsically connected to the logically necessary truth-makers of those

beliefs, and thereby constitute authentic objective a priori knowledge. This in turn yields Kantian Intuitionism for logic.

It seems to me, as it also seemed to Kant, that the primary cognitive mechanism for authoritative rational intuition, whether in mathematics, logic, morality, or philosophy, is the veridical schematizing imagination insofar as it builds on directly referential sense perception and minimal episodic memory, and not sense perception *alone*:

We will call this formal and pure condition of the sensibility, to which the use of the concept of the understanding is restricted, the **schema** of this concept of the understanding... The schema is in itself always only a product of the imagination; but since the synthesis of the latter has as its aim no individual intuition but rather only the unity in the determination of sensibility, the schema is to be distinguished from the image. Thus, if I place five points in a row,, this is an image of the number five. On the contrary, if I only think number in general, which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (i.e., a thousand) in accordance with a certain concept than the image itself, which in this case I could survey and compare with the concept only with difficulty. Now this representation of a general procedure of the imagination for providing a concept with its image is what I call the schema for this concept.

In fact it is not images of objects but schemata that ground our pure sensible concepts.... [T]he **image** (*Bild*) is a product of the empirical faculty of productive imagination, [but] the **schema** of sensible concepts (such as figures in space) is a product and as it were a monogram of pure a priori imagination, through which and in accordance with which the images first become possible... The schema of a pure concept of the understanding ... is something that can never be brought to an image at all, but rather is only the pure synthesis, in accord with a rule of unity according to concepts in general, which the category expresses, and is a transcendental product of the imagination, which concerns the determination of inner sense in general, in accordance with conditions of its form (time). (*CPR* A140-142/B180-181)

In turn, my Kant-inspired rationale for holding that the proper cognitive mechanism for authoritative rational intuition—whether in mathematics, logic, morality, or philosophy—is the veridical schematizing imagination insofar as it builds on directly referential sense perception and minimal episodic memory, and not sense perception alone, is that the veridical schematizing imagination has three basic features *not* also shared by sense perception on its own.¹⁷⁸

First, I can veridically schematically imagine an object O even though O is not uniquely located in spacetime, whereas I cannot veridically sense-perceive O unless O is uniquely located in spacetime.

Second, to generate a veridical schematic mental image of an object O is thereby to generate a figural or spatiotemporal image, distinct from O itself, that is directly available to introspective scanning and manipulation (for example, image-rotation, zooming in, pulling back, etc.) whereas to perceive O veridically is not *thereby*¹⁷⁹ to generate anything figural or spatiotemporal, distinct from O itself, that is directly available to introspective scanning and manipulation.

And **third**, I can generate a veridical schematic image of an objectively real object O_r (e.g., someone I know well) without its being the case that O_r stands either in any efficacious causal relation or in an effective “tracking” relation to my conscious image of O_r (such I can locate O_r in an egocentric phenomenal space relative to my body and also follow O_r ’s movements in this “centered” space over time), whereas it is plausible to think that I cannot veridically sense-perceive O_r without either an efficacious causal relation or an effective tracking relation obtaining between O_r and my conscious perceptual representation of O_r .

These three features of the veridical schematic imagination (i.e., that its objects can be abstract, that it generates figural or spatiotemporal images directly available to introspective scanning and manipulation, and that its veridicality-conditions are not based on either efficacious causation or effective tracking) all seem to me to be deeply relevant to authoritative rational intuition in logic.

It is obvious enough, I think, that authoritative rational intuition in logic will necessarily be such that its objects are abstract and that its veridicality-conditions are not *grounded on* either efficacious causation or effective tracking. That is what got us into the Original and Extended Benacerraf Dilemmas in the first place. But the other basic feature of the veridical schematic imagination, i.e., its generation of figural or spatiotemporal images directly available to introspective scanning and manipulation, may not be so obviously relevant. What I want to claim, however, is that it is this **second** of the three basic features that actually clinches the case for the necessary cognitive connection between authoritative rational intuition in logic and the veridical schematizing imagination.

This becomes clear when we ask ourselves about the conditions under which I generate a veridical schematic mental image of an objectively real object O_r or objectively real dynamic process DP_r . Here I am drawing directly on a body of recent work on mental imagery in cognitive psychology by Philip Johnson-Laird, Steven Kosslyn, and Roger Shepard.¹⁸⁰ According to these psychologists, the representation-relation between an image (Johnson-Laird regards images as paradigm examples of mental models) and a real object or real dynamic process is essentially *depictive or pictorial*, and not essentially *descriptive or propositional*. Here it should be noted that I am taking sides in a vigorous debate in cognitive science about the nature of mental imagery, with Johnson-Laird, Kosslyn, and Shepard on the depictivist side, and Zenon Pylyshyn and others on the descriptivist or propositionalist side.¹⁸¹ I am not saying that this debate is actually over, or that it has been decisively resolved, but rather only that it seems to me that the case for two irreducibly distinct types of mental representation and

representational content is at this point definitely *stronger* than the case for the thesis that all mental representations and representational content are at bottom descriptive or propositional. On the basis of that assumption, then, I will forge ahead.

Now a veridical depictive or pictorial relation is based on sharing the same configuration, figure, pattern, shape, or structure, and not based on satisfying some specific set of descriptive or propositional criteria. So a schematic image I veridically represents its corresponding real object O_r or dynamic process DP_r if and only if I is continuously isomorphic or spatiotemporal-structure-coincident with O_r or DP_r . When I form a veridical schematic mental image of some object or dynamic process, I consciously scan and manipulate my schematic mental image, mental model, mental diagram, or mental picture (or, in the case of a dynamic process, in effect a “mental movie”) until it apparently shares the same phenomenal configuration, figure, pattern, shape, or structure as the real object or real dynamic process I have imaged. In other words, I *mentally simulate* the structure of the schematically imaged object or dynamic process.

But here is the crucial part. Whenever, during this procedure of veridical mental simulation, I have actually reached the point of what *seems to me* to be the precise or one-to-one matching of the relevant elements of the structure of my schematic mental image, mental model, mental diagram, or mental picture (or “mental movie”) with the corresponding elements of the structure of the schematically imaged object or dynamic process, as I have consciously represented it (whether via minimal episodic memory, directly referential sense perception, judgment, or inference), then I *thereby* induce in myself a completely convincing, intrinsically compelling, self-evident, or rationally

unshakeable belief that the schematically imaged object or dynamic process really and truly *is* just as I have consciously represented it. That is because the criterion of veridicality for schematic images is exact continuous isomorphism or spatiotemporal-structure-coincidence with their objects or dynamic processes. So whenever my veridical schematic mental image is experienced from the inside, or phenomenally, as having the very same configuration, figure, pattern, shape, or structure as what is specified by the content of my conscious representation of the object or dynamic process, then necessarily I am thereby fully convinced that the schematically imaged object or dynamic process is just as I have represented it to be.

Of course, not every schematic mental image is veridical. The world can be otherwise than I have imagistically represented it to be. But the crucial thing for my purposes here is that *in cases of veridical schematic mental imaging*, the cognitive step *from* the consciously-experienced continuous isomorphism or spatiotemporal-structure-coincidence between my schematic mental image and what is specified by the content of my conscious representation of the schematically imaged object or dynamic process, *to* a completely convincing, intrinsically compelling, self-evident, or rationally unshakeable belief that the schematically imaged object or dynamic process is precisely as I have represented it by means of my cognition is *modally automatic, strictly underdetermined by sensory experiences and/or contingent facts*, and *self-contained*. Otherwise put, in veridical schematic mental imaging, the subjectively experienced “rightness of fit” between my schematic mental image and what is specified by the content of my conscious representation of the schematically imaged object or dynamic process is *cognitively optimal*. So I am thereby both *objectively and subjectively certain* that the

schematically imaged object or dynamic process is precisely as I have represented it to be. And in this way the phenomenal structure-matching activity of the veridical schematizing imagination adequately explains the “clarity, distinctness, and indubitability” of authoritative rational intuition.

It is crucial to emphasize here how sharply different this schematic imaginal account of authoritative rational intuition is from classical *conceptual-linguistic analysis* accounts of how rational intuition occurs, all the way from Arthur Pap,¹⁸² H.P. Grice, and Peter Strawson¹⁸³ in the 50s, 60s, 70s, and 80s, to Chalmers and Jackson¹⁸⁴ in the 1990s and 2000s.¹⁸⁵ On conceptual-linguistic analysis accounts, the rational act, process, or state of fully understanding the meanings of the constituent concepts or words of a sentence or statement cognitively *suffices* for an authoritative rational intuition. But this is clearly mistaken, since even conceptual-linguistic analysts who fully understand the meanings of the very same sentences or statements can diametrically disagree about them *because they are being guided by very different fundamental philosophical “pictures” in the later Wittgenstein’s sense of that term*—and they cannot *all* be right. But the real-world cognitive fact of diametric philosophical disagreement in conceptual-linguistic analysis, together with the full semantic understanding of all disagreeing parties, is perfectly consistent with the further fact that any or all of the disagreeing reasoners fail to have authoritative rational intuitions, *precisely because they have simply failed successfully to perform an intentional act of veridical schematic mental imaging*. If so, then even over and above full semantic understanding, they have simply failed successfully to *depict* or *picture* the truth. Here we can also play an illuminatingly relevant riff on the early Wittgenstein’s equally famous and notorious Tractarian

distinction between “saying” (*sagen*) and “showing” (*zeigen*).¹⁸⁶ In order to have an authoritative rational intuition, it is not enough just to be able *to say it to yourself*—you have to be able *to show it to yourself* too. Authoritative rational intuition requires a further successful intentional performance of veridical schematic imaging *over and above* the mere act, state, or procedure of full conceptual-linguistic understanding.

Part Three. This brings me to the third part of the theory. In section **VII**, we saw that the objective reality of truth plays an essential role in the Categorical Epistemology of knowledge, in that necessarily, High-Bar justified true belief includes an inherent or intrinsic connection between justification and objective truth. In the special case of authentic a priori knowledge based on authoritative rational intuitions *in logic*, then, High-Bar justified true belief thereby includes an intrinsic connection between *a priori* justification and *logically necessary* objective truth. In turn, the satisfaction of LOCKED-ONTO and STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC., by means of the successful operations of the schematizing imagination in logical cognition will guarantee that authoritative rational intuitions in logic are High-Bar justified, inherently or intrinsically connected to the logically necessary objective truth-makers of those beliefs, and thereby constitute authentic objective a priori knowledge.

This leads to another issue. We now know that in order for an authoritative rational intuition in logic to constitute authentic objective a priori knowledge, logical necessity must be objectively real and weakly or counterfactually transcendently ideal. But what *is* logical necessity? And for that matter, what *is necessity*? Obviously I cannot

even begin to address, much less answer, such a huge question within the fairly limited scope of this essay. But since I have already frequently helped myself to the concept of necessity, I should at least very briefly describe the general modal framework I am using.¹⁸⁷

For me, necessity is the truth of an interpreted sentence or statement in every member of a set of possible worlds, together with its non-falsity in every other possible world. A possible world is nothing more and nothing less than a maximally consistent set of different conceivable ways the actual world might have been: that is, the largest distinct set of mutually consistent concepts such that the addition of one more concept to that set would yield an inconsistency. Logical possibility, more generally, is the consistency of a sentence or statement with the laws of some classical or non-classical logic. Logical necessity is the truth of an interpreted sentence or statement in virtue of logical laws or intrinsic conceptual connections alone, hence the truth of a sentence or statement in all logically possible worlds. Put in traditional terms, logical necessity is *analyticity*.¹⁸⁸

Logical necessity is usually contrasted with physical or nomological necessity, that is, the truth of an interpreted sentence or statement in all logically possible worlds governed by our actual laws of nature; correspondingly, physical or nomological possibility is the joint consistency of a sentence with the laws of logic *and* our actual laws of nature. Physical or nomological necessity is also a form of “hypothetical” or “relative” necessity. More precisely, an interpreted sentence or statement S is hypothetically or relatively necessary if and only if it is logically necessary that $\Gamma \rightarrow S$, where Γ is some set

of special axioms or postulates, e.g., our actual laws of nature. Thus hypothetical or relative necessity is parasitic on logical necessity or analyticity.

