Three Infinities in Early Modern Philosophy
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1. Quantitative Infinity

⇒ A General Characterization:

(1) There is a measure or size that can be applied to finite magnitudes, i.e., finite size; and
(2) This measure or size can be extended so as to apply in the infinite case, i.e., infinite size.

1.1 Locke on Infinite Size

A. Locke: “Finite, and Infinite, seem to me to be looked upon by the Mind, as the Modes of Quantity, and to be attributed primarily in their first designation only to those things, which have parts, and are capable of increase and diminution, by the addition or subtraction of any the least part: and such are the Ideas of Space, Duration, and Number.” (Essay, II.xvii.1)

• Disambiguate “quantity”: magnitude vs. size.
• Magnitude: “those things, which have parts…”.
• What is size? How is it extended to the infinite case?

B. Locke: “This further is observable in number, that it is that which the mind makes use of in measuring all things that by us are measurable, which principally are expansion [space] and duration; and our idea of infinity, even when applied to those [expansion and duration], seems to be nothing but the infinity of number.” (Essay, II.xvi.8, emphasis added)

C. Locke: “The same happens also in Space, wherein conceiving ourselves to be as it were in the Centre, we do on all sides pursue those indeterminable Lines of Number; and reckoning any way from ourselves, a Yard, Mile, Diameter of the Earth, or Orbis Magnus, by the infinity of Number, we add others to them, as often as we will; and having no Reason to set Bounds to those repeated Ideas, than we have to set Bounds to Number, we have that indeterminable Idea of Immensity. (Essay II.xvii.11, emphasis added)

D. Locke: “[B]y being able to repeat the idea of any length of duration we have in our minds, with all the endless addition of number, we come by the idea of eternity.” (Essay, II.xvii.5)

1.2 Extension: Two Examples

⇒ Cavalieri’s Geometria (1635) and Exercitationes (1647):

• Homogeneous magnitudes can be ordered by a relation < which determines a total ordering; that is, given A and B in the same class of magnitudes, either A < B or A = B or A > B.
• Homogeneous magnitudes A and B can be added to and subtracted from each other. Subtraction can be performed only when B is greater than A. This is denoted by A + B and B – A.
• Homogeneous magnitudes A and B can have a ratio to each other denoted by A: B.

⇒ Set Theory

• Mancosu: “Cantor’s theory of cardinal numbers offers a way to generalize arithmetic from finite sets to infinite sets using the notion of one-to-one association between two sets.” (“Measuring the Size of Infinite Collections of Natural Numbers”, p. 612)
1.3 Principles of Comparison and Galileo’s Paradox

\(\Rightarrow\) **Bijection Principle:** There are as many Fs as Gs (i.e., the collection of Fs is equal in size to the collection of Gs) iff there is a bijection from the Fs to the Gs.

- Cf. quotations C and D.

\(\Rightarrow\) **Euclid’s Axiom:** A whole is greater in size than any of its proper parts.

- Cf. quotation A.

\(\Rightarrow\) **Galileo’s Paradox:**

1. The collection of squares of natural numbers is a proper part of the collection of natural numbers.
2. **Euclid’s Axiom:** A whole is greater in size than any of its proper parts.
3. There is a bijection from the squares of natural numbers to the natural numbers.
4. **Bijection Principle:** There are as many Fs as Gs (i.e., the collection of Fs is equal in size to the collection of Gs) iff there is a bijection from the Fs to the Gs.
5. Conclusion: The collection of natural numbers is both equal in size to and greater in size than the collection of squares of natural numbers.

2. Two Approaches to God’s Infinity

2.1 A Quantitative Approach

E. Gassendi: “[A]lthough every supreme perfection is normally attributed to God, it seems that such perfections are all taken from things which we commonly admire in ourselves, such as longevity, power, knowledge, goodness, blessedness and so on. By amplifying these things as much as we can, we assert that God is eternal, omnipotent, omniscient, supremely good, supremely blessed and so on.” (Fifth Objections, AT VII.287-88/CSM II.200-201)

F. Locke: “If I find that I know some few things, and some of them, or all, perhaps imperfectly, I can frame an idea of knowing twice as many; which I can double again, as often as I can add to number; and thus enlarge my idea of knowledge, by extending its comprehension to all things existing, or possible... The degrees or extent wherein we ascribe existence, power, wisdom, and all other perfections (which we can have any ideas of) to that sovereign Being, which we call God, being all boundless and infinite, we frame the best idea of him our minds are capable of: all which is done, I say, by enlarging those simple ideas we have taken from the operations of our own minds, by reflection; or by our senses, from exterior things, to that vastness to which infinity can extend them.” (Essay, II.xxiii.34)

G. Nolan and Nelson: “Numbers provide a good analogy here. Natural numbers are endlessly augmentable insofar as any specified natural number, no matter how large, has a successor... [I]t might be argued that if one understands that natural numbers have no limit, this induces the idea of the cardinality of the natural numbers. Something like this is indicated in the modern mathematical concept...of “omega,” which is, as it were, the set of natural numbers viewed as complete. In other words, the modern mathematical idea of the cardinality of the natural numbers functions in a way similar to the idea of complete infinity (God) in Descartes’ philosophy.” (“Proofs for the Existence of God”, p. 108)
2.2 A Qualitative Approach

