Solutions to exercises in Chapter 11

E11.1 *Show that fin(ω, ω_1) does not have ccc.*

Solution: $\langle\{(0,\alpha)\} : \alpha < \omega_1\rangle$ is a system of pairwise incompatible elements.

E11.2 *Show that fin(ω, ω_1) preserves cardinals $\geq \omega_2$.*

Solution: Since $|\text{fin}(\omega, \omega_1)| = \omega_1$, obviously $\text{fin}(\omega, \omega_1)$ satisfies the ω_2-cc. Hence it preserves cardinals $\geq \omega_2$ by Proposition 4.4.

E11.3 *Here we work only in ZFC (or in a fixed model of it). Suppose that $(X, <)$ is a linear order. Let $\mathbb{P}$ be the set of all pairs (p, n) such that $n \in \omega$ and $p \subseteq X \times \mathscr{P}(n)$ is a finite function. Define $(p, n) \leq (q, m)$ iff $m \leq n$, $\text{dmn}(q) \subseteq \text{dmn}(p)$, $\forall x \in \text{dmn}(q)[p(x) \cap m = q(x)$, and*

$$\forall x, y \in \text{dmn}(q), \text{ if } x < y \text{ then } p(x)\backslash p(y) \subseteq m.$$

Show that $\mathbb{P}$ has ccc.

Solution: Suppose that $\mathscr{A}$ is an uncountable subset of P. By the Δ-system theorem, we may assume that $\langle\text{dmn}(p) : (p, n) \in \mathscr{A}\rangle$ is a Δ-system, say with root r. We may also assume that $p \restriction r = q \restriction r$ whenever $(p, n), (q, m) \in \mathscr{A}$. Now suppose that $(p, n), (q, m) \in \mathscr{A}$. Let s be the maximum of m and n. Clearly $p \cup q$ is a function, and so $(p \cup q, s) \in P$. We claim that $(p \cup q, s) \leq (p, n), (q, m)$, as desired. By symmetry it suffices to show that $(p \cup q, s) \leq (q, m)$. Suppose that $x \in \text{dmn}(q)$. Then $(p \cup q)(x) \cap m = q(x) \cap m = q(x)$. If $x, y \in \text{dmn}(q)$ and $x < y$, then $(p \cup q)(x)\backslash(p \cup q)(y) = q(x)\backslash q(y) \subseteq m$.

E11.4 *Continuing exercise E11.3, suppose that we are working in a c.t.m. M of ZFC. Let G be $\mathbb{P}$-generic over M. For each $x \in X$ let*

$$a_x = \bigcup\{p(x) : (p, n) \in G \text{ for some } n \in \omega, \text{ with } x \in \text{dmn}(p)\}.$$

Thus $a_x \subseteq \omega$. Show that if $x < y$, then $a_x\backslash a_y$ is finite.

Solution: For each $z \in X$ let $D_z = \{(p, n) : z \in \text{dmn}(p)\}$. Given any $(q, m) \in P$, if $z \notin \text{dmn}(q)$ clearly $(q \cup \{(z, 0)\}, m) \in P$, $(q \cup \{(z, 0)\}, m) \in D_z$, and $(q \cup \{(z, 0)\}, m) \leq (q, m)$. So D_z is dense.

Choose $(p, n) \in D_x \cap G$ and $(q, m) \in D_y \cap G$. Say $(p, n), (q, m) \geq (r, s) \in G$. We claim then that $a_x\backslash a_y \subseteq s$. Let $i \in a_x\backslash a_y$. Say $i \in u(x)$ with $(u, t) \in G$ and $x \in \text{dmn}u$. Say $(u, t), (r, s) \geq (v, z) \in G$. Thus $v(x)\backslash v(y) \subseteq s$. Now $i \in u(x)$, so $i \in v(x)$. Also, $i \notin v(y)$ since $i \notin a_y$. So $i \in s$, as desired.