In addition to logical or analytic necessity and physical or nomological necessity, there is also metaphysical necessity. Metaphysical necessity is either

(i) necessity as defined over the set of all logically possible worlds (in which case it is also logical necessity, analyticity, or “weak” metaphysical necessity), or

(ii) necessity as defined over a set of possible worlds that is definitely smaller than the set of all logically possible worlds and determined by whatever it is that constitutes the underlying essence or nature of our actual world (in which case it is “strong” or “essentialist” metaphysical necessity).

More precisely, an interpreted sentence or statement *S* is strongly or essentialistically metaphysically necessary if and only if

(i) *S* is true in every member of a set *K* of logically possible worlds.,

(ii) *K* is smaller than the set of all logically possible worlds.,

(iii) *K* is larger than the set of all physically possible worlds.,

(iv) *K* includes the class of physically possible worlds.,

(v) *K* is the class of logically possible worlds consistent with the underlying essence or nature of our actual world, including its basic spatiotemporal structure, its basic dynamical structure, and its basic mathematical structure., and

(vi) *S* takes no truth-value in every logical possible world not belonging to *K*.

Put in traditional terms, strong or essentialist metaphysical necessity is *synthetic a priori* necessity.¹⁸⁹

Needless to say, the distinction between logical or analytic a priori necessity and synthetic a priori necessity is highly philosophically controversial.¹⁹⁰ It is not my specific aim in *Objectivity Regained* either to defend the analytic-synthetic distinction or to demonstrate the existence of the synthetic a priori—although I have, of course, been *using* the notions of the analytic-synthetic distinction and the synthetic a priori pretty

liberally as explanatory notions; and here I am, doing it again. For better or worse, I have attempted to defend the analytic-synthetic distinction and also to demonstrate the existence of the synthetic a priori in *The Rational Human Condition*, ch. 2.4. My appeal to it in this particular context is intended only to indicate that

(i) I take the concept of necessity to extend beyond the concept of logical or analytic necessity, hence my modal framework is *modally dualistic.*, and that

(b) The modally dualistic possible worlds framework I have adopted is directly and ultimately based on Weak or Counterfactual Transcendental Idealism, i.e., WCTI, via The L-is-T Thesis.

The crucial take-away for my purposes here, then, is that *that logical or analytic necessity is objectively real and also transcendentally ideal.*

Part Four. Now for the fourth and final part of the theory. I have proposed that logical objects are, essentially, distinct roles, positions, or offices in logical structures, that is, logics construed as non-platonically, weakly or counterfactually ideally abstract formal relational systems consisting of coherent sets of interlinked patterns of linguistic types. I have also proposed that the primary cognitive mechanism of logical intuition is the ability for consciously scanning and manipulating linguistic schematic mental images. And I have also proposed that the objective reality and transcendental ideality of logical necessity is an essential part of logical knowledge, construed as High-Bar a priori justified logically necessarily true belief. Given the philosophical picture of authoritative rational intuition I developed in section **VII**, then my claim is that I have authentic a priori logical knowledge via my logical rational intuition that *S* if and only if

(1) I authoritative logically rationally intuit that *S.*, and

(2) It is an objectively real and weakly counterfactually transcendentally ideal fact that logically necessarily *S.*

More precisely now with respect to (1), I authoritatively logically rationally intuit that *S* if and only if

(1.1) I rationally intuit that *S*., hence

(1.2) I take it to be logically necessary and a priori that *S*., and

(1.3) I consciously scan and manipulate my linguistic schematic mental image ‘*S*’ of the sentence or statement *S* to the point of phenomenal continuous isomorphism or spatial-structure-coincidence with what is specified by the semantic content of my rational intuition that (logically necessarily) *S*.

So, most explicitly, my claim is that I have logical knowledge that *S* if and only if

(1.1) I rationally intuit that *S*., hence

(1.2) I take it to be logically necessary and a priori that *S*., and

(1.3) I consciously scan and manipulate my linguistic schematic mental image ‘*S*’ of the sentence or statement *S* to the point of phenomenal continuous isomorphism or spatial-structure-coincidence with what is specified by the semantic content of my rational intuition that (logically necessarily) *S*., and

(2) it is an objectively real and weakly or counterfactually transcendently ideal fact that logically necessarily *S*.

Let me now try to make this more concrete with a simplified¹⁹¹ example. Consider the following:

(*) Either Barack Obama is president of the USA in 2011 or I’m the man in the moon. I’m not the man in the moon. Therefore Barack Obama is president of the USA in 2011.

Now, assuming my knowledge of English and of classical propositional logic, this text is read and understood by me as a simple disjunctive syllogism, in the form of a single sentence or statement: ‘Either Barack Obama is president of the USA in 2011 or I’m the man in the moon, and I’m not the man in the moon, therefore Barack Obama is president of the USA in 2011’. But not only do I read and fully understand this argument in the form of a single sentence or statement: I also rationally cannot help believing it to be both valid and sound. This is because insofar as I formulate (*) to myself, thereby

representing a logical object (in this case an argument in the form of a single sentence), I also generate a visual mental image that looks more or less like this:

$$P \vee Q, \sim Q \vdash P$$

Let us call this symbolic sequence ‘(#)’. In turn, I will label the visual schematic mental image of the symbolic sequence (#), ‘ $I(\#)$ ’. (#) is of course a straightforward translation of (*) into the fairly standard symbolism I learned for classical propositional logic as an undergraduate. Then $I(\#)$ is used by me to intuit the argument expressed by (*) as a valid and sound argument carried out according to the rules for classical negation, disjunction, and disjunctive syllogism. This in turn happens precisely insofar as I use $I(\#)$ as a linguistic schematic image of what is semantically represented by (*), which is a logical fact, and then consciously scan and manipulate $I(\#)$ so as to bring it into a phenomenal continuous isomorphism or spatiotemporal-structure-coincidence with that fact, which in turn is specified by the semantic content of (*). Finally, this authoritative logical rational intuition counts as *authentic logical a priori knowledge* because not only is the rational intuition completely convincing, intrinsically compelling, or self-evident, it is also the case that (*) semantically represents an *objectively real and weakly or counterfactually transcendentally ideal* logically necessary fact, namely a genuinely valid and sound argument in classical propositional logic in the form of a single interpreted sentence or statement.

This completes my transcendental solution to the Extended Benacerraf Dilemma. I have accepted the standard uniform semantics of logical truth (“Truth is uniform and Tarskian”), and also the causal and sensory anchoring of all human cognition and knowledge including logical cognition and knowledge (“All human knowledge begins in causally-triggered sensory experience”), as well as the authentic a priori human

knowability of objectively real and weakly or counterfactually transcendently ideal abstract logical objects, construed as linguistic objects of a special humanly-cognizable kind. I have asserted the thesis of Logical Structuralism, and also the thesis that logical objects and their constitutive structures are non-platonically abstract and weakly or counterfactually transcendently ideal (i.e., The L-is-T Thesis), and therefore causally relevant. But I have denied that humans need to stand in an efficacious causal relation to objectively real and weakly or counterfactually transcendently ideal logical abstract objects in order authentically to know them a priori, because I have denied that authoritative rational intuition in logic should be cognitively *grounded on* sense perception, even if, necessarily, all human cognition whatsoever is cognitively *anchored in* causally-triggered directly referential sense perception. Instead I have proposed that the primary cognitive mechanism for authoritative rational intuition in logic is the veridical schematizing imagination and not veridical directly referential sense perception alone, and that linguistic veridical schematic mental images (whether of ordinary natural language inscriptions or of formal-logical symbols) are the mental vehicles of this special kind of authoritative rational intuition. Now a veridical schematic mental image need not stand in any sort of efficacious causal relation to its corresponding object or real dynamic process in order to be veridical. Instead, it need only be continuously isomorphic or spatiotemporal-structure-coincident with its object in order to be veridical. Hence my successful intentional act of authoritative rational intuition in logic can adequately represent its logical object by virtue of the fact that its mental vehicle, a linguistic veridical schematic mental image, is continuously isomorphic or structure-coincident with the schematically-represented objectively real and weakly or counterfactually

transcendentally ideal object of my logical intuition. Furthermore, the veridical schematic imaginal cognitive mechanism of authoritative rational intuition in logic is a process of phenomenal spatiotemporal-structure-matching between

(i) the linguistic mental model, mental diagram, mental picture, structural image, or schema of a single (perhaps fairly long and complex) sentence or statement that I use to express my logical intuition, and

(b) what is specified by the semantic content of that logical intuition, which in turn represents logical objects and their constitutive structures, which in turn take the very same form of (perhaps fairly long and complex) sentences or statements in some classical or non-classical logical system.

So the thesis that authoritative rational intuition in logic is a special type of veridical schematic imaginal cognition squares perfectly with Logical Structuralism. Finally, I have also thereby extended Kantian Structuralism and Kantian Intuitionism to logic. For all these reasons, then, I think that we now authentically know a priori why logic *must be* transcendental.

IX. Intuitions, Mathematics, Logic, and Morality

Like ethics, logic can also be called a normative science. How must I think in order to reach the goal, truth?... [T]he task we assign logic is only that of saying what holds with the utmost generality for all thinking, whatever its subject matter. We must hold that the rules for our thinking and for our holding something to be true are prescribed by the laws of truth.

--G. Frege¹⁹²

That an act *qua* fulfilling a promise, or *qua* effecting a just distribution of good, or *qua* returning services rendered, or *qua* promoting the virtue or insight of the agent, is *prima facie* right, is self-evident; not in the sense that it is evident from the beginning of our lives, or as soon as we attend to the proposition for the first time, but in the sense that when we have reached sufficient mental maturity and have given sufficient reflection to the proposition it is evident without any need of proof, or evidence beyond itself. It is self-evident just as a mathematical axiom, or the validity of a form of inference, is evident. The moral order expressed in these propositions is just as much a part of the fundamental nature of the universe (and, we may add, of any possible universe in which there are moral agents at all) as the spatial or numerical structure expressed in the axioms of geometry or arithmetic. In our confidence that these propositions are true there is involved the same trust in our reason that is involved in our confidence in mathematics; and we should have no justification for trusting it in the latter sphere and distrusting it in the former.

--W.D. Ross¹⁹³

XI.1

In this section, I want to develop some substantive analogies between a rational intuition-based epistemology of mathematics and logic on the one hand, and a rational intuition-based epistemology of morality on the other. In so doing, I also want to extend some of the basic ideas of Kantian Intuitionism and Kantian Structuralism to the contemporary Kantian moral theory that I spelled out and defended in part 3 of *The Rational Human Condition*, a.k.a. *Existential Kantian Ethics*.

XI.2

The classical locus of moral intuitionism is W.D. Ross's *The Right and the Good*. There Ross famously argues that we have rational, self-evident, non-inferential, infallible a priori intuitions about an irreducibly plural class of co-basic moral principles he calls "*prima facie* duties," which include the seven duties of "fidelity," "reparation," "gratitude," "justice," "beneficence," "self-improvement," and "non-maleficence."¹⁹⁴

These seven principles, purportedly, are knowable by *any* mature, reflective rational human animal. Prima facie duties are sharply distinguished from *actual duties*, or *duties proper*, which

(1) are the objectively real moral obligations binding on moral agents or persons in particular act-contexts, and

(2) are objectively determined by their being the moral principles that, in that act-context, have the greatest balance of prima facie rightness over prima facie wrongness, of all possible acts for that agent in that context, when the act is taken in “its whole nature.”¹⁹⁵

At the same time, however, according to Ross, it is not possible to intuit, or authentically know, actual duties—at best, it is possible to cognize actual duties with “right opinion,” not sufficiently justified true belief.¹⁹⁶ Ross’s main goals in combining the classical theory of rational intuition with his theory of prima facie duties vs. actual duties are three:

(i) to provide a secure, realistic, and a priori, hence non-naturalistic, but also non-monistic/pluralistic foundation for moral theory, yet also at the same time

(ii) to accommodate the obvious empirical fact of conflicts of duties—i.e., *moral contradictions* or *moral dilemmas*—that seem to arise directly from the foundational fact of a plurality of basic duties together with our actual “human, all too human” existence in our *desperately* nonideal world, as well as

(iii) to incorporate some measure of commonsensical or real-world fallibilism about our moral judgments in particular contexts in this desperately nonideal world.

