Real Analysis Syllabus

The real analysis prelim will be based on the material related to the topics listed below. This list is not meant to be exhaustive, but is intended to be a guide to subjects to be studied thoroughly.

1. **Metric Spaces:** Completeness, compactness, connectedness, Ascoli-Arzela Theorem, Baire Category Theorem, Stone-Weierstrass Theorem.

2. **Measure and Integration:** Integration theory on general measure spaces including Lebesgue integral and Lebesgue-Stiel jes integral on the line, Lusin's Theorem, Egoroff's Theorem, Fatou's Lemma, the Monotone and Dominated Convergence Theorems, convergence in measure, dense subspaces of L^1, e.g., simple functions, continuous functions with compact support..., Radon-Nikodym Theorem, Tonelli and Fubini Theorems.

3. L^p **spaces:** Hoelder and Minkowski Inequalities, Riesz Representation Theorem for L^p spaces, completeness of L^p , dense subsets of L^p.

4. **Differentiation:** Differentiation of monotone functions, differentiation of an indefinite integral, functions of bounded variation, absolute continuity, Fundamental Theorem of Lebesgue Calculus, Lebesgue points.

   References:

   G. Folland, Real Analysis

   W. Rudin, Real and Complex Analysis

   H.L. Royden, Real Analysis (2nd Ed.)

   C. Apostol, Mathematical Analysis (2nd Ed.)

Real Analysis Prelim January 2008

1. Let (X, B, μ) be a measure space. Suppose f and g are non-negative integrable functions such that $A \in B$ implies
$$\int_A f d\mu = \int_A g d\mu.$$
Prove that $f = g$ a.e.

2. Suppose $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Suppose $f \in L^1(\mathbb{T}) =$ "'the absolutely Lebesgue integrable functions on $\mathbb{T}$."' If
$$\hat{f}(n) = \frac{1}{2\pi} \int_{\pi}^{\pi} f(x) e^{-int} dx,$$
prove that
$$\lim_{|n| \to \infty} \hat{f}(n) = 0.$$

3. Let $A \subset [0, 1]$ be a measurable set of positive measure. Prove that there exist $x, y \in A$ such that $x \neq y$ and $x - y$ is rational.

4. Let f, g be two nonnegative measurable functions on $[0, 1]$ such that $\int_0^1 f(x)dx = 1$, $\int_0^1 g(x)dx = 2$, $\int_0^1 (g(x))^2 dx = 5$. Let $E = \{x | g(x) \geq f(x)\}$. Show that $m(E) \geq \frac{1}{5}$ where m is Lebesgue measure.

5. Set
$$h(x) = \int_0^\infty (x + y)^{-1} f(y) dy$$
for $f \in L^1(0, \infty)$. Show that h is C^∞ away from the origin. Show as well that $h' \in L^1[a, \infty)$ for all $a > 0$.

6. Let $f : [0, 1] \mapsto \mathbb{R}$ be absolutely continuous and satisfy $f(0) = 0$ as well as
$$\int_0^1 |f'(x)|^2 dx < \infty.$$
Show that
$$\lim_{x \downarrow 0} x^{-\frac{1}{2}} f(x)$$
exists and find the limit.

Real Analysis Prelim August 2007

1. Let $\{f_n\}$ be a sequence of positive Lebesgue measurable functions defined on $[0,1]$ such that $\sum_{n=1}^{\infty} m(\{x \in [0,1] : f_n(x) > 1\}) < \infty$. Show that

$$\limsup_{n\to\infty} f_n(x) \leq 1$$

for almost all $x \in [0,1]$. Here m denotes Lebesgue measure on $[0,1]$.

2. Let $f \in L^p[0,1]$, where $2 < p \leq \infty$. Show that

$$\int_{[0,1]} \frac{|f(x)|}{\sqrt{x}} dx < \infty.$$

3. If $1 < p < \infty$, let $\{f_n\}$ be a sequence of elements of $L^p[0,1]$ such that there exists $M > 0$ with $||f_n||_p \leq M$ for all $n > 0$. Suppose that the sequence $\{f_n\}$ converges pointwise almost everywhere to a function f defined on $[0,1]$.

   (a) Prove that $f \in L^p[0,1]$.

   (b) Let q be the conjugate exponenet to p. Prove that for each function $g \in L^q[0,1]$,

   $$\lim_{n\to\infty} \int_{[0,1]} f_n(x)g(x)dx = \int_{[0,1]} f(x)g(x)dx.$$

   [Hint: you might find Egoroff's Theorem to be helpful.]

