Analysis Prelim **August 2008**

1) (17 pts) Suppose f is a real valued Lebesgue measurable function defined on the real line. Suppose that $y \cdot f(y)$ is a Lebesgue integrable function on $[1, \infty)$. Let $g(x) = \int_1^\infty f(y)\sin(xy)dy$.

a) (7 pts) Explain why $g(x)$ exists for all real x.

b) (10 pts) Show $g'(x)$ exists for all real x and show

$$g'(x) = \int_1^\infty y \cdot f(y)\cos(xy)dy.$$

2) (17 pts) Let ν be a real valued signed measure on the measurable space (X, β). Then there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \emptyset$. Such a decomposition is called a Hahn decomposition. Define ν^+ and ν^- by $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$. Then ν^+ and ν^- are measures and $\nu = \nu^+ - \nu^-$. Define $|\nu|(E) = \nu^+(E) + \nu^-(E)$ for all $E \in \beta$.

a) (3 pts) Show $\nu^+(A) = \nu^-(B) = 0$.

b) (3 pts) Show $-\nu^-(E) \leq \nu(E) \leq \nu^+(E)$ and $\nu(E) \leq |\nu|(E)$ for all $E \in \beta$.

c) (6 pts) If f is measurable and bounded on X define $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$ and show that $|\int_E f d\nu| \leq \kappa |\nu|(E)$ for all $E \in \beta$. Here $|f(x)| \leq \kappa$ for all $x \in X$.

d) (5pts) Prove for all $E \in \beta$, $|\nu|(E) = \sup(|\int_E f d\nu| : |f(x)| \leq 1 \text{ for all } x \in E)$.

3) (17 pts) a) (3 pts) State Fatou's Lemma for a general measure space (X, β, μ). (No proof required.)

b) (10 pts) Let $\{f_n\}$ be a sequence of non-negative measurable extended real-valued functions defined on the measure space (X, β, μ). Suppose there exists an integrable function g defined on X such that $f_n(x) \leq g(x)$ for all $x \in X$ and for all positive integers n . Prove that $\int_X \limsup f_n d\mu \geq \limsup \int_X f_n d\mu$.

c) (4 pts) Construct a sequence $\{f_n\}$ of non-negative real-valued Lebesgue measurable functions such that $\limsup f_n(x) = \infty$ for almost all $x \in [0, 1]$ but $\int_{[0,1]} f_n dx \to 0$. Does your sequence of functions satisfy the hypothesis of part (b)?

4) (17 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be an element of $L^1(\mathbb{R})$ and suppose in addition that f is absolutely continuous on every interval of the form $[a, b]$, for $a.b \in \mathbb{R}$ and $a < b$. Assume also that the derivative $f' \in L^1(\mathbb{R})$. Prove there exists a constant C such that $|\int_{\mathbb{R}}(f(x+h) - f(x))dx| \leq C \cdot h$ for all $h > 0$.

5) (17 pts) Let E be a Lebesgue measurable set with $m(E) > 0$ where m is Lebesgue measure on $\mathbb{R}$. Given α with $0 < \alpha < 1$ show there exists an open interval I such that $m(E \cap I) > \alpha m(I)$.

(*Hint*: First prove the case when $m(E) < \infty$.)

6) (17 pts) Let f be a Lebesgue measurable function on $[0, \infty)$ with Lebesgue measure m. Define $\lambda_f(a) = m(\{x : |f(x)| > a\})$. Define $f^*(t) = \inf\{a : \lambda_f(a) \leq t\}$. Here f^* is called the decreasing rearrangement of f.

a) (4 pts) Define $f(t) = [t]$, where $[t]$ is the integer part of t for $0 < t < 3$ and $f(t) = 0$ for $t \geq 3$. Graph f^* for this special case.

b) (3 pts) In general, prove f^* is non-increasing.

c) (5 pts) in general, prove $\lambda_f(f^*(t)) \leq t$ for all $t > 0$ and $f^*(\lambda_f(a)) \leq a$ for all $a > 0$.

d) (5 pts) Show that in general, $\int_0^\infty |f| dt = \int_0^\infty f^* dt$.