So Ross’s moral intuitionism is designed precisely in order to accommodate the pluralistic and desperately nonideal character of our moral lives, while also postulating a secure, realistic, a priori, and non-naturalistic foundations for ethics. But for these very reasons, his intuitionism is also philosophically *notorious*. Here are the four classical critical objections to it:

(1) It seems that Ross postulates a mysterious and “queer” (in J.L. Mackie’s sense¹⁹⁷) faculty for intuitively knowing the prima facie duties that has no independent plausibility or empirical support whatsoever.

(2) Ross's infallibilism about moral intuition seems to fly in the face of the highly plausible thesis of fallibilism about a priori knowledge, as well as fallibilism about empirical knowledge.

(3) The obvious empirical fact of widespread disagreement, even amongst mature, reflective rational human animals, about precisely which moral principles are true and which are false seems to undermine completely Ross's claim that even *some* moral principles are known intuitively.

(4) Ross's unhappy combination of moral realism and anti-naturalism is clearly false for two reasons: **first**, because it entails platonism about basic moral principles, which in turn is undermined by Benacerraf's Dilemma or BD, and **second**, because the only possible adequate foundation for moral realism is Ethical Naturalism, i.e., the strong supervenience of all moral facts on scientifically natural facts.

In addition to these four classical worries, here is a fifth but *non*-classical critical objection, which in certain respects is similar to John Rawls's basic worry about Ross's theory, to the effect that Ross cannot ultimately avoid a theory of the *lexical ordering* and *weighting* of the supposedly equally morally binding, lexically unordered, and unweighted prima facie duties:¹⁹⁸

(5) Ross's theory of prima facie duties explicitly postulates an irreducible pluralism of basic moral principles, yet at the same time he implicitly presupposes a monistic structural deontological scale in explicating the advance from prima facie duties to actual duties—otherwise how could there be an objective determination of one moral principle's being the one which, in a given context, expresses the greatest balance of prima facie rightness over prima facie wrongness, of all possible acts for that agent in that context, when the act is taken in "its whole nature"?

Now in view of what I have already argued in sections **I** to **X**, it seems to me that the four classical critical objections to Ross's version of moral intuitionism can be adequately handled by a smooth extension of Kantian Intuitionism about mathematical and logical intuitions, as follows:

Re (1): No queer or mysterious faculty of moral intuition *whatsoever* need be postulated. Rational intuition is just another application of the basic human cognitive faculties which are assumed and also fully empirically confirmed in both contemporary cognitive psychology and also philosophical psychology: reason, imagination, sense perception, consciousness, self-consciousness, memory, conceptualization, and essentially non-conceptual representation.,

Re (2): Fallibilism about a priori knowledge is perfectly consistent with the existence of authentic a priori knowledge via authoritative or self-evident intuition that satisfies LOCKED-ONTO and STRONG DISJUNCTIVISM ABOUT VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC. Indeed, Ross's fully appropriate analogy between moral a priori knowledge and mathematical a priori knowledge of simple arithmetic axioms implicitly invokes this crucial point.

Re (3): The threefold distinction between (i) authoritative (fully reliable) intuitions, (ii) constructed (fairly reliable) intuitions, (ii) *prima facie* (fairly unreliable) intuitions, and makes it possible to explain

(3a) the widespread disagreement amongst even mature, reflective rational human animals about *many or most* moral principles, as a function of *prima facie intuition*,

(3b) the moderate convergence of agreement amongst mature, reflective rational human animals about *a non-trivial number* of moral principles, as a function of *constructed intuition*, and

(3c) the strictly universal agreement amongst mature, reflective rational human animals about *a certain special class* of basic moral principles, as a function of *authoritative intuition*.

Re (4): My Kantian Intuitionist solution to BD clearly shows that both the reality and also the authentic knowability of objectively necessary a priori moral principles requires neither platonism nor Ethical Naturalism, but instead is metaphysically grounded on Weak Transcendental Idealism or WCTI, which, when it is applied to mathematics and logic, also entails the denial of the strong supervenience of robustly normative facts of any kind, including moral facts, on scientifically natural facts.¹⁹⁹

In order to deal with the fifth or non-classical objection to Ross's moral intuitionism, however, I have extended not only Kantian *Intuitionism* but also Kantian *Structuralism* from mathematics and logic to morality. Mathematical Structuralism, we will remember, says that mathematical entities such as numbers or sets are not ontologically autonomous or substantially independent objects, but instead are, essentially, *positions* or *roles* in a mathematical structure, where a mathematical structure is a complete set of formal relations and operations that defines a mathematical system; and also that what counts as an individual object of the system is thereby uniquely determined by the system as a whole, hence that any such individual object is identical to whatever possesses a specific set of intrinsic structural system-dependent properties.

In the text quoted as the epigraph for this chapter, it seems clear enough that Ross himself had a Moral Structuralist idea in mind—

The moral order expressed in these propositions is just as much a part of the fundamental nature of the universe (and, we may add, of any possible universe in which there are moral agents at all) as the spatial or numerical structure expressed in the axioms of geometry or arithmetic.

—although he never systematically developed that important thought.

In any case, here are the six basic ideas behind the Kantian Moral Structuralism that I developed and defended in *The Rational Human Condition*, chapter 3.1:

- (i) There is a three-levelled structural hierarchy of moral principles.,
- (ii) Moral principles are not ontologically autonomous or substantially independent objects, but instead are, essentially, positions or roles in a moral structure, where a moral structure is a complete set of semantic relations and normative forces that defines a moral system.,
- (iii) The semantic content and normative force of any individual moral principle is thereby determined by the moral system as a whole—that is, any such individual principle is identical to whatever possesses a specific set of intrinsic structural system-dependent properties.,
- (iv) Authoritative moral intuition applies only to the top level in the hierarchy, which are absolutely universal objectively necessary a priori procedural meta-principles, and neither to intermediate-level first-order substantive *ceteris paribus* moral principles, nor to bottom-level actual duties.,
- (v) The rational advance from the authoritatively intuited top-level meta-principles to the intermediate-level first-order substantive *ceteris paribus* principles to the bottom-level actual duties is a process of cognitive and volitional *construction*, significantly analogous to constructivist methods of proof in intuitionist mathematics and logic.and finally
- (vi) Real conflicts of first-order substantive *ceteris paribus* moral principles at the intermediate level of the hierarchy are automatically resolved by a special set of level-theoretic structural constraints, taken together with one other absolutely universal objectively necessary a priori moral meta-principle called **The Lesser Evil Principle**, which collectively fully preserve the absolutely universal objective truth and reality of the authoritatively-intuited meta-principles at the top level of the hierarchy.

XI.3

So much for the extension of Kantian Structuralism to the Kantian theory of moral principles. Now I need to complete my extension of Kantian Intuitionism in the

philosophy of mathematics and the philosophy of logic to moral intuitionism, by saying something about the internal structure of the cognitive process of *authoritative moral intuition* in relation to the internal structure of the cognitive process of *authoritative mathematical or logical intuition*.

In the case of mathematical or logical intuitions that constitute authentic a priori mathematical knowledge, it will be remembered, I proposed that phenomenological self-evidence be necessarily combined with the cognitive generation and manipulation of veridical schematic images, by means of which we are pre-reflectively essentially non-conceptually consciously aware of proper parts of *mathematical* structures, in such a way that this satisfies both LOCKED-ONTO and STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC. So too in the case of moral intuitions that constitute authentic a priori moral knowledge, I want to propose that phenomenological self-evidence be also necessarily combined with the cognitive generation and manipulation of veridical schematic images, by means of which we are pre-reflectively essentially non-conceptually consciously aware of proper parts of *moral* structures, again in such a way that this satisfies both LOCKED-ONTO and STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC.

What kind of moral structures? In the case of my *Mathematical Structuralism*, the relevant structures are number-systems, the primitive and non-primitive recursive functions over them, and logical constructions on those systems and functions, which in

turn are immanently realized in the manifestly real natural spacetime world of *material objects*. In the case of my *Logical Structuralism*, the relevant structures are complete classical or non-classical logical systems cognitively constructed by rational human animals in natural languages. Now in the case of my *Moral Structuralism*, the relevant structures are moral principles at any of the levels of The Hierarchy of Kantian Moral Principles, which in turn are immanently realized in the manifestly real natural spacetime world of *moral forms of life* and rational human animal world or *practically constructive intentional agency*. Authoritative moral intuition is then necessarily mediated by the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata of a specifically *ethical* kind, in such a way as to lock onto basic patterns in moral forms of life and in the dynamic processes of practically constructive intentional agency.

In order to make this line of thinking about moral intuition clearer and more distinct, I can appeal to Kant's striking idea, spelled out in section 59 of *The Critique of the Power of Judgment*, that *the beautiful is the symbol of the morally good*. Here is Paul Guyer's excellent analysis of this section:

Kant begins Section 59 with the blunt statement that "intuitions are always required to demonstrate the reality of our concepts" (5:351); stated without restriction. This would seem to apply to moral concepts along with all others. But Kant adds that although examples can be given for empirical concepts and schemata for pure concepts, it is impossible to produce intuitions which directly confirm the reality of concepts of reason, that is, ideas. However, he next argues that "hypotyposis," or presentation of concepts to the senses (*Versinnlichung*), can also be accomplished "symbolically." In this case, although the concept is one which "only reason can think and to which no sensible intuition is adequate," there is nevertheless still some form of sensible intuition that counts as a representation of the concept. In the case of such a symbolic representation, what agrees with the concept is not the actual content of the intuition but "merely the form of reflection" on it (5: 351). Kant illustrates this claim with an analogy between a "despotic state" and a "handmill": Although an observation of the latter is not itself an observation of the former, there are similarities between the ways these two things behave, and thus in the structure of our thought about them, even if not in the direct empirical content of our observation of them. In light of this, one can serve as a symbol of the other. Kant then applies this analysis of symbolism to the claim that "the beautiful is a symbol of the morally good." His argument is that the experience of aesthetic judgment furnishes us with some experience of both

our capacity for autonomy and the possibility of harmony between this capacity and nature outside it, which must be grounded in a supersensible basis for both. Because of partial parallels between the structure of aesthetic judgment ... and moral judgment, aesthetic experience can serve as a symbol of moral judgment. The experience of both autonomy and the supersensible basis of agreement between our autonomy and nature that we enjoy in the aesthetic case can count as some sensible representation (*Versinnlichung*) of those as moral ideas as well.²⁰⁰

So Kant's striking idea is that pattern-matching cognition, via a priori intuition and the productive or schematizing imagination, is not only a basic feature of mathematical a priori intuition, where the pattern-matching activity runs directly from veridical schematic images to *mathematical structures*, but also a basic feature of moral a priori intuition, where the pattern-matching activity runs directly from veridical schematic images or mental models to *ethical structures*. Kant's own example is that a "handmill" could function as a symbol of a "despotic state." Of course Kant lived and died a full century before the invention of movies and the discovery of cinematic aesthetics. But it seems to me that a perfect example of an authoritative moral intuition that pre-reflectively consciously and essentially non-conceptually employs the symbolic structures of aesthetic schematic imagery in just the way that Kant's account proposes, would be the following: In order to intuit the meaning and necessary truth of the FHE—

So act that you use humanity, whether in your own person or in the person of any other, always at the same time as an end, never merely as a means. (*GMM* 4: 429)

—authoritatively, and thus with self-evidence, now call up before your mind any of the searing, unforgettable photographs or film footage of the Holocaust presented in Alain Resnais's brilliant 1955 documentary *Night and Fog*, as veridical images or mental models of *direct violations* of the FHE, i.e., of people being considered and treated as mere things, as nothing but garbage or offal, by the Nazis. In the 18th century it made good sense that a handmill could morally symbolize a despotic state. But the titanicly evil and cruelly despotic Nazi state, the experience of which, had he had the misfortune

to live through it, would have utterly horrified and sickened Kant, and which undoubtedly would have caused him to change his moral philosophy in several fundamental ways (e.g., by making lying seem a fairly low-level sort of wickedness, by comparison) was almost literally a human *incinerator*, not a handmill.