4. Let $[a,b]$ and $[c,d]$ be two closed and bounded intervals in $\mathbb{R}$.

   (a) If f_1 is defined on $[c,d]$ and satisfies a Lipschitz condition there, and f_2 is a function that is absolutely continuous on $[a,b]$ whose range is contained in $[c,d]$, show that the composite function $f_1 \circ f_2$ is absolutely continous on $[a,b]$. (Recall a function $f : [c,d] \to \mathbb{R}$ is said to satisfy a Lipschitz condition on $[c,d]$ if there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|, \forall x, y \in [c,d]$.)

   (b) Let $f : [a,b] \to \mathbb{R}$ be a function. For each of the following statements, either prove it is correct, or show it is false by providing a counterexample.

      i. If f is of bounded variation over $[a,b]$, then f is absolutely continuous on $[a,b]$.

      ii. If f is absolutely continuous on $[a,b]$, then f is of bounded variation over $[a,b]$.

5. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a bounded function, and suppose that for each fixed $t \in [0,1]$ the function $f(\cdot, t)$ is a Lebesgue measurable function of x. For each $(x,t) \in [0,1] \times [0,1]$ suppose the partial derivative $\frac{\partial f}{\partial t}$ exists. Suppose also that $\frac{\partial f}{\partial t}$ is bounded over $[0,1] \times [0,1]$. Prove that for all $t_0 \in [0,1]$

$$\frac{d}{dt}[\int_0^1 f(x,t)dx]_{t=t_0} = \int_0^1 \frac{\partial f}{\partial t}(x,t_0)dx.$$

6. Let f be a non-negative Lebesgue integrable function defined on $\mathbb{R}$, and let m_2 denote two-dimensional Lebesgue measure on $\mathbb{R}^2$.

   (a) Prove that the sets $\{(x,y) : 0 < y < f(x)\}$ and $\{(x,y) : 0 \leq y \leq f(x)\}$ are measurable with respect to the Lebesgue product measure on $\mathbb{R}^2$, and establish the equalities

   $$m_2(\{(x,y) : 0 < y < f(x)\}) = m_2(\{(x,y) : 0 \leq y \leq f(x)\}) = \int_{\mathbb{R}} f(x)dx.$$

(b) Define $\phi : [0, \infty) \to [0, \infty]$ by $\phi(t) = m(\{x \in \mathbb{R} : f(x) \geq t\})$. Prove that ϕ is a decreasing function of t and that

$$\int_{[0,\infty)} \phi(t)dt = \int_{\mathbb{R}} f(x)dx.$$

Real Analysis Prelim January 2006

1. For $1 \le p < \infty$, write L^p for the classical Banach space of all Lebesgue-integrable functions on $\mathbb{R}$ that satisfy $\int_{-\infty}^{\infty} |f|^p < \infty$, and where the norm is defined by

$$||f||_p = \left(\int_{-\infty}^{\infty} |f|^p dx \right)^{\frac{1}{p}}.$$

   (a) Suppose a sequence $\{f_n\}$ of elements of L^p converges to an element f of L^p, i.e., $\lim_{n\to\infty} ||f - f_n||_p = 0$. Prove that $\lim_{n\to\infty} ||f_n||_p = ||f||_p$.

   (b) Suppose p and q are numbers strictly between 1 and ∞, and assume that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose a sequence $\{f_n\}$ of elements of L^p converges to an element f of L^p, and that a sequence $\{g_n\}$ of elements of L^q converges to an element g of L^q. Prove that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n g_n dx$ converges to $\int_{-\infty}^{\infty} fg dx$.

2. For each $t \in \mathbb{R}$ and $f \in L^1$, where L^1 is as defined in Question 1, define

$$U_t(f)(x) = f(x - t), x \in \mathbb{R}.$$

   Fix $f \in L^1$. Prove that the map $t \to U_t(f)$ is uniformly continuous as a map from $\mathbb{R}$ to L^1.

3. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = \int_0^1 \frac{1}{x^x} dx.$$

   You may use the Gamma function evaluation $\int_0^{\infty} e^{-t} t^k dt = k!$.

4. If f is a real-valued function on $\mathbb{R}$, write S'_f for the set of all x such that f is differentiable at x. Call a set S a *differentiation* set if there exists a function f for which $S = S'_f$.

   (a) Show that S'_{f+g} contains $S'_f \cap S'_g$. Show by example that it can happen that $S'_{f+g} = \mathbb{R}$ but $S'_f \cap S'_g = \emptyset$.

   (b) Show that every open interval I is a differentiation set.

   (c) Let E be a Lebesgue-measurable subset of $\mathbb{R}$ with $m(E) < \infty$. Show that, for each $\epsilon > 0$, there exists a differentiation set S for which $m(E \Delta S) < \epsilon$, where m denotes Lebesgue measure, and $E \Delta S$ is the symmetric difference of the sets E and S.

5. Let f and g be real-valued Lebesgue measurable functions on $[0, 1]$, not assumed to be integrable. Let $E = \{(x, y) \in [0, 1] \times [0, 1] : f(x) = g(y)\}$.

