Math 2300 – Calculus II – University of Colorado

Fall 2010 – Final exam problems

1 Short-answer type questions

Chapter 7 – Integration

1. Decide whether the following statements are true or false. Give a brief justification for your answer.

   (a) If f is continuous for all x and $\int_0^\infty f(x)dx$ converges, then so does $\int_a^\infty f(x)dx$ for all positive a.

   TRUE. The two integrals differ only by the constant $\int_0^a f(x)dx$ (which is a finite number, since f is continuous). So, if $\int_0^\infty f(x)dx$ is finite, then so $\int_a^\infty f(x)dx$.

   (b) if $f(x)$ is continuous and positive for $x > 0$ and if $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)dx$ converges.

   FALSE. $\lim_{x\to\infty} f(x) = 0$, does not necessarily imply convergence of $\int_0^\infty f(x)dx$, even when f is continuous and positive. For example: $f(x) = \frac{1}{x+1}$ is continuous and positive for $x \geq 0$, but $\int_0^\infty \frac{1}{x+1}dx = \lim_{b\to\infty} \ln|b+1| = \infty$ is obviously divergent. (The function f does not decrease to zero fast enough to insure convergence of the integral.

   (c) If $\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ both converge then $\int_0^\infty (f(x)+g(x))dx$ converges.

   TRUE. $\int_0^\infty (f(x)+g(x))dx = \int_0^\infty f(x)dx + \int_0^\infty g(x)dx$, and the sum of two finite numbers is a finite number.

   (d) If $\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ both diverge then $\int_0^\infty (f(x)+g(x))dx$ diverges.

   FALSE. One of the terms can be divergent and positive ($+\infty$), while the other can be divergent and negative ($-\infty$). For example: $f(x) = x, g(x) = -x$. Then $\int_0^\infty xdx = \infty$ and $\int_0^\infty -xdx = -\infty$, but $\int_0^\infty [f(x) - g(x)]dx = \int_0^\infty 0dx = 0$

2. For the next four problems, let a be any positive number, and suppose $f(x)$ is continuous and $\int_0^\infty f(x)dx$ converges.

   (a) $\int_0^\infty af(x)dx$ converges.

   TRUE. $\int_0^\infty af(x)dx = a\int_0^\infty f(x)dx$ converges.

   (b) $\int_0^\infty f(ax)dx$ converges.

   TRUE. With the substitution $w = ax$, the integral becomes: $\int_0^\infty f(ax)dx = \int_0^\infty f(w)\frac{dw}{a} = \frac{1}{a}\int_0^\infty f(w)dw$, which converges.

(c) $\int_0^\infty f(a+x)dx$ converges.

TRUE. With the substitution $w = x + a$, the integral becomes: $\int_0^\infty f(x+a)dx = \int_a^\infty f(w)dw$, which converges, since $\int_0^\infty f(w)dw$ converges.

(d) $\int_0^\infty (a + f(x))dx$ converges.

FALSE. $\int_0^\infty (a+f(x))dx = \int_0^\infty a dx + \int_0^\infty f(x)dx$. The first term is infinite and the second is finite (since $\int_0^\infty f(w)dw$ converges), so their some is infinite.

3. For the following problems, state which of the integration techniques you would use to evaluate the integral, but **DO NOT** evaluate the integrals. If your answer is **substitution**, also list w and dw; if your answer is **integration by parts**, also list u, dv, du and v; if your answer is **partial fractions**, set up the partial fraction decomposition, but do not solve for the numerators; if your answer is **trigonometric substitution**, write which substitution you would use.

(a) $\int \tan x dx$

Use substitution $w = \cos x$, so that $dw = \sin x$, and the integral becomes: $\int w^{-1} dw$.

(b) $\int \frac{dx}{x^2-9}$

Factor out the denominator as $x^2 - 9 = (x-3)(x+3)$, and use partial fractions to rewrite the integrand as $\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$.

(c) $\int e^x \cos x dx$

Start by using an integration by parts, with $u = e^x$ and $dv = \cos x dx$. (Then use another integration by parts with $U = e^x$ and $dV = \sin x dx$, and solve the resulting equation for the integral.)

(d) $\int \frac{\sqrt{9-x^2}}{x^2} dx$

Start by using a trig substitution $x = 3\sin t$, so that $dx = 3\cos t dt$. (Then the integral becomes: $\int \frac{3\cos t}{\sin^2 t} 3\cos t dt = \int \frac{\cos^2 t}{\sin^2 t} dt = \int \frac{1-\sin^2 t}{\sin^2 t} dt = \int (\sec^2 t - 1) dt$.)

(e) $\int \frac{\sin(\ln x)}{x} dx$

A substitution $w = \ln x$, with $dw = \frac{1}{x} dx$, will transform the integral into $\int \sin w dw$.

(f) $\int x^{3/2} \ln x dx$

Use integration by parts, with $u = \ln x$ and $dv = x^{3/2} dx$, so that $du = \frac{1}{x}$ and $v = \frac{2}{5} x^{5/2}$.