So just as mathematical or logical rational intuition is ultimately a matter of *mathematical or logical pattern-matching* insofar as it is authoritative, followed by *logico-mathematical construction* insofar as it constructed and fairly reliable, so too moral intuition is ultimately a matter of *moral pattern-matching* insofar as it is authoritative, followed by *logico-practical construction* insofar as it constructed and fairly reliable.

By sharp contrast to authoritative moral intuition in this Kantian Structuralist and Kantian Intuitionist sense, and as contemporary Experimental Philosophy or X-Phi clearly shows,²⁰¹ ordinary people's or even philosophers' shoot-from-the-hip so-called "moral intuitions" under controlled psychological experimental conditions, just like their shoot-from-the-hip so-called "mathematical intuitions" and "logical intuitions" under controlled psychological experimental conditions,²⁰² are mostly unreliable and "human, all too human." But as I argued in section **IV**, these experimental results are essentially irrelevant to the modal epistemology of mathematical and logical rational intuition. So too, correspondingly, these experimental results are essentially irrelevant to the modal epistemology of moral rational intuition.

For example, it is a basic authoritative objectively necessarily true a priori and High-Bar justified moral intuition that

It is morally wrong to torture completely innocent people to death, e.g., like the Nazis did, whatever the consequences.

Let us call this **The Anti-Torture Principle**. It is crucial to note that nowadays familiar thought-experiments involving torturing one innocent person in order to prevent a ticking time bomb from killing or maiming hundreds or thousands or millions of other innocent people, are essentially irrelevant to the truth of **The Anti-Torture Principle**, for three reasons. **First**, even if a moral agent did freely choose to torture one innocent person under such extreme conditions, in order to save hundreds or thousands or millions of other innocent people, that would clearly be a case of a *moral dilemma*, and the act of torturing *itself* would still be morally wrong and impermissible, whatever the consequences. So the agent would still have to take complete responsibility for doing it, even if a first-order substantive *ceteris paribus* moral principle to the effect that, other things being equal, one ought to prevent harm to other people, were still morally *directly relevant*. The agent did the wrong thing, *even though* her dilemma-driven and tragic choice to torture the one innocent person produced good results for many people.

Second, torturing one innocent person under such extreme conditions, although it is categorically wrong even despite the benevolent intentions of the agent, would clearly and distinctly *not* be torturing people *like the Nazis did*. The agent did the wrong thing, *even though* her dilemma-driven and tragic choice to torture the one innocent person was in all other respects nothing like the act of a Nazi. **Third**, choosing to torture one innocent person under such extreme conditions, although it is categorically wrong even despite the benevolent intentions of the agent, would not be in any sense *arbitrary*. The agent did the wrong thing, *even though* her dilemma-driven and tragic choice to torture the one innocent person was the very opposite of an arbitrary choice. Hence such

thought-experiments do not in any way impugn or undermine **The Anti-Torture Principle**.

Indeed, seriously denying and violating **The Anti-Torture Principle** would be to impugn and undermine human rationality and real human personhood *itself*, including one's *own* human rationality and real human personhood, and would be just like seriously denying that $3+4=7$ or **The Minimal Logical Meta-Principle of Non-Contradiction**, only much, much worse. That the Nazis *did* seriously deny and violate **The Anti-Torture Principle** with horrifyingly and sickeningly evil results, and that such unspeakably wicked torture and murder has, in human history, happened over and over and over again both before that time and also since that time, has no direct bearing at all on the alethic force and epistemic full reliability of that principle. Nazis and other deniers and violaters of **The Anti-Torture Principle** *did* and still *do* turn thereby themselves into *devils*, or moral monsters. Moral catastrophe, moral tragedy, and the most horrifying and sickening forms of moral evil more generally,²⁰³ as much as they may cry out for moral explanation, responsibility, reparation, or theodicy, do *not* in themselves constitute a challenge to moral modal epistemology.

XII. Conclusion

How do we authentically know objectively a priori that $3+4=7$, and more generally, how do we authentically know mathematical truths objectively a priori? The answer I have proposed in this essay is that we authentically know Primitive Recursive Arithmetic objectively a priori—including of course the simple objective and necessary arithmetical truth that $3+4=7$ —by means of *mathematical intuition via Hilbert's "objects of finitary reasoning,"* i.e., by cognitive generating and manipulating sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata, and then matching self-evident phenomenological patterns with corresponding truth-making parts of naturally realized mathematical structures, in such a way that LOCKING-ONTO and STRONG DISJUNCTIVISM ABOUT THE COGNITIVE CONSTRUCTION AND MANIPULATION OF VERIDICAL SENSIBLE FORMS IN KANTIAN A PRIORI INTUITION, ETC. are both satisfied, which in turn yields High-Bar justification via basic or non-basic authoritative rational intuitive knowledge of PRA or simple arithmetic. Then we know the rest of elementary Peano arithmetic, especially including its infinitary and universally quantified part, as well as all the other parts of mathematics, *constructively* and/or *inferentially*, with as much justification as can be provided by conceptual and logical reasoning that is necessarily anchored in the authentically objectively a priori knowable and mathematically intuitable finitary primitive recursive arithmetic base. All this, in turn, jointly vindicates two respectively basic and non-basic authoritative *philosophical* rational intuitions, **The Reliability of Rational Intuitions in Simple Arithmetic**—

At least some of the truths of Primitive Recursive Arithmetic, or PRA, are authentically knowable a priori by fully reliable basic authoritative rational intuitions on the basis of

Hilbert-style “objects of finitary reasoning,” i.e., on the basis of our cognitive construction and manipulation of sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata.,

and the Kantian-Brouwerian-Hilbertian epistemic principle, a.k.a. **The KBH**—

The KBH: Nothing will count as mathematical knowledge of any kind unless it presupposes our innately-specified rational human ability to know at least some of the finitist sub-structures of PRA or simple arithmetic in authoritative rational intuition, via the cognitive construction and manipulation of sensible forms in Kantian a priori intuition, mental models, mental diagrams, mental pictures, structural imagery, or schemata.

That brings us, finally, back again to Benacerraf’s Dilemma or BD. If Kantian Structuralism and Kantian Intuitionism are true, then both of Benacerraf’s preliminary philosophical assumptions about (1) a “standard, uniform” natural-language semantics of truth and (2) a “reasonable epistemology” of cognizing true statements—i.e.,

(I) Truth is uniform and Tarskian, and

(II) All human knowledge begins in causally-triggered sense experience

—are themselves necessarily true and also express basic authoritative philosophical rational intuitions, and the other four steps of BD are also true under plausible interpretations of them, but the unacceptably skeptical conclusion does *not* follow.

Mathematical objective a priori knowledge in the classical sense still *is* really possible, at the very least with respect to the theorems of PRA or simple arithmetic like our old friend “ $3+4=7$,” but in other fundamental parts of mathematics too. Kantian Structuralism and Kantian Intuitionism *also* jointly solve the classical application problem for mathematics. They *also* solve Benacerraf’s other problem about what the numbers could not be. They *also* explain why classical Logicism failed. They *also* account for the synthetic necessity of mathematical truth. And finally, they *also* provide a possible solution to the classical Problem of the Continuum. All of these very important individual theoretical virtues then

seem to me to add up very naturally to a single big sufficient reason for accepting my positive intuition-based solution to BD.

And that is not all. As I argued in sections **IX** and **X**, BD can also be extended to logic, and solved in essentially the same way.

Given Kantian Structuralism and Kantian Intuitionism about mathematics and logic, what is required for both mathematical and logical objective a priori truth and authentic knowledge of them is just a linguistically competent, healthy, developmentally normal, and (relatively) mature rational human animal, who can grasp both the essentially non-conceptual content of perception and also the conceptual and propositional content of statements or judgments, who has also learned the basics of PRA or simple arithmetic, who has also learned the basics of simple logic, and who is thus primed and ready for speaking her own natural language, and for essentially non-conceptually and pre-reflectively or first-order consciously but also conceptually and self-consciously intaking her manifestly real world through direct sense perception. And that is *all* that is required. For she is thereby capable of performing High-Bar justified necessarily true authoritative rational intuitions in mathematics, logic, morality, and philosophy, and thus capable of achieving authentic objective a priori knowledge according to the highest robustly normative principles of theory and practice. Mathematics, just like logic, and, as I have *also* argued, *just like morality* insofar as it is inherently governed by the hierarchical structuralist system of moral principles in Existential Kantian Ethics, and finally *just like philosophy itself*, is an *objective science*, and yet also inherently a *human science*. They are, all of them, robustly normative objective rational Moral Sciences.

In this way, by plausibly rejecting *both* classical Platonism *and* post-Bencerrafian skepticism about mathematical truth and knowledge, by plausibly *also* rejecting the skepticism and eliminativism of Experimental Philosophy, and then by *decisively* adopting the thesis that objectivity is the same as synthetically a priori necessary counterfactual universal rational human intersubjectivity (= Weak or Counterfactual Transcendental Idealism, or WCTI), *together with* a Kantian philosophy of mathematics and logic, we *thereby also* vindicate the metaphysical force and the full epistemic reliability of some basic authoritative intuitions in morality and philosophy, and find

Eden raised in the waste wilderness.

Or in other words: If my overall argument in this essay is sound, then Mathematical Psychologism is false, Scientific Naturalism is false, Radical Skepticism about Rational Intuitions and Radical Skepticism about Philosophical Intuitions Only are both false, Experimental Philosophy or X-Phi is not only irrelevant to modal epistemology but also false, Kantian Structuralism about mathematics and logic is true, Kantian Intuitionism about mathematics and logic is true, WCTI is true, and Preservationism about rational intuitions is true, and we thereby achieve the happy philosophical condition of *Objectivity Regained*.

So now let us go forth and multiply. And of course also add, subtract, divide, and correctly perform the other primitive recursive functions over the natural numbers too.²⁰⁴

NOTES

¹ For convenience I refer to Kant's works infratextually in parentheses. The citations include both an abbreviation of the English title and the corresponding volume and page numbers in the standard "Akademie" edition of Kant's works: *Kants gesammelte Schriften*, edited by the Königlich Preussischen (now Deutschen) Akademie der Wissenschaften (Berlin: G. Reimer [now de Gruyter], 1902-). For references to the first *Critique*, I follow the common practice of giving page numbers from the A (1781) and B (1787) German editions only. I generally follow the standard English translations from the German texts, but have occasionally modified them where appropriate. Here is a list of the abbreviations and English translations of the works cited:

- BL* "The Blomberg Logic." In *Immanuel Kant: Lectures on Logic*. Trans. J.M. Young. Cambridge: Cambridge Univ. Press, 1992. Pp. 5-246.
- CPJ* *Critique of the Power of Judgment*. Trans. P. Guyer and E. Matthews. Cambridge: Cambridge Univ. Press, 2000.
- CPR* *Critique of Pure Reason*. Trans. P. Guyer and A. Wood. Cambridge: Cambridge Univ. Press, 1997.
- GMM* *Groundwork of the Metaphysics of Morals*. Trans. M. Gregor. In *Immanuel Kant: Practical Philosophy*. Cambridge: Cambridge Univ. Press, 1996. Pp. 37-108.
- JL* "The Jäsche Logic," in *Immanuel Kant: Lectures on Logic*. Pp. 519-640.
- PC* *Immanuel Kant: Philosophical Correspondence, 1759-99*. Trans. A. Zweig. Chicago: Univ. of Chicago Press, 1967.
- Prol* *Prolegomena to any Future Metaphysics*. Trans. J. Ellington. Indianapolis, IN: Hackett, 1977.