E11.5 *Continuing exercises E11.3 and E11.4, show that if $x < y$, then $a_y\backslash a_x$ is infinite. Hint: for each $i < \omega$ let*

$$E^i = \{(p, n) : x, y \in \text{dmn}(p) \text{ and } |p(y)\backslash p(x)| \geq i\},$$

and show that E^i is dense.

Solution: For each $i < \omega$ let

$$E^i = \{(p, n) : x, y \in \text{dmn}(p) \text{ and } |p(y)\backslash p(x)| \geq i\}.$$

We claim that E^i is dense. Let (q,n) be given. Wlog $x,y\in \text{dmn}(q)$. Say dmn(q) is

$$u_0<\cdots<u_j=x<\cdots<u_{m-1}.$$

Let $\text{dmn}(r)=\text{dmn}(q)$, $r(u_t)=q(u_t)$ for $t\le j$; choose $w>n$ with $|w-n|=i$, and let $r(u_t)=q(u_t)\cup(w\backslash n)$ for $j<t$. Then $(q,n)\ge(r,w)\in E^i$, as desired.

Now for any $i\in\omega$ we show that $|a_y\backslash a_x|\ge i$. Choose $(p,n)\in E^i\cap G$. We claim that $p(y)\backslash p(x)\subseteq a_y\backslash a_x$ (as desired). Let $j\in p(y)\backslash p(x)$. So $j\in a_y$, and $j<n$ since $p(y)\subseteq n$. Suppose that $j\in a_x$. Say $y\in q(x)$, with $(q,v)\in G$ and $x\in\text{dmn}(q)$. Say $(p,n),(q,v)\ge(r,s)\in G$. Then $j\in r(x)$ since $j\in q(x)$. Hence $j\in r(x)=p(x)\cap n$, so $j\in p(x)$, contradiction.

E11.6 *Suppose that M is a ctm of ZFC, and $(X,<)\in M$ is a linear order. Let $\mathbb{P}$ be as in the preceding exercises, and suppose that G is $\mathbb{P}$-generic over M. Show that in $M[G]$, if $x<y$ then $a_x\backslash a_y$ is finite and $a_y\backslash a_x$ is infinite, where $\langle a_x:x\in X\rangle$ is as in E11.4.*

Solution: This is immediate from the above exercises.

E11.7 *Suppose that M is a ctm of ZFC, $\mathbb{P}\in M$ is a quasi-order, G_1,G_2 and $\mathbb{P}$-generic over M, and $G_1\subseteq G_2$. Show that $G_1=G_2$.*

Solution: Suppose that $p\in G_2\backslash G_1$. By 9.3(i), choose $q\in G_1$ such that $q\perp p$ or $q\le p$. Since $p\notin G_1$, it follows that $q\perp p$. But $p,q\in G_2$, contradiction.

For the next few exercises, suppose that M is a ctm of ZFC, and $\mathbb{P}$ and $\mathbb{Q}\in M$ are quasi-orders. A dense embedding of $\mathbb{P}$ into $\mathbb{Q}$ is a function $f:P\to Q$ such that the following conditions hold:

(1) If $p_1,p_2\in P$ and $p_1\le p_2$, then $f(p_1)\le f(p_2)$.

(2) If $p_1,p_2\in P$ and $p_1\perp p_2$, then $f(p_1)\perp f(p_2)$.

(3) For any $q\in Q$ there is a $p\in P$ such that $f(p)\le q$.

Now assume that $f:P\to Q$ is a dense embedding.

E11.8 *Show that if H is $\mathbb{Q}$-generic over M, then $f^{-1}[H]$ is $\mathbb{P}$-generic over M.*

Solution: First we show that $f^{-1}[H]$ is a filter. If $p_1\in f^{-1}[H]$ and $p_1\le p_2$, then $f(p_1)\in H$ and $f(p_1)\le f(p_2)$, so $f(p_2)\in H$ and hence $p_2\in f^{-1}[H]$. Now we show that any two members of $f^{-1}[h]$ are compatible. (See Lemma 9.7.) Suppose that $p_1,p_2\in f^{-1}[H]$. Thus $f(p_1),f(p_2)\in H$. So they are compatible. Hence p_1 and p_2 are compatible.