   (a) Prove that E is measurable with respect to the Lebesgue product measure $m \times m$ defined on $[0, 1] \times [0, 1]$.

   (b) Suppose in addition that $m \times m(E) = 1$. Prove that there is a real constant c such that $f \equiv g \equiv c$, m a.e. on $[0, 1]$.

6. A sequence of functions $\{f_n\} \in L^1[0, 1]$ is said to be *uniformly integrable* if

$$\lim_{c\to\infty} \sup_n \int_{\{x\in[0,1]:|f_n(x)|<c\}} |f_n(x)| dx = 0.$$

   Show that for such a sequence, if $f_n \to f$ almost everywhere for some f, then $f_n \to f$ in $L^1[0, 1]$.

Step 1: First show that $\sup_n \int_{[0,1]} |f_n| dx < \infty$.
Step 2: Next show that the (almost-everywhere) limit f is indeed in $L^1[0,1]$.
Step 3: Last, estimate $\int_{[0,1]} |f - f_n| dx$, employing Egorov's Theorem to finish the job.

Real Analysis Prelim   August 2006

All problems are worth the same number of points.
Clearly explain the reasoning necessary to achieve your answers.

1. Let $\{f_n\}$ be a sequence of real-valued measurable functions defined on the real line. Let $E = \{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$. Show E is a measurable set.

2. Suppose (X, S, μ) is a measure space. Let g and h be real-valued measurable functions defined on X. Let $\{f_n\}$ be a sequence of real-valued measurable functions defined on X. Suppose that for all n, $g \le f_n \le h$, μ-almost everywhere.

3. Let $f \in L_1(R)$. Show $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\sin(nx)dx = 0$.

4. Let f be a real-valued measurable function defined on $[a, b]$. Then given $\epsilon > 0$ show there exists a continuous function g on $[a, b]$ such that $\mu\{x : f(x) \ne g(x)\} < \epsilon$. Here μ is Lebesgue measure.

5. Let f and g be real-valued absolutely continuous functions defined on $[0, 1]$.

   (a) Show that the product fg is also absolutely continous on $[0, 1]$.

   (b) If $g(0) = f(1) = 0$ then show $\int_0^1 f'(x)g(x)dx = -\int_0^1 f(x)g'(x)dx$. Also, explain why these integrals exist.

   (c) Suppose f and h are in $L_1(R)$. Suppose furthur that $\int_0^1 h(x)g(x)dx = -\int_0^1 f(x)g'(x)dx$ for all absolutely continous functions g with $g' \in L_\infty[0, 1]$ and $g(0) = 0$. Show f is absolutely continous and $h = f'$, μ-almost everywhere.
   Here μ is Lebesgue measure.

6. Prove the following. Let (X, S, μ) and $(Y, T\nu)$ be σ-finite measure spaces. Suppose $K(x, y)$ is a real-valued measurable function with respect to $\mu \times \nu$. Suppose there exists a $C > 0$ such that
$$\int_Y |K(x, y)|d\nu \le C, \text{ for all } x \in X$$
$$\int_X |K(x, y)|d\mu \le C, \text{ for all } y \in Y.$$
For all $f \in L_p(Y, T, \nu)$, $1 < p < \infty$ the function $T(f)$ defined on X by $T(f)(x) = \int_Y K(x, y)d\nu$ exists μ-almost everywhere. Show $||T(f)||_p \le C||f||_p$ where $||f||_p$ is the $L_p(Y, T, \nu)$ norm of f.

Real Analysis Prelim August 2005

1. Show that, if $f \in L^1(\mathbb{R})$, then

$$\lim_{\delta \to 0} \int_{\mathbb{R}} |f(x+\delta) - f(x)| dx = 0.$$

You may use the fact that the space $C_C(\mathbb{R})$ of continuous, complactly supported functions on $\mathbb{R}$ is dense in $L^1(\mathbb{R})$, with respect to the norm $||g||_1 = \int_{\mathbb{R}} |g(x)|dx$.

2. For this problem, you may take it as given that, if f and g are measurable functions on $\mathbb{R}$, then $F(x,y) = f(x-y)g(y)$ defines a measurable function on $\mathbb{R}^2$. The *convolution* $f * g$ of measurable functions f, g on $\mathbb{R}$ is the function $\int f * g$, of a real variable x, defined by

$$f * g(x) - \int_{\mathbb{R}} f(x-y)g(y)dy.$$

The domain of $f * g$ is the set of all x such that $f * g(x)$ is defined (that is, such that $F(x,y) = f(x-y)g(y)$ defines a Lebesgue integrable function of y).