² W.D. Ross, *Foundations of Ethics* (Oxford: Oxford Univ. Press, 1939), p. 144.

³ L. Wittgenstein, *Philosophical Investigations*, trans. G.E.M. Anscombe (3rd edn., New York: Macmillan, 1953), §81, p. 38^e.

⁴ A. Pap, *Semantics and Necessary Truth* (New Haven, CT: Yale Univ. Press, 1958), p. 422.

⁵ S. Kripke, *Naming and Necessity* (2nd edn., Cambridge, MA: Harvard Univ. Press, 1980), p. 42.

⁶ W. Tait, "Finitism," *Journal of Philosophy* 78 (1981): 524-546, at p. 546.

⁷ C. Parsons, *Mathematical Thought and its Objects* (Cambridge: Cambridge Univ. Press, 2008), p. 166.

⁸ The notion of objectivity covers both (i) knowledge, belief, or perception, and also (ii) what is known, believed, or perceived: so I will sometimes let 'objectively' qualify acts or states of knowing, believing, or perceiving, and sometimes let it qualify propositions, statements, states-of-affairs, objects, or other intentional targets of knowing, believing, or perceiving.

⁹ See R. Descartes, "Rules for the Direction of the Mind," in Descartes, *The Philosophical Writings of Descartes*, trans. J. Cottingham et al. (3 vols.; Cambridge: Cambridge Univ. Press, 1984), vol. I, pp. 9-78; and Descartes, "Meditations on First Philosophy" and "Objections and Replies," in Descartes, *The Philosophical Writings of Descartes*, vol. II, pp. 3-62 and 63-383, at pp. 24 and 103-105. Significantly, in the *Rules*, Descartes closely associates clear and distinct intuition and its indubitability with *imaginative visualization*. See S. Gaukroger, *Descartes: An Intellectual Biography* (Oxford: Clarendon/Oxford Univ. Press, 1995), pp. 115-124, and 158-181.

¹⁰ D. Hilbert, "Über das Unendliche," *Mathematische Annalen*, 95 (1926): 161-90, translated as "On the Infinite," in J. van Heijenoort (ed.), *From Frege to Gödel. A Source Book in Mathematical Logic, 1897-1931* (Cambridge, MA: Harvard Univ. Press, 1967), pp. 367-392, at p. 376.

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- ¹¹ Later, in section III, I will argue that some non-basic intuitions are also fully reliable and authoritative. But that refinement is not necessary for the point I am making right here.
- ¹² See, e.g., P. Johnson-Laird, *Mental Models* (Cambridge, MA: Harvard Univ. Press, 1983).
- ¹³ For an attempt at this, see R. Hanna, *The Rational Human Condition* (Unpublished MS, Fall 2011-Winter 2012 version).
- ¹⁴ W. Sellars, "Philosophy and the Scientific Image of Man," in Sellars, *Science, Perception, and Reality* (New York: Humanities Press, 1963), pp. 1-40, at p. 1.
- ¹⁵ See J.L. Mackie, *Ethics: Inventing Right and Wrong* (Oxford: Oxford Univ. Press, 1977).
- ¹⁶ J. Milton, "Paradise Regained," in J. Milton, *The Poems of John Milton* (2nd edn., New York: Ronald Press, 1953), pp. 495-544, at p. 495, book I, lines 1-7.
- ¹⁷ Benacerraf, "Mathematical Truth," at pp. 672-673.
- ¹⁸ See P. Benacerraf, "Mathematical Truth," *Journal of Philosophy* 70 (1973): 661-680, at pp. 672-673.
- ¹⁹ See A. Tarski, "The Concept of Truth in Formalized Languages," in A. Tarski, *Logic, Semantics, and Metamathematics*, trans. J.H. Woodger (2nd edn., Indianapolis, IN: Hackett, 1983), pp. 152-278; and A. Tarski, "The Semantic Conception of Truth and the Foundations of Semantics," *Philosophy and Phenomenological Research* 5 (1943-44): 341-375.
- ²⁰ Tarski, "The Concept of Truth in Formalized Languages," pp. 156-157.
- ²¹ Tarski, "The Concept of Truth in Formalized Languages," p. 155, underlining added.
- ²² See C. Parsons, "Kant's Philosophy of Arithmetic," in C. Parsons, *Mathematics in Philosophy* (New York: Cornell Univ. Press, 1983), pp. 110-149; R. Hanna, "Mathematics for Humans: Kant's Philosophy of Arithmetic Revisited," *European Journal of Philosophy* 10 (2002): 328-353; and R. Hanna, *Kant, Science, and Human Nature* (Oxford: Clarendon/Oxford Univ. Press, 2006), ch. 6.
- ²³ For a compelling argument against accepting a multiform semantics of truth, see T. Williamson, *The Philosophy of Philosophy* (Oxford: Blackwell, 2007), ch. 3.
- ²⁴ See, e.g., J. Katz, "What Mathematical Knowledge Could Be," *Mind* 104 (1995): 491-522.
- ²⁵ See, e.g., J. Divers and A. Miller, "Arithmetical Platonism: Reliability and Judgment-Dependence," *Philosophical Studies* 95 (1999): 277-310; and B. Hale and C. Wright, "Benacerraf's Dilemma Revisited," *European Journal of Philosophy* 10 (2002): 101-129.
- ²⁶ See, e.g., E. Sosa, "Reliability and the A Priori," in T. Szabó Gendler and J. Hawthorne (eds.), *Conceivability and Possibility* (Oxford: Clarendon/Oxford Univ. Press, 2002), pp. 369-384. In *Kant, Science, and Human Nature*, chs. 6-7, I work out Kant's idea that mathematical knowledge is grounded on reflective self-consciousness together with the imagination.
- ²⁷ One way of doing this would be via "plenitudinous platonism": For every consistently imaginable mathematical statement, there is a corresponding mathematical object. See, e.g., M. Balaguer, *Platonism and Anti-Platonism in Mathematics* (Oxford: Oxford Univ. Press, 1998). This construes imaginability as conceivability. But there are other ways of thinking about the imagination, e.g., Kant's conception of the productive imagination as a "schematizing" (i.e., mental modelling) capacity (*CPR* A84-147/B116-187, and esp. A120 n.). In *Rationality and Logic* (Cambridge, MA: MIT Press, 2006), ch. 6., I extended BD to logical knowledge, and then developed a strategy for solving the extended BD that starts with the thesis

that a reasonable epistemology should be modelled on the imagination, not on perception. So by the classification scheme described here, strictly speaking, that earlier solution counts as a pre-emptive negative or skeptical solution. But to the extent that the present solution postulates the innate presence of mental modelling abilities in sense perception, it also postulates the innate presence of the capacity for *imagination* within the capacity for sense perception. So in that sense, the present positive or anti-skeptical solution is really only an extension and refinement of the earlier solution.

²⁸ See, e.g., S. Shapiro, *Thinking about Mathematics* (Oxford: Oxford Univ. Press, 2000), chs. 6, 7, and 9.

²⁹ Many thanks to Catherine Legg for pushing me critically on this point.

³⁰ E. Sosa, "Minimal Intuition," in DePaul and Ramsey (eds.), *Rethinking Intuition*, pp. 257-269, at p. 268, underlining added.

³¹ See, e.g., R. Hanna, "Kant and Nonconceptual Content," *European Journal of Philosophy* 13 (2005): 247-290; R. Hanna, "Kantian Non-Conceptualism," *Philosophical Studies* 137 (2008): 41-64; R. Hanna and M. Chadha, "Non-Conceptualism and the Problem of Perceptual Self-Knowledge," *European Journal of Philosophy* 19 (2011); R. Hanna, "Beyond the Myth of the Myth: A Kantian Theory of Non-Conceptual Content," *International Journal of Philosophical Studies* 19 (2011): 321-396; and R. Hanna, *The Rational Human Condition* (Unpublished MS, Winter 2012 version), ch. 2.2.

³² See, e.g., J. Russell and R. Hanna, "A Minimalist Approach to the Development of Episodic Memory," *Mind and Language* 27 (2012): 29-54.

³³ See, e.g., R. Hanna, "Mathematics for Humans: Kant's Philosophy of Arithmetic Revisited," *European Journal of Philosophy* 10 (2002): 328-353. See also note 21.

³⁴ See, e.g., G. Bealer, "Intuition and the Autonomy of Philosophy," in DePaul and Ramsey (eds.), *Rethinking Intuition*, pp. 201-239, at pp. 207-214; and G. Bealer, "Modal Epistemology and the Rationalist Renaissance," in Szabó Gendler and Hawthorne (eds.), *Conceivability and Possibility*, pp. 71-125.

³⁵ See, e.g., Williamson, *The Philosophy of Philosophy*, esp. chs. 1, 2, and 7.

³⁶ See, e.g., Sosa, "Minimal Intuition," p. 259.

³⁷ I am indebted to Toni Kannisto for the idea that transcendental arguments and transcendental explanations are based on subjunctive conditionals (a.k.a. "counterfactuals").

³⁸ For other recent or contemporary conceptions of the a priori, see: L. Bonjour, *In Defense of Pure Reason* (Cambridge: Cambridge Univ. Press, 1998); A. Casullo, *A Priori Justification* (Oxford: Oxford Univ. Press, 2003); and P. Tidman, "The Justification of A Priori Intuitions," *Philosophy and Phenomenological Research* 56 (1996): 161-171.

³⁹ See C.I. Lewis, "A Pragmatic Conception of the A Priori," *Journal of Philosophy* 20 (1923): 169-177; C.I. Lewis, *Mind and the World Order* (New York: Dover, 1956); and W.V.O. Quine, "Two Dogmas of Empiricism," in W.V.O. Quine, *From a Logical Point of View* (2nd edn., New York: Harper and Row, 1961), pp. 2-46. Quine's conception of the a priori is essentially the same as Lewis's pragmatic conception, but adds confirmation/disconfirmation holism.

⁴⁰ See M. Menkin, "Stop Alien Abductions," at URL = <<http://www.stopabductions.com/>>.

⁴¹ See also T. Williamson, "How Deep is the Distinction Between A Priori and A Posteriori Knowledge," (Unpublished MS, December 2011 version), available online at URL = <http://www.philosophy.ox.ac.uk/__data/assets/pdf_file/0005/24395/Casullo.pdf>. Interestingly, Williamson regards the compatibility between a priority and empirical anchorage as decisive evidence of

the superficiality of the distinction between a priori and a posteriori knowledge. On the contrary, I think that it is decisive evidence of the robustness of *the classical Kantian distinction*, although I do agree that, incidentally, it *also* shows the superficiality of the distinction as it is handled in recent and contemporary work on the a priori. See, e.g., the work cited in notes 34, 38, and 39.

⁴² See, e.g., J. Kim, *Supervenience and Mind* (Cambridge: Cambridge Univ. Press, 1993), esp. part 1; Chalmers, *The Conscious Mind*, chs. 2-3; and T. Horgan, "From Supervenience to Superdupervenience: Meeting the Demands of a Material World," *Mind* 102 (1993): 555-586.

⁴³ Many thanks to Lloyd Humberstone for raising this objection in conversation.

⁴⁴ The philosophical trick is to show how the necessary and the a priori are necessarily equivalent without also conflating them. For two different ways of doing this, see R. Hanna, *Kant and the Foundations of Analytic Philosophy* (Oxford: Clarendon/Oxford Univ. Press, 2001), section 5.2; and for an alternative view, see N. Stang, "Did Kant Conflate the Necessary and the A Priori?," *Noûs* 44 (2011): 1-29.

⁴⁵ R. Cummins, "Reflections on Reflective Equilibrium," in M.R. DePaul and W. Ramsey (eds.), *Rethinking Intuition: The Psychology of Intuition and Its Role in Philosophical Inquiry* (Lanham, MD: Rowman and Littlefield, 1998), pp. 113-127, at p. 125, underlining added.

⁴⁶ J. Weinberg, "How to Challenge Intuitions Empirically Without Risking Skepticism," *Midwest Studies in Philosophy* 31 (2007): 318-343, at p. 340, underlining added.