Now suppose that D is dense in $\mathbb{P}$. We claim that $f[D]$ is dense in $\mathbb{Q}$. For, suppose that $q\in Q$. Choose $p_1\in P$ such that $f(p_1)\le q$. Then choose $p_2\in D$ such that $p_2\le p_1$. Then $f(p_2)\le f(p_1)\le q$, as desired. Since $f[D]$ is dense in $\mathbb{Q}$, choose $q\in f[D]\cap H$. Say $q=f(p_3)$ with $p_3\in D$. Thus $p_3\in f^{-1}[H]\cap D$, as desired.

E11.9 *For G $\mathbb{P}$-generic over M, define $g(G)=\{q\in Q:\exists p\in G[f(p)\le q]\}$. Show that $g(G)$ is $\mathbb{Q}$-generic over M.*

Solution: Suppose that $q_1\in g(G)$ and $q_1\le q_2$. Clearly then $q_2\in g(G)$.

Suppose that $q_1, q_2 \in g(G)$. Choose, accordingly, $p_1, p_2 \in G$ such that $f(p_1) \leq q_1$ and $f(p_2) \leq q_2$. Choose $p_3 \in G$ such that $p_3 \leq p_1, p_2$. Then $f(p_3) \in g(G)$ and $f(p_3) \leq q_1, q_2$. So $g(G)$ is a filter on $\mathbb{Q}$.

Now suppose that D is dense in $\mathbb{Q}$. Let $E = \{p \in P : \exists q \in D[f(p) \leq q]\}$. We claim that E is dense in $\mathbb{P}$. For, let $p_1 \in P$ be given. Choose $q \in D$ such that $q \leq f(p_1)$. Then choose $p_2 \in P$ such that $f(p_2) \leq q$. Thus $p_2 \in E$. Now $f(p_2) \leq f(p_1)$. It follows that p_1 and p_2 are compatible; say $p_3 \leq p_1, p_2$. Now $f(p_3) \leq f(p_2) \leq q$, so $p_3 \in E$. This finishes the proof that E is dense in $\mathbb{P}$.

Choose $p \in E \cap G$. Say $f(p) \leq q \in D$. Thus $q \in D \cap g(G)$.

E11.10 *Continuing E11.9, show that* $G = f^{-1}[g(G)]$ *and* $M[G] = M[g(G)]$. *Also show that if* H *is* $\mathbb{Q}$*-generic over* M, *then* $H = g(f^{-1}[H])$ *and* $M[H] = M[f^{-1}[H]]$.

Solution: By the definition of g, if $p \in G$ then $f(p) \in g(G)$; hence $p \in f^{-1}[g(G)]$. Thus $G \subseteq f^{-1}[g(G)]$. Now by E11.9, $g(G)$ is $\mathbb{Q}$-generic over M, and then by E11.8, $f^{-1}[g((G)]$ is $\mathbb{P}$-generic over $\mathbb{P}$. Now $G = f^{-1}[g(G)]$ by E11.7.

We have $f \in M$ and $g(G) \in M[g(G)]$. Now $g(G)$ is $\mathbb{Q}$-generic over M by E11.9, so $M[g(G)]$ is a ctm of ZFC. Also, $G = f^{-1}[g(G)] \in M[g(G)]$. Hence $M[G] \subseteq M[g(G)]$ by Theorem 9.11. Also, $G = f^{-1}[g(G)]$ by the preceding paragraph, and $G \in M[G]$, so also $g(G) = f[G] \in M[G]$. Hence by Theorem 9.11 again, $M[g(G)] \subseteq M[G]$. So $M[G] = M[g(G)]$.

If $q \in g(f^{-1}[H])$, then there is a $p \in f^{-1}[H]$ such that $f(p) \leq q$. So $f(p) \in H$, and hence also $q \in H$. So this shows that $g(f^{-1}[H]) \subseteq H$. By E11.7 again, $g(f^{-1}[H]) = H$.

Finally, using what has already been shown, $M[H] = M[g(f^{-1}[H])] = M[f^{-1}[H]]$.