(a) Show that, if f, g are continuous on $\mathbb{R}$ and vanish outside $\mathbb{R}^+ = (0, \infty)$ then

$$f * g(x) = \int_0^x f(x-y)g(y)dy,$$

and then use this to show that $f * g$ is continuous and vanishes outside $\mathbb{R}^+$. Hint for continuity: Note that

$$f * g(x) - f * g(x_0) = \int_0^{x_0} (f(x-y) - f(x_0 - y))g(y)dy + \int_{x_0}^x f(x-y)g(y)dy.$$

(b) Show that, if $f, g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$ (in particular, $f * g(x)$ is defined for almost all $x \in \mathbb{R}$. , and $||f * g|| - 1 \leq ||f||_1 ||g||_1$.

3. For this problem, you may assume that for any function f on $\mathbb{R}$ to $\mathbb{R}$ the set of all points of discontinuity is an F_σ set.
For the following three questions, if your answer is yes, explicitly construct such an example. If your answer is no, give an argument that no such function exists.

(a) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous at exactly one point?

(b) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous at every irrational point and discontinuous at every rational point?

(c) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous at every rational point, and discontinuous at every irrational point?

4. (a) Prove *Tchebyschev's inequality*, which says the, if f is nonnegative and integrable on $[0,1]$ and μ is Lebesgue measure on $[0,1]$, then

$$\mu(\{x \in [0,1] : f(x) \geq c\}) \leq \frac{1}{c} \int_{[0,1]} f(x)dx.$$

(b) We define an *ultraCauchy* sequence in a metric space (X, d) to be a sequence $a_1, a_2, ...$ in X such that, for some $c > 0$,

$$d(a_k, a_{k+1}) \leq \frac{c}{k^3}$$

for all $k \in \mathbb{Z}^+$.
Let $h_1, h_2, ...$ be an ultraCauchy sequence in $L^2(0,1)$, the space of square-integrable functions on $[0,1]$, with metric

$$d(f,g) = ||f - g||_2 = \sqrt{\int_{[0,1]} |f(x) - g(x)|^2 dx}.$$

Show that the h_k's converge pointwise almost everywhere: that is, show that $\lim_{k\to\infty} h_k(x)$ exists for almost all $x \in [0,1]$.
Hint: For each $k \geq 1$, define $G_k = \{x \in [0,1] : |h_k(x) - h_{k+1}(x)|^2 \geq \frac{1}{k^4}\}$. Use part (a) to show that, given $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $H_N = \cup_{k=N}^{\infty} G_k$ has a measure less than ϵ. Then show that, if $x \notin \cap_N H_N$, $\{h_k(x)\}$ is a Cauchy sequence.

Real Analysis Prelim January 2004

1. Let λ be Lebesgue measure on $[0, 1]$ and let $\phi : [0, 1] \to [a, b]$ be measurable. Define a set function μ on the Borel sets of $[a, b]$ by $\mu(A) = \lambda(\phi^{-1}(A))$.
   (a) Prove that μ is a measure on the Borel sets of $[a, b]$.
   (b) If $f : [a, b] \to \mathbb{R}$ is a bounded measurable function, probe that
   $$\int_{[0,1]} f(\phi(x))d\lambda(x) = \int_{[a,b]} f(x)d\mu(x).$$

2. Let $f : \mathbb{R} \to [0, \infty)$ be an integrable function such that
   $$\int_{\mathbb{R}} |x|f(x)dx < \infty.$$
   Prove that $\frac{d}{dt} \int_{\mathbb{R}} \cos(tx)f(x)dx = -\int_{\mathbb{R}} \sin(tx)xf(x)dx$.

3. Let f and g be bounded continuous functions on $[0, \infty)$. Define
   $$f * g(t) = \int_0^t f(t - x)g(x)dx, t \geq 0.$$
   If ϕ is a measurable function on $[0, \infty)$ define
   $$L(\phi)(s) = \int_0^\infty e^{-sx}\phi(x)dx$$
   for all $s > 0$ for which $\int_0^\infty e^{-sx}|\phi(x)|dx < \infty$.
   Prove that for all $s > 0$, $L(f * g)(s) = L(f)(s)L(g)(s)$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function that is an element of $L^1(\mathbb{R})$. For $n > 0$ set
   $$A_n = \{x \in \mathbb{R} : |f(x)| > n\}.$$
   (a) Prove that
   $$\lim_{n\to\infty} \lambda(A_n) = 0,$$
   where λ is Lebesgue measure on $\mathbb{R}$.
   (b) Prove that
   $$\lim_{n\to\infty} n\lambda(A_n) = 0.$$

5. (a) Suppose f is an integrable function on $\mathbb{R}$. Suppose further that f is absolutely continuous on each closed and bounded interval $[a, b]$, and that its derivative f' also is integrable over $\mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = 0$.
   (b) Give an example of an integrable, absolutely continuous function $f \in L^1(\mathbb{R})$ for which $\lim_{x\to\infty} f(x)$ does not exist.