⁴⁷ See D. Hume, *Treatise of Human Nature* (Oxford: Clarendon/Oxford Univ. Press, 1978); W. Sellars, "Empiricism and the Philosophy of Mind," in Sellars, *Science, Perception, and Reality*, pp. 127-196; and W.V.O. Quine, "Epistemology Naturalized," in Quine, *Ontological Relativity and Other Essays* (New York: Columbia Univ. Press, 1969), pp. 69-90.

⁴⁸ In addition to Cummins's and Weinberg's important papers (see notes 43-44), see also T. Szabó Gendler, *Intuition, Imagination, and Philosophical Methodology* (Oxford: Clarendon/Oxford Univ. Press, 2010); A. Goldman and J. Pust, "Philosophical Theory and Intuitional Evidence," in DePaul and Ramsey (eds.), *Rethinking Intuition*, pp. 178-197; A. Goldman, "Philosophical Naturalism and Intuitional Methodology," Romanell Lecture, in *Proceedings and Addresses of the American Philosophical Association* 84, 2 (2010): 115-150; also J. Knobe and S. Nichols (eds.), *Experimental Philosophy* (Oxford: Oxford Univ. Press, 2008); S. Stich, *The Fragmentation of Reason*. Cambridge, MA: MIT Press, 1990; and S. Stich, "Reflective Equilibrium, Analytic Epistemology, and the Problem of Cognitive Diversity," in DePaul and Ramsey (eds.), *Rethinking Intuition*, pp. 95-112.

⁴⁹ See Hanna, *Rationality and Logic*, ch. 5.

⁵⁰ My thoughts on this issue have been focused by reading Brian Talbot's interesting paper-in-progress, "The Dilemma of Calibrating Intuitions" (Unpublished MS, February 2011 version).

⁵¹ This particular assumption is skeptically deployed by S. Hales in "The Problem of Intuition," *American Philosophical Quarterly* 37 (2000): 135-147.

⁵² See Talbot, "The Dilemma of Calibrating Intuitions," section 2.

⁵³ See Sellars, "Philosophy and the Scientific Image of Man," especially sections II to V.

⁵⁴ L. Wittgenstein, *Culture and Value*, trans. P. Winch (Chicago, IL: Univ. of Chicago Press, 1980), p.56e.

⁵⁵ See Sellars, "Empiricism and the Philosophy of Mind."

⁵⁶ P. Maddy, *Second Philosophy* (Oxford: Oxford Univ. Press, 2007), p. 367, underlining added.

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- ⁵⁷ Sellars, "Empiricism and the Philosophy of Mind," p. 173.
- ⁵⁸ See, e.g., Maddy, *Second Philosophy*, part IV.
- ⁵⁹ See "West Brain, East Brain: What a Difference Culture Makes," *Newsweek* (Feb 18, 2010).
- ⁶⁰ In this case, my colleague Robert Rupert. Many thanks to him for formulating this application of Mathematical Psychologism.
- ⁶¹ See T. Skolem, "The foundations of elementary arithmetic established by means of the recursive mode of thought, without the use of apparent variables ranging over infinite domains," in J. v. Heijenoort (ed.) *From Frege to Gödel* (Cambridge, MA: Harvard Univ. Press, 1967), pp. 302-333; Parsons, *Mathematical Thought and its Objects*, chs. 5 and 7; Tait, "Finitism"; and A.S. Troelstra and D.V. Dalen, *Constructivism in Mathematics: An Introduction*, vol. 1 (Amsterdam: North Holland, 1988), pp. 120-126.
- ⁶² See, K. Gödel, "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems," in Van Heijenoort (ed.), *From Frege to Gödel*, pp. 596-617.
- ⁶³ Tait, "Finitism," p. 546.
- ⁶⁴ H.M. Sheffer, "Review of *Principia Mathematica*, Volume I, second edition," *Isis* 8 (1926): 226-231, at p. 228.
- ⁶⁵ See Hanna, *Rationality and Logic*, ch. 3.
- ⁶⁶ See, e.g., S. Shapiro, *Thinking about Mathematics* (Oxford: Oxford Univ. Press, 2000), pp. 212-220; and H. Putnam, *Philosophy of Logic* (New York: Harper Torchbooks, 1971), ch. 5.
- ⁶⁷ See, e.g., G. Boolos and R. Jeffrey, *Computability and Logic* (3rd edn., Cambridge: Cambridge Univ. Press, 1989), chs. 1-8.
- ⁶⁸ Parsons, "Kant's Philosophy of Arithmetic," p. 140.
- ⁶⁹ See Hanna, *Kant, Science, and Human Nature*, esp. chs. 1-4 and 6.
- ⁷⁰ P. Benacerraf, "What Numbers Could Not Be," *Philosophical Review* 74 (1965): 47-73.
- ⁷¹ M. Resnick, *Mathematics as a Science of Patterns* (Oxford: Clarendon/Oxford Univ. Press, 1997).
- ⁷² See S. Shapiro, *Philosophy of Mathematics: Structure and Ontology* (New York: Oxford Univ Press, 1997); and Shapiro, *Thinking about Mathematics*, ch. 10.
- ⁷³ See Parsons, *Mathematical Thought and its Objects*, esp. chs. 3, 5-6, and 9.
- ⁷⁴ See, e.g., D. Armstrong, *A Materialist Theory of the Mind* (London: Routledge, 1968); N. Block, "Troubles with Functionalism," in N. Block (ed.), *Readings in the Philosophy of Psychology*, 2 vols. (Cambridge: Harvard Univ. Press, 1980), vol. 1, pp. 268-305; D. Braddon-Mitchell and F. Jackson, *Philosophy of Mind and Cognition* (2nd edn.; Oxford Blackwell, 2007), esp. chs. 3, 5, 7, and 15; Kim, *Philosophy of Mind*, chs. 5-6; D. Lewis, "An Argument for the Identity Theory," *Journal of Philosophy* 63 (1966): 17-25; D. Lewis, "Psychophysical and Theoretical Identifications," *Australasian Journal of Philosophy* 50 (1972): 249-258; and D. Lewis, "Reduction of Mind," in S. Guttenplan (ed.), *A Companion to the Philosophy of Mind* (Oxford: Blackwell, 1994), pp. 412-431,
- ⁷⁵ See Block, "Troubles with Functionalism"; see also J. Searle, *Minds, Brains, and Science*. (Cambridge: Harvard Univ. Press, 1984).

⁷⁶ See, e.g., F. Jackson, “Mental Causation,” *Mind* 105 (1996): 377-413.

⁷⁷ This is not to say that I am a Functionalist about the mind—I’m not—although I do also defend a version of non-reductive Functionalism about *the body*. See Hanna and Maiese, *Embodied Minds in Action*, ch. 8. But if I *were* a Functionalist about the mind, then I *would* also adopt an interpretation of it that equally emphasizes functional roles and role-players.

⁷⁸ This is also not to say that I think that qualia exist—I don’t, and in fact I’m a qualia Eliminativist—although I do also defend the existence of intrinsic structural *phenomenal characters*. See Hanna and Maiese, *Embodied Minds in Action*, chs. 1-2.

⁷⁹ The standard responses to the epiphenomenalism problem are Causal Overdeterminationism and Reductionism. I reject both of these, and defend a non-reductive *jointly sufficient cause* solution to the problem of mental causation. See Hanna and Maiese, *Embodied Minds in Action*, chs. 6-7.

⁸⁰ See, e.g., R. Hanna, “Logic, Mathematics, and the Mind: A Critical Study of Richard Tiezen’s *Phenomenology, Logic, and the Philosophy of Mathematics*,” *Notre Dame Journal of Formal Logic* 50 (2009): 339-361.

⁸¹ See, e.g., URL = <http://en.wikipedia.org/wiki/Pikes_Peak>. I visited the summit of Pike’s Peak in summer 2010, and confirmed this claim by direct perception.

⁸² See R. Hanna, “The Inner and the Outer: Kant’s ‘Refutation’ Reconstructed,” *Ratio* 13 (2000): 146-174; and Hanna, *Kant, Science, and Human Nature*, ch. 1.

⁸³ Parsons, *Mathematical Thought and its Objects*, pp. 100-116.

⁸⁴ See Tait, “Finitism”; and also W. Tait, “Remarks on Finitism,” in W. Sieg, R. Sommer, and C. Talbott (eds.), *Reflections on the Foundations of Mathematics: Essays in honor of Solomon Feferman* (Urbana, IL: Association for Symbolic Logic, Lecture Notes in Logic, 2002), vol. 15, pp. 407-16.

⁸⁵ See, e.g., H. Field, *Science without Numbers: A Defense of Nominalism* (Princeton, NJ: Princeton Univ. Press, 1980); and H. Field, *Realism, Mathematics, and Modality* (Oxford: Blackwell, 1989).

⁸⁶ See, e.g., Maddy, *Second Philosophy*, part IV. Maddy’s philosophy of logic is, in effect, the reversed image of Kantian Structuralism. Her thesis is that rational human minds cognitively conform to the logical structures of the non-microphysical or manifest parts of natural “Kant-Frege” worlds (*Second Philosophy*, part III). By contrast, my thesis is that there are no such things as natural Kant-Frege worlds unless rational human animals are really possible. More precisely, a necessary condition of the existence and specific character of any natural Kant-Frege world is that if rational human animals *were* to exist in that world, then they *would* be able to perceive it veridically, judge it truly, and believe true propositions about it with sufficient justification (i.e., know it). Hence all K-F worlds manifestly and necessarily conform to the mental structures of the innately-specified cognitive capacities of rational human animals, whether or not any rational human animals happen to exist at any given time. Again, that is what I call Weak or Counterfactual Transcendental Idealism (WCTI).

⁸⁷ Parsons, *Mathematical Thought and its Objects*, pp. 80-100.

⁸⁸ See, e.g., L. Wittgenstein, *Remarks on the Foundations of Mathematics*, trans. G.E.M. Anscombe (2nd edn., Cambridge, MA: MIT Press, 1983).

⁸⁹ The Löwenheim-Skolem theorem, together with the Upward Löwenheim-Skolem theorem proved by Tarski, shows that Cantorian or transfinite arithmetic is a conservative extension of Peano arithmetic, especially including PRA, by showing (i) that a first-order mathematical theory has non-denumerably infinite models if and only if it has denumerably infinite models, and (ii) that a first-order mathematical

theory has denumerably infinite models only if it has denumerably finite models. See, e.g., G. Hunter, *Metalogic* (Berkeley and Los Angeles, CA: Univ. of California Press, 1996), pp. 189-190 and 201-208.

⁹⁰ See M. Potter, *Reason's Nearest Kin* (Oxford: Oxford Univ. Press, 2000).

⁹¹ L. Wittgenstein, *Philosophical Investigations*, trans. G.E.M. Anscombe (New York: MacMillan, 1953), §§112-115, pp. 47^e-48^e.

⁹² See Benacerraf, "What Numbers Could Not Be." This problem, in turn, is closely connected to Frege's "Caesar" problem. See G. Frege, *Foundations of Arithmetic*, trans. J.L. Austin (2nd edn.; Evanston, IL: Northwestern Univ. Press, 1953), p. 68.

⁹³ Parsons, *Mathematical Thought and its Objects*, p. 48.

⁹⁴ See, e.g., Parsons, *Mathematical Thought and its Objects*, pp. 272-293.

⁹⁵ P. Benacerraf, "What Mathematical Truth Could Not Be—I," in A. Morton and S. Stich (eds.), *Benacerraf and his Critics* (Oxford: Blackwell, 1996), pp. 9-59.

⁹⁶ See C. Wright, *Frege's Conception of Numbers as Objects* (Aberdeen: Aberdeen Univ. Press, 1983); B. Hale, *Abstract Objects* (Oxford: Blackwell, 1987); and B. Hale and C. Wright, *The Reason's Proper Study* (Oxford: Clarendon/Oxford Univ. Press, 2001).