H. Descartes: “I say that the notion I have of the infinite is in me before that of the finite because, by the mere fact that I conceive being or that which is [l'être ou ce qui est], without thinking whether it is finite or infinite, what I conceive is infinite being; but in order to conceive a finite being, I have to take away something [retranche quelque chose] from this general notion of being, which must accordingly be there first.” (Letter to Clerselier of 23 April 1649, AT V.356/CSMK.377)

I. Descartes: “It should be observed that I never use the word ‘infinite’ to signify the mere lack of limit (which is something negative, for which I have used the term ‘indefinite’) but to signify a real thing which is incomparably greater than all those which are in some way limited.” (Letter to Clerselier of 23 April 1649, AT V.356/CSMK.377)

3. Descartes on Indefinite Infinity

J. Descartes: “There is, for example, no imaginable extension which is so great that we cannot understand the possibility of an even greater one; and so we shall describe the size of possible things as indefinite. Again, however many parts a body is divided into, each of the parts can still be indefinitely divisible. Or again, no matter how great we imagine the number of stars to be, we still think that God could have created even more; and so we will suppose the number of stars to be indefinite.” (Principles, I.26, AT VIIIA.15/CSM I.202, my emphasis)

K. Descartes: “Our reason for using the term ‘indefinite’ rather than ‘infinite’ in these cases is [that] in the case of God alone, not only do we fail to recognize any limits in any respect [omni ex parte], but our understanding positively tells us that there are none. Secondly, in the case of other things, our understanding does not in the same way positively tell us that they lack limits in some respect [alia ex parte]; we merely acknowledge in a negative way that any limits which they may have cannot be discovered by us.” (Principles, I.27, AT VIIIA.27/CSM I.201)

L. Beyssade: “…God is positively infinite in all respects and not just with respect to one kind of being [whereas] extended substance…is not [i.e., not unlimited] in all respects or absolutely infinite…” (La philosophie première de Descartes, p. 313)

⇒ A Metaphysical Distinction:

(Beyssade’s Proposal) The infinite is unlimited in each and every respect; indefinite things are unlimited in some but not all respects.

(My Proposal) The infinite is completely unlimited in that it is being unqualified; indefinite things are unlimited in some respect(s) in which they merely lack a limit (i.e., an upper bound).

4. The Relation between Indefinite Infinity and Quantitative Infinity

4.1 Leibniz’s Resolution of Galileo’s Paradox

M. Leibniz: “Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one [unum] and not a whole [totum].” (A 6.3.158)

N. Leibniz: “It is perfectly correct to say that there is an infinity of things, i.e., there are always more
than one can specify.” (New Essays, 2.17.1)

O. Leibniz: “When it is said that there are infinitely many terms, it is not being said that there is some specific number of them, but that there are more than any specific number.” (Letter to Bernoulli of 13 January 1699, GM 3:566)

⇒ Arthur’s Two Formulae: (where \( Fx := x \) is finite, and \( m \) and \( n \) stand for natural numbers)

(i) \( \exists m \forall n (F_n \rightarrow m > n) \)
(ii) \( \forall n \exists m (F_n \rightarrow m > n) \)

- Arthur’s Proposal: Leibniz rejects (i) and accepts (ii).

4.2 Indefinite vs. Quantitative Infinity

⇒ A Lockean Extension of Size:

\[ k \text{ is the size of some quantity } Q \text{ iff } k \text{ is a mode of } Q \text{ and:} \]
1) \( k = n \) for some natural number \( n \), or
2) For every quantity \( P \) with some size \( s \), \( Q \) is either greater than, equal to, or less than \( P \) (either \( k > s \), \( k = s \), or \( k < s \)).

⇒ A Third Formula (where \( Fx := x \) is finite, and \( k \) and \( l \) stand for sizes, on this notion of size):

(iii) \( \exists k \forall l (F_l \rightarrow k > l) \)

- Leibniz rejects (iii).
  - See quotations M and P.

P. Leibniz: “There is an actual infinite in the mode of a distributive whole, not of a collective whole. Thus something can be enunciated concerning all numbers, but not collectively. So it can be said that to every even <number> corresponds its odd <number>, and vice versa; but it cannot be accurately said that the multitudes of odds and evens are equal.” (GP II, 315, my emphasis)

- Hypothesis: Leibniz accepts the following constraint on size.

⇒ Measurability Constraint: \( x \) can have a size only if \( x \) is one \([unum]\) and a whole \([totum]\).

  - Cf. quotation M.

5. Conclusion

(ii) \( \approx \) indefinite infinity
(iii) \( \approx \) quantitative infinity
being unqualified = qualitative infinity

Locke rejects (i), accepts (ii), and arguably accepts (iii).
Leibniz rejects (i), accepts (ii), and arguably rejects (iii).
Descartes accepts (ii) and reserves qualitative infinity for God alone.