Real Analysis Prelim August 2004

1. Let (X, B, μ) be a measure space. Suppose f and g are non-negative integrable functions such that $A \in B$ implies
$$\int_A f d\mu = \int_A g d\mu.$$
Prove that $f = g$ a.e.

2. Suppose $\mathbf{T}= \{z \in \mathbb{C} : |z| = 1\}$. Suppose $f \in L^1(\mathbf{T})$="The absolutely Lebesgue integrable functions on $\mathbf{T}$." If
$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-int} dx,$$
prove that
$$\lim_{|n|\to\infty} \hat{f}(n) = 0.$$

3. Let $A \subset [0, 1]$ be a measurable set of positive measure. Prove that there exist $x, y \in A$ such that $x \neq y$ and $x - y$ is rational.

4. Let f, g be two nonnegative measurable functions on $[0, 1]$ such that $\int_0^1 f(x)dx = 1$, $\int_0^1 g(x)dx = 2$, $\int_0^1 (g(x))^2 dx = 5$. Let $E = \{x|g(x) \geq f(x)\}$. Show that $m(E) \geq \frac{1}{5}$ where m is Lebesgue measure.

5. Set
$$h(x) = \int_0^\infty (x + y)^{-1} f(y)dy$$
for $f \in L^1(0, \infty)$. Show that h is C^∞ away from the origin. Show as well that $h' \in L^1[a, \infty)$ for all $a > 0$.

6. Let $f : [0, 1] \to \mathbb{R}$ be absolutely continuous and satisfy $f(0) = 0$ as well as
$$\int_0^1 |f'(x)|^2 dx < \infty.$$
Show that
$$\lim_{x\downarrow 0} x^{-\frac{1}{2}} f(x)$$
exists and find the limit.

Real Analysis Prelim January 2002

1. Let (X, ρ) be a separable metric space. Suppose $S \subseteq X$. Show that there exists a countable set $F \subseteq S$ such that F is dense in S, i.e., $S \subseteq \overline{F} =$ closure of F.

2. Let (X, S, μ) be a measurable space for which $\mu(X) < \infty$ and $E \in S$ implies $\mu(E) \geq 0$. Let $f : X \to \mathbb{R}$ be S-measurable. Suppose $g : \mathbb{R} \to \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ is Borel measurable. For a Borel set $E \subseteq \mathbb{R}$, define $\nu(E) = \mu(f^{-1}(E)$.

   (a) Show that ν is a measure.

   (b) Show that
   $$\int_{\mathbb{R}} g d\nu = \int_X g \circ f d\mu.$$

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a derivative at every point of $\mathbb{R}$. Is it always true that f' is bounded on $[-1, 1]$? It it is, give a proof. If not, give a counterexample.

4. Let $f : [0, \infty) \to \mathbb{R}$ be a bounded measurable function such that $\lim_{x\to\infty} x^2 f(x) = 1$. Find an integral expression for
   $$\lim_{\lambda \to 0^+} \frac{\int_0^\infty (1 - \cos(x)) f(\frac{x}{\lambda}) dx}{\lambda^2}.$$
   Justify your answer.

5. Let $\{f_n\}$ and f be measurable functions on $[0, 1]$ with Lebesgue measure. Suppose that f_n converges in measure to f, and that for some $p > 1$, $\sup_n ||f_n||_p < \infty$.

   (a) Prove that f_n converges to f in L^1.

   (b) Find a counterexample to the statement in part (a) if $p = 1$.

6. Let (X, S, μ) be a measure space such that $\mu(X) > 0$. Recall that $L^\infty(X, S, \mu)$ is a metric space under the L-infinity norm. An element E of S is called an *atom* if $\mu(E) > 0$ and whenever $F \in S$ satisfies $F \subseteq E$, then either $\mu(F) = 0$ or $\mu(E \setminus F) = 0$.

   (a) Suppose there is a sequence $E_1, E_2, ...$ in S such that $m \neq n$ implies $E_m \cap E_n = \emptyset$ and $\mu(E_n)$ is always strictly positive. Show that $L^\infty(X, S, \mu)$ is not separable.

   (b) Show that if $L^\infty(X, S, \mu)$ is separable, there there are finitely many atoms, $E_1, ..., E_n$ in S, such that
   $$\mu\left(X \setminus \cup_{k=1}^\infty\right) = 0.$$

Real Analysis Prelim August 2002

1. (16 pts) Let (X,d) be a metric space, and let $A \subseteq X$. Recall that a point $x \in X$ is said to be an *accumulation point* of the set A if every neighborhood of x contains a point $a \in A$ with $a \neq x$.