⁹⁷ See C. Parsons, "Kant's Philosophy of Arithmetic," p. 131; and S. Shapiro, "Induction and Indefinite Extensibility: The Gödel Sentence is True, But Did Someone Change the Subject?," *Mind* 107 (1998): 597-624, at p. 604.

⁹⁸ See, e.g., S. Kripke, *Wittgenstein on Rules and Private Language* (Cambridge, MA: Harvard Univ. Press, 1982).

⁹⁹ See Hanna and Maiese, *Embodied Minds in Action*, esp. chs. 1-2 and 6-8.

¹⁰⁰ See also Hanna, *Kant and the Foundations of Analytic Philosophy*, chs. 3-5.

¹⁰¹ Many thanks to Jon Shaheen for helping me to get clearer on this point.

¹⁰² See, e.g., D. Struik, *A Concise History of Mathematics* (New York: Dover, 1967), p. 160.

¹⁰³ E. Husserl, *Logical Investigations*, 2 vols., trans. J.N. Findlay (London: Routledge and Kegan Paul, 1970), vol. 2, pp. 765 and 787, texts combined.

¹⁰⁴ L. Wittgenstein, *Tractatus Logico-Philosophicus*, trans. C.K. Ogden (London: Routledge and Kegan Paul, 1981), prop. 5.4731, p. 129.

¹⁰⁵ E. Gettier, "Is Justified True Belief Knowledge?," *Analysis* 23 (1963): 121-123.

¹⁰⁶ T. Williamson, *Knowledge and its Limits* (Oxford: Oxford Univ. Press, 2000), p. v.

¹⁰⁷ See Hanna, *The Rational Human Condition*, part 2/volume 2.

¹⁰⁸ See, e.g., C. Korsgaard, *Self-Constitution: Agency, Identity, and Integrity* (Oxford: Oxford Univ. Press, 2009).

¹⁰⁹ F. Dostoyevsky, *The Brothers Karamazov*, 2 vols., trans. D. Magarshack (Harmondsworth, Middlesex: Penguin Books, 1958), vol. 2, p. 743.

¹¹⁰ Bealer, "Intuition and the Autonomy of Philosophy," esp. pp. 205-206, and 218-221.

¹¹¹ See also C. Parsons, "Arithmetic and the Categories," *Topoi* 3 (1984): 109-121; C. Parsons, "Intuition and Number," in A. George (ed.), *Mathematics and Mind* (Oxford: Oxford Univ. Press, 1994), pp. 141-157; Parsons, "Kant's Philosophy of Arithmetic"; C. Parsons, "Mathematical Intuition," *Proceedings of the Aristotelian Society* 80 (1979-1980): 145-168; and C. Parsons, "Reason and Intuition," *Synthese* 125 (2000): 299-315.

¹¹² See R. Hanna, "Transcendental Idealism, Phenomenology, and the Metaphysics of Intentionality," in K. Ameriks and N. Boyle (eds.), *The Impact of Idealism* (Cambridge: Cambridge Univ. Press, forthcoming).

¹¹³ For a full development of this interpretation, see R. Hanna, *Kant and the Foundations of Analytic Philosophy* (Oxford: Clarendon/Oxford Univ. Press, 2001), chs. 1-2.

¹¹⁴ I think that Kant was mistaken that mental representations can be nonconscious, and on the contrary that necessarily, all mental representations are, at least, pre-reflectively conscious in some salient way. See Hanna and Maiese, *Embodied Minds in Action*, pp. 28-34. It is also possible that when Kant writes here that "synthesis in general is ... the mere effect of the imagination, of a blind though indispensable function of the soul, without which we would have no cognition at all, but of which we are seldom even conscious" he is confusing consciousness with self-consciousness or apperception.

¹¹⁵ G. Frege, "On Sense and Reference," in *Translations from the Philosophical Writings of Gottlob Frege*, trans. M. Black (Oxford: Blackwell, 1952), pp. 56-78.

¹¹⁶ See also R. Hanna, "Kant's Theory of Judgment," *The Stanford Encyclopedia of Philosophy* (Summer 2009 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2009/entries/kant-judgment/>.

¹¹⁷ See D. Schacter, "Perceptual Representation Systems and Implicit Memory: Towards a Resolution of the Multiple Memory Systems Debate," *Annals of the New York Academy of Science* 608 (1990): 543-571.

¹¹⁸ See (and hear) M.A. Numminen, "Wovon Man Nicht Sprechen Kann, Darüber Muss Man Schweigen," at URL = <<http://www.youtube.com/watch?v=57PWqFowq-4>>.

¹¹⁹ L.E.J. Brouwer, *Brouwer's Cambridge Lectures on Intuitionism*, (Cambridge: Cambridge Univ. Press, 1981), pp. 4-5.

¹²⁰ See also M. Giaquinto, *Visual Thinking in Mathematics* (Oxford: Oxford Univ. Press, 2007).

¹²¹ Wittgenstein, *Tractatus Logico-Philosophicus*, prop. 4.002, pp. 61-63, translation slightly modified.

¹²² See, e.g., N. Chomsky, *Knowledge of Language* (Westport, CN: Praeger, 1986).

¹²³ See Hanna, *Rationality and Logic*, esp. chs. 4-7.

¹²⁴ Parsons, *Mathematical Thought and its Objects*, p. 150.

¹²⁵ See B. Russell, *The Problems of Philosophy* (Oxford: Oxford Univ. Press, 1991), ch. 5; and B. Russell, "Knowledge by Acquaintance and Knowledge by Description," in B. Russell, *Mysticism and Logic* (Totowa, NJ: Barnes & Noble, 1981), pp. 152-167.

¹²⁶ See, e.g., Quine, *Philosophy of Logic*, p. 82.

¹²⁷ See, e.g., W. van Stigt, *Brouwer's Intuitionism* (Amsterdam: North Holland, 1990).

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- ¹²⁸ See, e.g., Tait, “Finitism”; and R. Zach, *Hilbert’s Finitism: Historical, Philosophical, and Meta-Mathematical Perspectives*, available online at URL = <http://people.ucalgary.ca/~rzach/static/hilbert.pdf>.
- ¹²⁹ See note 31.
- ¹³⁰ See note 31.
- ¹³¹ W. Tait, “Finitism,” *Journal of Philosophy* 78 (1981): 524-546, at p. 546.
- ¹³² See, e.g., R. Kanigel, *The Man Who Knew Infinity* (New York: Washington Square Press, 1991).
- ¹³³ See Hanna, *Rationality and Logic*, esp. chs. 2-4 and 6.
- ¹³⁴ See Hanna, *The Rational Human Condition*, part 2/volume 2, ch. 2.3.
- ¹³⁵ See, e.g., M. Potter, *Sets: An Introduction* (Oxford: Clarendon/Oxford Univ Press, 1990), ch. 3; and Potter, *Set Theory and its Philosophy*.
- ¹³⁶ See, e.g., Boolos and Jeffrey, *Computability and Logic*, ch. 25. In section X, Argument 1, I work out an argument for what is, in effect, the analogue of **The KBH** with respect to first-order monadic logic.
- ¹³⁷ J. Milton, “Paradise Lost,” in Milton, *The Poems of John Milton*, pp. 204-487, at p. 487, book XII, lines 641-649.
- ¹³⁸ Wittgenstein, *Philosophical Investigations*, §89, p. 42^c.
- ¹³⁹ L. Wittgenstein, *Tractatus Logico-Philosophicus*, trans. C.K. Ogden (London: Routledge & Kegan Paul, 1981), prop. 6.13, p. 169.
- ¹⁴⁰ See, e.g., C.I. Lewis, *Survey of Symbolic Logic* (Berkeley, CA: Univ. of California Press, 1918), pp. 1-2; and B. Russell, *Introduction to Mathematical Philosophy* (London: Routledge, 1993).
- ¹⁴¹ Wittgenstein, *Tractatus Logico-Philosophicus*, prop. 6.13, p. 169.
- ¹⁴² Wittgenstein, *Tractatus Logico-Philosophicus*, prop. 6.3751, p. 181.
- ¹⁴³ L. Wittgenstein, “Some Remarks on Logical Form,” *Proceedings of the Aristotelian Society*, suppl. vol. 9 (1929): 162-171.
- ¹⁴⁴ F. Waismann, *Wittgenstein and the Vienna Circle*, trans. J. Schulte and B. McGuinness (New York: Harper and Row, 1979), pp. 67-68.
- ¹⁴⁵ See also Hanna, *Kant, Science, and Human Nature*, ch. 6.
- ¹⁴⁶ I am counting non-Euclidean geometry as a conservative extension of Euclidean geometry, on the two-part ground that (i) the parallel postulate is logically independent of the basic Euclidean postulates and (ii) substituting either of the classical Riemannian or Lobachevskian alternatives for the parallel postulate does not entail the denial of any other Euclidean postulates.
- ¹⁴⁷ See, e.g., B. Russell, “Mathematical Logic as Based on the the Theory of Types,” in B. Russell, *Logic and Knowledge* (New York: G.P. Putnam’s Sons, 1971), pp. 59-102, at p. 63.
- ¹⁴⁸ See, e.g., M. Potter, *Reason’s Nearest Kin* (Oxford: Clarendon/Oxford Univ. Press, 2000), ch. 5.

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- ¹⁴⁹ See, e.g., B. Mates, *Elementary Logic* (2nd edn., New York: Oxford Univ. Press, 1972).
- ¹⁵⁰ A. Tarski, “The Semantic Conception of Truth and the Foundations of Semantics,” p. 371.
- ¹⁵¹ See also, e.g., C. Parsons, “Kant’s Philosophy of Arithmetic,” in C. Parsons, *Mathematics in Philosophy* (New York: Cornell Univ. Press, 1983), pp. 110-149, at p. 131; and S. Shapiro, “Induction and Indefinite Extensibility: The Gödel Sentence is True, But Did Someone Change the Subject?,” *Mind* 107 (1998): 597-624, at p. 604.
- ¹⁵² W.V.O. Quine, *Philosophy of Logic* (2nd edn., Cambridge, MA: Harvard Univ. Press, 1986), p. 64.
- ¹⁵³ W.V.O. Quine, “Carnap and Logical Truth,” in W.V.O. Quine, *The Ways of Paradox and Other Essays* (2nd edn., Cambridge, MA: Harvard Univ. Press, 1975), pp. 107-132, at p. 111.
- ¹⁵⁴ See also N. Denyer, “Pure Second-Order Logic,” *Notre Dame Journal of Formal Logic* 33 (1992): 220-224; and A. Paseau, “Pure Second-Order Logic with Second-Order Identity,” *Notre Dame Journal of Formal Logic* 51 (2010): 351-360. Pure second-order logic is second-order monadic logic without any functional or first-order variables, i.e., with systematic insensitivity as to whether domains are empty or non-empty. In this respect it is formally very similar to Kant’s pure general logic, although pure general logic does contain first-order variables ranging over comprehensions (*Umfangen*) of actual and possible individuals.
- ¹⁵⁵ See, e.g., Boolos and Jeffrey, *Computability and Logic*, chs. 10, 22, and 25, and esp. pp. 250-255.
- ¹⁵⁶ Quine, *Philosophy of Logic*, p. 81.
- ¹⁵⁷ See G. Priest, *In Contradiction*. (Dordrecht: Martinus Nijhoff, 1987); and G. Priest, “What is So Bad About Contradictions?,” *Journal of Philosophy* (1998): 410-426.
- ¹⁵⁸ See, e.g., Tarski, “The Semantic Conception of Truth and the Foundations of Semantics.”
- ¹⁵⁹ See, e.g., Gödel, “On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems.”
- ¹⁶⁰ Quine, *Philosophy of Logic*, p. 82.
- ¹⁶¹ Quine, “Two Dogmas of Empiricism,” pp. 22-23.
- ¹⁶² Quine, *Philosophy of Logic*, p. 81.
- ¹⁶³ See, e.g., N. Chomsky, *Knowledge of Language* (Westport, CT: Praeger, 1986); and R. Hanna, *Rationality and Logic* (Cambridge, MA: MIT Press, 2006).
- ¹⁶⁴ See, e.g., S. Haack, *Deviant Logic* (Cambridge: Cambridge Univ. Press, 1974); and G. Priest, *An Introduction to Non-Classical Logic* (Cambridge: Cambridge Univ. Press, 2001).
- ¹⁶⁵ See G. Priest, *In Contradiction*. (Dordrecht: Martinus Nijhoff, 1987); and G. Priest, “What is So Bad About Contradictions?,” *Journal of Philosophy* (1998): 410-426.
- ¹⁶⁶ In *Rationality and Logic*, ch. 3—see esp. p. 45—I did not adequately recognize the crucial difference between The Minimal Meta-Logical Principle of Non-Contradiction on the one hand, and other weak principles of classical logic on the other. Only The Minimal Metalogical Principle of Non-Contradiction is obeyed by *every possible* non-classical logic, e.g., by dialethic paraconsistent logics. The other weak principles of classical logic, by contrast, are undermined by logics that are either not truth-preserving or not consistency-preserving. Many thanks to Richard Grandy and Jeffrey Rowlands for pointing this out to me.

