On $\mathbf{GL}_2(\mathbf{Z}_2)$

See page 121. We show that the other 10 matrices are not invertible. The proof here is elementary; a proof using more matrix theory is also possible, but is not really shorter. Let Z be the zero matrix. Of course it is not invertible. If

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I,$$

then $a + c = 1$ and also $a + c = 0$, contradiction. So $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not invertible. We show that the remaining 8 matrices are not invertible by finding, for each A in the remaining 8, a nonzero matrix B such that AB is the zero matrix, or BA is the zero matrix. [Hence A could not be invertible, since otherwise, if $AB = Z$ then $B = A^{-1}AB = ZB = Z$, where Z is the zero matrix; similarly for $BA = Z$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

so $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ are not invertible;

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

so $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ are not invertible;

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

so $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are not invertible;

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

so $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not invertible;

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

so $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not invertible.