   (a) Let $x \in X$. Prove that the set $\{x\}$ is nowhere dense in X if and only if x is an accumulation point of X.

   (b) State, but do not prove, the Baire Category Theorem.

   (c) Prove that if (X,d) is complete and each $x \in X$ is an accumulation point of X, then X is uncountable.

   (d) Using part (iii), deduce that $\mathbb{R}$ is uncountable.

2. (17 pts)

   (a) Define simple functions, taking their domain to be a bounded interval $[a,b]$ in $\mathbb{R}$.

   (b) Show that sums and products of simple functions are simple.

   (c) Define step functions, taking their domain to be a bounded interval $[a,b]$ in $\mathbb{R}$.

   (d) Let f be a bounded measurable function on a bounded interval $[a,b]$. Define the Lebesgue integral of f on $[a,b]$.

   (e) Let f be a bounded measurable function on a bounded interval $[a,b]$, which is Riemann integrable. Show that the Riemann and Lebesgue integrals of f on $[a,b]$ coincide.

3. (16 pts) Let E be a Lebesgue-measurable subset of $\mathbb{R}$, and assume that the measure of E is 0. Let A_E be the subset of $\mathbb{R}^2$ consisting of the points (x,y) for which $x^2+y \in E$.

   (a) Assuming that A_E is a measurable subset of $\mathbb{R}^2$, compute its measure.

   (b) Prove that A_E is a Lebesgue-measurable subset of $\mathbb{R}^2$.

4. (17 pts) Let $f \in L^1([0,1])$, and suppose that $f_1, f_2, ...$ is a sequence of absolutely continuous functions on $[0,1]$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in [0,1]$.

   (a) Suppose the sequence $f_1', f_2', ...$ converges in L^1 to a function g. Prove that f is absolutely continuous.

   (b) Suppose that there exists an L^1 function g for which $|f_n'(x)| \leq g(x)$ for every $x \in [0,1]$ and for every $n \in \mathbb{N}$. Prove that f is absolutely continuous.

   (c) Show by giving an example, that the L^1 norms of the functions f_n' can be uniformly bounded, and yet f is not absolutely continuous.

5. (9 pts) Prove that it is impossible to find bounded sequences of numbers, $a_1, a_2, ...$ and $b_1, b_2, ...$ such that $\lim_{n\to\infty}[a_n \cos(nx) + b_n \sin(nx)] = 1$a.e. on $[0, 2\pi]$.

6. (16 pts) Prove Egoroff's Theorem, as stated here (m denotes Lebesgue measure): Let $f_1, f_2, ...$ be a sequence of measurable functions from $[0,2\pi]$ to $\mathbb{R}$ that converges a.e. to a real-valued measurable function f. Then for every $\epsilon > 0$, there is a subset $A_\epsilon \subset [0,2\pi]$ with $m(A_\epsilon) < \epsilon$, such that $f_1, f_2, ...$ converges uniformly to f on $[0,2\pi]\backslash A_\epsilon$.

7. (9 pts) Prove the following form of the Riemann-Lebesgue Lemma. If $f \in L_1(\mathbb{R})$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\cos(nx)dx = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\sin(nx)dx = 0.$$

Hint: As a first step, prove the statements for the case in which f is the characteristic function of a finite interval.

Real Analysis Prelim January 2001

1. Let m be Lebesgue measure on [0,1] and let $\{f_n\}$ be a sequence of measurable functions on [0,1]. Either prove or give a counter-example to each of the following statements:

   a) If f_n converges to f *a.e.* (m) then f_n converges to f in measure (m).

   b) If f_n converges to f in measure (m) then f_n converges to f *a.e.* (m).

2. Let $\alpha \in \mathbb{R}$ and set $f(x) = x \sin(\frac{1}{x^\alpha})$ if $x > 0$ and $f(0) = 0$. For which α is f of bounded total variation on [0,1]? Prove that your answer is correct.

3. Let m be Lebesgue measure on $\mathbb{R}$, and let $\{f_n\}$ be a sequence of functions in $L^2(m)$ with $||f_n||_2 \leq 1$ for all n.

   a) If f_n converges to f in measure (m) prove that $f \in L^2(m)$ and that $||f||_2 \leq 1$.

   b) If in addition $||f_n||_2 \to ||f||_2$ prove that $f_n \to f$ in $L^2(m)$.

4. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and $f : X \to [0, \infty)$ be a $\mathcal{B}$ measurable function. Let $\mathcal{A}$ be a sub σ-algebra of $\mathcal{B}$ and define $\nu : \mathcal{A} \to \mathbb{R}$ by $\nu(E) = \int_E f \, d\mu$.

   a) Prove that ν is a measure on $\mathcal{A}$.

   b) Prove that there is an $\mathcal{A}$ measurable function g on X such that for all $E \in \mathcal{A}$

$$\nu(E) = \int_E g \, d\mu.$$

5. Let $X = Y = [0, 1]$ and let B be the Borel sets in [0,1]. Let (X, B, μ) be Lebesgue measure on B and let (Y, B, ν) be counting measure on B. Let $D = \{(x, y) \in X \times Y : x = y\}$. Prove that $\int(\int \chi_D \, d\mu) \, d\nu$, $\int(\int \chi_D \, d\nu) \, d\mu$, and $\int \chi_D \, d(\mu \times \nu)$ are all unequal. Here χ_D is the indicator (characteristic) function of the set D.

6. a) Give an example of a measure space (X_0, B_0, μ) for which $1 \leq p < q \leq \infty$ implies that $L^p(\mu) \subset L^q(\mu)$. (No proof required)

   b) Give an example of a measure space (X_1, B_1, ν) for which $1 \leq p < q \leq \infty$ implies that $L^q(\nu) \subset L^p(\nu)$. (No proof required)

   c) Give an example of a measure space (X_2, B_2, λ) for which for any $1 \leq p < q \leq \infty$ both $L^p(\lambda) \not\subseteq L^q(\lambda)$ and $L^q(\lambda) \not\subseteq L^p(\lambda)$. (No proof required)

   d) Choose one of your examples in a), b), or c) and prove that it is an example having the claimed property.

Real Analysis Prelim August 2001

1. Let $f_1, f_2, ...$ be a sequence of continuous real-valued functions on a metric space (X, d) and suppose that the sequence converges uniformly to a function f on X. Prove that f is continuous. Is this result still true if the sequence is only assumed to converge pointwise to f? If so give a proof; if not give a counter-example and explain it.

2. Let (X, d) be a metric space and let $x_1, x_2, ...$ be a Cauchy sequence in X. Prove that if the sequence has a cluster point, x, then it converges to x.

3. A function $f : [0, 1] \to \mathbb{R}$ is said to be *Holder continuous of order 1/2* provided there exists a constant C such that $|f(x) - f(y)| \leq |x - y|^{\frac{1}{2}}$ for all x and y in $[0, 1]$. Let $f_1, f_2, ...$ be a sequence of Holder continuous functions of order 1/2, taking [0,1] to [0,1], such that the same constant C suffices for all of them. Show that there is a subsequence of $f_1, f_2, ...$ that converges uniformly on [0,1] to a continuous function f.

4. A function $f : [0, 1] \to \mathbb{R}$ is said to satisfy a Lipschitz condition provided there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all x, y in [0,1].

   (a) Show that a function f satisfying a Lipschitz condition on [0,1] is absolutely continuous.

   (b) Show that an absolutely continuous function f satisfies a Lipschitz condition iff $|f'|$ is bounded.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and of period 2π, i.e. $f(x + 2\pi) = f(x)$ for all x. Show that for every $\epsilon > 0$, there is a finite Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx))$$

   such that $|\phi(x) - f(x)| < \epsilon$ for all x.

6. Let $f \in L^1(\mathbb{R})$ and suppose that $g : \mathbb{R} \to \mathbb{R}$ has compact support and is infinitely differentiable. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \int_{\mathbb{R}} f(t)g(x - t)dt,$$

   for $x \in \mathbb{R}$. Show that this makes sense and that the resulting function is everywhere differentiable, with

$$h'(x) = \int f(x)g'(x - t)dt$$

   for $x \in \mathbb{R}$.

7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions such that $f(x) = g(y)$ for almost all pairs (x, y) relative to Lebesgue measure in the plane. Prove that f and g agree almost everywhere with constant functions.

Real Analysis Prelim August 2000

1. Suppose $f \in L_\infty[0,1]$.
   (a) Show that $f \in L_p[0,1]$ for all $1 \leq p < \infty$.
   (b) Show that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

2. Suppose f is a measurable real-valued function defined on $\mathbb{R}$. Suppose for all pairs of real numbers $a < b$, $0 \leq \int_a^b f(x)dx \leq b - a$. Show that $0 \leq f(x) \leq 1$ for almost all x.

3. Suppose f and g are positive measurable on [0,1] such that $f(x)g(x) \geq 1$ for all x. Prove that $1 \leq \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right)$.

4. Let f be a continuous function on $[0,\infty)$. Suppose $\lim_{x\to\infty} f(x)$ exists (and is finite). Prove that f is uniformly continuous on $[0,\infty)$.

5. A function f is continuously differentiable on (0,1). Let $f_n(x) = n[f(x+\frac{1}{n}) - f(x)]$ and let $0 < a < b < 1$. Show that $f_n(x)$ converges uniformly on (a,b). Find out the limit function.