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- ¹⁶⁷ H. Putnam, "There is At Least One A Priori Truth," in H. Putnam, *Realism and Reason: Philosophical Papers, Vol. 3* (Cambridge: Cambridge Univ. Press, 1983), pp. 98-114, at pp. 100-101.
- ¹⁶⁸ H.M. Sheffer, "Review of *Principia Mathematica*, Volume I, second edition," *Isis* 8 (1926): 226-231, at p. 228.
- ¹⁶⁹ L. Carroll, "What the Tortoise Said to Achilles," *Mind* 4 (1895): 278-280.
- ¹⁷⁰ Quine, "Truth by Convention," in Quine, *The Ways of Paradox and Other Essays*, pp. 77-106, at p. 104.
- ¹⁷¹ S. Haack, "The Justification of Deduction," *Mind* 85 (1976): 112-119.
- ¹⁷² See Hanna, *Rationality and Logic*, ch. 3.
- ¹⁷³ O. O'Neill, "Vindicating Reason," in P. Guyer (ed.), *The Cambridge Companion to Kant* (Cambridge: Cambridge Univ. Press, 1992), pp. 280-308, at p. 305.
- ¹⁷⁴ This is not, however, to say that pure general logic is a "transcendental logic" in Kant's technical sense of that term.
- ¹⁷⁵ This sub-section draws directly on Hanna, *Rationality and Logic*, section 6.6.
- ¹⁷⁶ See, e.g., Shapiro, *Philosophy of Mathematics: Structure and Ontology*, chs. 3-5. For an extension of structuralism to logic, see e.g., A. Koslow, *A Structuralist Theory of Logic* (New York: Oxford Univ. Press, 1992).
- ¹⁷⁷ See Hanna, *Rationality and Logic*, chs. 4-5.
- ¹⁷⁸ See also Parsons, "Mathematical Intuition."
- ¹⁷⁹ Of course in perceiving an object we often generate an image of it too. But this is not, I think, absolutely necessary. Otherwise it would have to be the case that absolutely everything I perceive, I can in principle remember. But surely there is some sort of "representational paring-down" that occurs in the transition from perceptual content to memory content.
- ¹⁸⁰ See P. Johnson-Laird, *Mental Models* (Cambridge: Harvard Univ. Press, 1983); S. Kosslyn, *Image and Mind* (Cambridge: Harvard Univ. Press, 1980); S. Kosslyn, *Image and Brain* (Cambridge: Harvard Univ. Press, 1994); R. Shepard, "The Mental Image," *American Psychologist* 33 (1978): 125-137; R. Shepard and S. Chipman, "Second Order Isomorphisms of Internal Representations: Shapes of States," *Cognitive Psychology* 1 (1970): 1-17; R. Shepard and L. Cooper, *Mental Images and their Transformations* (Cambridge: MIT Press, 1982); and R. Shepard and J. Metzler, "Mental Rotation of Three-Dimensional Objects," *Science* 171 (1971): 701-703.
- ¹⁸¹ See, e.g., N. Block (ed.), *Imagery* (Cambridge: MIT Press, 1981); N. Block (ed.), *Readings in the Philosophy of Psychology*, 2 vols., (Cambridge: Harvard Univ. Press, 1980), vol. 2, part 2; and N. Block, "The Photographic Fallacy in the Debate about Mental Imagery," *Noûs* 17 (1983): 651-661.
- ¹⁸² See A. Pap, *Semantics and Necessary Truth* (New Haven, CT: Yale Univ. Press, 1958); and A. Pap, *Elements of Analytic Philosophy* (2nd edn., New York: Hafner, 1972).
- ¹⁸³ See, e.g., H.P. Grice, *Studies in the Way of Words* (Cambridge: Harvard Univ. Press, 1989); H.P. Grice and P.F. Strawson, "In Defense of a Dogma," *Philosophical Review* 65 (1956); and P.F. Strawson, *Analysis and Metaphysics* (New York: Oxford Univ. Press, 1992).

¹⁸⁴ See, e.g., Chalmers, “Foundations of Two-Dimensional Semantics; D. Chalmers and F. Jackson, “Conceptual Analysis and Reductive Explanation,” *Philosophical Review* 110 (2001): 315-360; and F. Jackson, *From Metaphysics to Ethics: A Defense of Conceptual Analysis* (Oxford: Oxford Univ. Press, 1998).

¹⁸⁵ Many thanks to Kevin White for urging me to make this contrast more explicit.

¹⁸⁶ See, e.g., Wittgenstein, *Tractatus Logico-Philosophicus*, prop. 4.022, p. 67.

¹⁸⁷ This modal framework is basically the same (with a few important differences, such as the general gloss on the notion of necessity, and the positive inclusion of synthetic necessity or “strong” metaphysical necessity) as that used by Chalmers in *The Conscious Mind*, pp. 52-71, and 136-138. See also S. Kripke, “Semantical Considerations on Modal Logic,” *Acta Philosophica Fennica* 16 (1963): 83-94; R. Montague, “Logical Necessity, Physical Necessity, Ethics, and Quantifiers,” in R. Montague, *Formal Philosophy* (New Haven, CT: Yale Univ. Press, 1974), pp. 71-83; and T. Smiley, “Relative Necessity,” *Journal of Symbolic Logic* 28 (1963): 113-134. For a closely related historical discussion of the analytic-synthetic distinction, see Hanna, *Kant and the Foundations of Analytic Philosophy*, chs. 3-5.

¹⁸⁸ Chalmers’s conception of logical or weak metaphysical necessity is also “two-dimensional,” a conception based mainly on earlier work by Kripke, David Kaplan, Robert Stalnaker, Gareth Evans, Martin Davies, and Lloyd Humberstone. See Chalmers, “The Foundations of Two-Dimensional Semantics,” in M. Garcia-Carpintero and J. Macia (eds.), *Two-Dimensionalism: Foundations and Applications* (Oxford: Oxford Univ. Press, 2004), pp. 55-140. The basic idea behind two-dimensionalism is that there are two distinct types of semantic functions from worlds to extensions, depending on the type of concept or intension one uses: (1) the “primary” intension (a function from subject-centered worlds considered as actual, to extensions) and (2) the “secondary” intension (a function from worlds considered as counterfactual variants on the indexically fixed actual world, to extensions). To each function or intension corresponds a different type of logical necessity. Analytic necessity corresponds to the primary intension; and a posteriori necessity corresponds to the secondary intension. For the notion of a posteriori necessity, see Kripke, *Naming and Necessity*. Of course two-dimensional modal semantics is controversial. The crucial point here for my purposes is that logical or analytic necessity in my sense will, in Chalmers’s framework, count as logical necessity according to the primary intension.

¹⁸⁹ Chalmers objects to strong or essentialist metaphysical necessity on the following three grounds: (a) that it is an ad hoc addition to the roster of modalities, (b) that it is brute and inexplicable, and (c) that the defenders of strong metaphysical necessity fail to provide an account of how humans get epistemic access to this modality. All of these objections may apply to conceptions of strong metaphysical necessity that take it to be a form of a posteriori necessity, and in particular identify it with physical necessity. But none of them apply to, e.g., Kant’s conception of strong metaphysical necessity as synthetic a priori necessity; see Hanna, *Kant and the Foundations of Analytic Philosophy*, ch. 5; and also Hanna and Maiese, *Embodied Minds in Action*, section 7.4. Leaving aside whatever worries one may have about my Kantian metaphysics of Weak or Counterfactual Transcendental Idealism, the crucial point here is simply that Chalmers’s objections do not generalize. Indeed, it is even arguable that strong or essentialist metaphysical necessity is more basic than logical necessity, since in the modal framework I have sketched there are going to be logical possibilities that are not real possibilities. For a similar idea, see S. Shalkowski, “Logic and Absolute Necessity,” *Journal of Philosophy* 101 (2004): 55-82.

¹⁹⁰ See, e.g., Hanna, *Kant and the Foundations of Analytic Philosophy*, chs. 3-5.

¹⁹¹ The simplification consists in separating the linguistic mental image I use in my intuition (in the example, *I* (#)) from the linguistic text (in the example, (*)) I use to represent the logical object. In most cases, the shape of the linguistic image and the shape of the linguistic text used to represent the logical object would be the same. Nevertheless the simplification is justified by psychological research strongly indicating that linguistic mental imagery is processed separately from the processing of either syntax or semantic content. See D. Schacter, “Perceptual Representation Systems and Implicit Memory: Toward a

Resolution of the Multiple Memory Systems Debate,” *Annals of the New York Academy of Science* 608 (1990): 543-571.

¹⁹² G. Frege, “Logic [1897],” in G. Frege, *Posthumous Writings*, trans. P. Long et al. (Chicago, IL: Univ. of Chicago Press, 1979), pp. 127-152, at p. 128.

¹⁹³ Ross, *The Right and the Good*, pp. 29-30, underlining added.

¹⁹⁴ Ross, *The Right and the Good*, p. 21.

¹⁹⁵ Ross, *The Right and the Good*, pp. 20, 33, and 41.

¹⁹⁶ Ross, *The Right and the Good*, pp. 30-31, and 145-148.

¹⁹⁷ See J.L. Mackie, *Ethics: Inventing Right and Wrong* (Oxford: Oxford Univ. Press, 1977).

¹⁹⁸ J. Rawls, *A Theory of Justice* (Cambridge, MA: Harvard Univ. Press, 1971), p. 41. Rawls correctly points out that unless Ross explicitly works out a monistic objective deontological lexical ordering or weighting scale for his prima facie duties in relation to actual act-contexts, then his view is a dead end. My objection is just the flip side of this, namely that Ross is clearly implicitly presupposing a monistic objective deontological lexical ordering or weighting scale for mapping his prima facie judgments to actual duties in actual contexts, without either admitting it or telling us precisely what it is. Many thanks to Eric Lee for pointing this out to me, and also to Kevin White for drawing my attention to the Rawls parallel.

¹⁹⁹ I also work out a fully explicit argument against Ethical Naturalism in *The Rational Human Condition*, section 3.04.

²⁰⁰ P. Guyer, *Kant and the Experience of Freedom* (Cambridge: Cambridge Univ. Press, 1993), pp. 40-41, underlining added.

²⁰¹ See, e.g., Appiah, *Experiments in Ethics*.

²⁰² See, e.g., Hanna, *Rationality and Logic*, ch. 5.

²⁰³ See, e.g., S. Neiman, 2002. *Evil in Modern Thought: An Alternative History of Philosophy* (Princeton: Princeton Univ. Press, 2002).

²⁰⁴ Many thanks to audiences at the Univ. of Tampere Finland, the Univ. of Colorado at Boulder USA, Monash Univ. Australia, and the Univ. of Victoria Canada, for very helpful critical comments on presentations of earlier versions of this material. I am also very grateful to the other members of The Foundations of Analytic Philosophy Research Group, a.k.a. The FAPRG (Andrew Chapman, Addison Ellis, Tyler Hildebrand, and Henry Pickford) for many discussions of issues in or around the topics of this essay, and to Catherine Legg for her detailed critical comments on an earlier version of section X.