6. Suppose f_n,g_n are non-negative finite functions on [0,1] related to each other by the (non-linear) integral equation

$$f_n(x) + \int_0^1 k(x,y)f_n^2(y)\,dy = g_n(x), 0 \leq x \leq 1,$$

   where $k(x,y)$ is a non-negative continuous function on $[0,1]\times[0,1]$. Suppose f_n are monotone decreasing in n (and so are g_n). Put $f = \lim_{n\to\infty} f_n$, $g = \lim_{n\to\infty} g_n$. Prove that

$$f(x) + \int_0^1 k(x,y)f^2(y)\,dy = g(x).$$

Real Analysis Prelim January 6, 1999

Each problem is worth 14 points.

1. If $f(x)$ is differentiable on $\mathbb{R}$, prove that $f'(x)$ is Borel measurable.

2. Let X be a set equipped with a σ-algebra $\mathcal{A}$. If m_i is a measure on $(X, \mathcal{A})$ for each i, such that $m_i(X) \leq 1$. Define $m : \mathcal{A} \to [0, +\infty)$ by $m(E) = \displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i} m_i(E)$ for all $E \in \mathcal{A}$. Prove that m is a measure on $(X, \mathcal{A})$.

3. Compute
$$\lim_{n\to\infty} \int_0^1 \frac{1}{(1+\frac{t}{n})^n t^{\frac{1}{n}}} dt.$$
Justify your answer

4. If $f(x)$ is a Lipschitz function on $\mathbb{R}$, i.e. there exists M such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$, prove that $f'(x)$ exists almost everywhere.

5. Let $I = [0, 1]$. Prove that the set of all polynomials with rational coefficients is dense in $L^p(I)$, $1 \leq p < \infty$.

6. Suppose that $I = [0, 1]$, $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$. $\int |f(x)|^p dx < \infty$, $g(x) \neq 0$ for all $x \in I$ and $\int |g(x)|^q dx < \infty$. Prove $\displaystyle\int_I |f(x)g(x)| dx \geq \left(\int_I |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_I |g(x)|^q dx\right)^{\frac{1}{q}}$.

7. Assume that f is Lebesgue integrable on $[0, 2\pi]$. Prove that
$$\lim_{n\to\infty} \frac{\pi}{2} \int_0^{2\pi} f(x)|\sin(nx)| dx = \int_0^{2\pi} f(x) dx.$$

Real Analysis Prelim   August 1999

1. If $1 < p < \infty$ and $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f_n \to f$ in L^p, $g_n \to g$ in L^q, then show that $f_n g_n \to fg$ in L^1.

2. Assume f is continuous on $\mathbb{R}$. Put

$$f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(x + \frac{k}{n})$$

Prove that $f_n(x)$ converges uniformly on every closed finite interval $[a, b]$. Find the limiting function.

3. If $A \subset \mathbb{R}^2$ is measurable and every vertical line meets A in a countable set, what is the Lebesgue measure of A?

4. (a) Give an example of a sequence of integrable functions f_n on $\mathbb{R}$ that are continuous and converge pointwise to an integrable function f and are such that

$$\int f_n \not\to \int f.$$

(b) Let f_n be a sequence of nonnegative measurable functions on $[0, 1]$ that converge to f. If f is integrable and $f_n \leq f + 1$ $\forall n$ show that

$$\int_0^1 f = \lim \int_0^1 f_n.$$

5. Let f be a positive continuous function such that $\int_{-\infty}^{\infty} f(x)\, dx$ is finite. Assume that, for some $T > 0$, f is decreasing on $[T, \infty)$ and it is increasing on $(-\infty, T]$. Show that

$$F(x) = \sum_{k=-\infty}^{\infty} f(x + k)$$

is a continuous periodic function with the period 1.

6. Let ϕ_n, $n = 1,2,\ldots$ be a complete orthonormal system in the space $L^2(\mathbb{R}, \lambda)$, where λ stands for Lebesgue measure.

   (a) Show that, for every Borel set $B \subset \mathbb{R}$ of strictly positive Lebesgue measure, one has

   $$1 \leq \int_B \sum_1^\infty |\phi_n(x)|^2 dx.$$

   Hint: Assuming $\lambda(B) < \infty$, apply the Parceval identity to 1_B. Then use Schwartz's inequality.

   (b) Show that $\sum_1^\infty |\phi_n(x)|^2 = \infty$ a.e.

7. Every real number $x \in [0, 1)$ has the unique binary representation

   $$x = \sum_{n=1}^\infty s_n(x) 2^{-n}$$

   where $s_n(x) = 0$ or 1 and $s_n(x) = 0$ for infinitely many n.

   (a) Show that $s_n(x)$ are Borel measurable functions.

   (b) Evaluate

   $$\int_0^1 (s_1 + \cdots + s_n)\, dx \text{and } \int_0^1 (s_1 + \cdots + s_n)^2\, dx.$$