(g) $\int \frac{1}{\sqrt{x^2+4}} dx$

Trig substitution $x = 2\tan(t)$, so that $dx = 2\sec^2 t$. (The integral becomes $\int \frac{\cos^2 t}{2} 2\sec^2 t dt = \int 1 dt = t + C = \tan^{-1}(t/2) + C$.)

(h) $\displaystyle\int \frac{e^x+1}{e^x+x}dx$

The numerator is the derivative of the denominator. Use substitution $w = e^x + x$, so that $dw = (e^x+1)dx$, and the integral becomes $\displaystyle\int w^{-1}dw$.

4. True or False: For $x > 0$, $e^x = 1 + \displaystyle\int_0^x e^t\,dt$

TRUE. $\displaystyle\int_0^x e^t\,dt = e^t\Big|_0^x = e^x - e^0 = e^x - 1.$

Chapter 9 – Sequences and series

1. The sequence $\left(\frac{1}{2}\right)^n$

   (a) Converges, but not monotonically

   (b) Converges, monotonically increasing

   (c) Converges, monotonically decreasing

   (d) Diverges

   The sequence decreases monotonically to zero.

2. Determine the general term of the following series if the starting value is $n = 1$.

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{4} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{16} - \cdots$$

   (a) $\dfrac{(-1)^n(x-1)^n}{2n}$

   (b) $\dfrac{(-1)^{n+1}(x-1)^n}{2^n}$

   (c) $\dfrac{(-1)^n(x-1)^n}{2^n}$

   (d) $\dfrac{(-1)^{n+1}2(x-1)^n}{2^n}$

3. The power series $\sum C_n x^n$ diverges at $x = 7$ and converges at $x = -3$. At $x = -9$, the series is

   (a) Conditionally convergent

   (b) Absolutely convergent

   (c) Divergent

   (d) Cannot be determined.

   This is a power series around $a = 0$, with an unknown radius of convergence $0 \le R \le \infty$. This means that: the series is absolutely convergent when $|x| < R$, the series is divergent when $|x| > R$, and the convergence is undecided for $x = \pm R$. From the information we have, we can tell that $7 \ge R$ (since $x = 7$ is outside of the convergence interval), and that $-R \le -3$ (since $x = -3$ is in the convergence interval). So $3 \le R \le 7$. This means that $x = 9$ is definitely outside of the convergence interval, so that the series is divergent at $x = 9$.

4. The power series $\sum C_n(x-5)^n$ converges at $x = -5$ and diverges at $x = -10$. At $x = 11$, the series is

(a) Conditionally convergent
(b) Absolutely convergent
(c) Divergent
(d) Cannot be determined.

This is a power series around $a = 5$, with an unknown radius of convergence $0 \leq R \leq \infty$. This means that: the series is absolutely convergent when $|x-5| < R$ (i.e. $5-R < x < 5+R$), the series is divergent when $|x-5| > R$ (i.e., $x < 5-R$ and $x > 5+R$), and the convergence is undecided for $x = 5 \pm R$. From the information we have, we can tell that $-5 \geq 5-R$ (since $x = -5$ is in the convergence interval), and that $-10 \leq 5-R$ (since $x = -10$ is outside of the convergence interval). It follows that $10 \leq R \leq 15$. This means that $x = 11$ (which is $11 - 5 = 6$ units away from the center $a = 5$, is definitely inside the convergence interval (an interior point, not an endpoint). So the series is absolutely convergent at $x = 11$.

5. The power series $\sum C_n x^n$ diverges at $x = 7$ and converges at $x = -3$. At $x = -4$, the series is

(a) Conditionally convergent
(b) Absolutely convergent
(c) Divergent
(d) Cannot be determined.

Same as in problem (3), The convergence radius is $3 \leq R \leq 7$. So there is not enough information to tell whether $x = -4$ is inside or outside the convergence interval.

6. In order to determine if the series converges or diverges, the comparison test can be used. Decide which series provides the best comparison.

$$\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+1}$$

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{2k}}{k^2}$

If we decide to use the Limit Comparison Test for two series with positive terms, the most appropriate series to compare with is: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$. Indeed, we consider the limit of the ratio sequence:

$$\lim_{k\to\infty} \frac{\sqrt{k+1}}{k^2+1} \Big/ \frac{1}{k^{3/2}} = \lim_{k\to\infty} \frac{k^{3/2}\sqrt{k+1}}{(k^2+1)} = \lim_{k\to\infty} \frac{\sqrt{k+1}}{(k^{1/2}+k^{-3/2})} = 1$$

Since the limit is > 0 and finite, the Limit Comparison Test implies that the two series have the same convergence type. We know that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges, as a p-series with $p = 3/2 > 1$. So the original series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+1}$ must converge as well.

If we decide to use the Direct Comparison Test for two series with positive terms, comparison with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ does not tell us anything, because $\dfrac{\sqrt{k+1}}{k^2+1} \geq \dfrac{1}{k^{3/2}}$ for k large enough; so our series would have larger terms than the terms of the convergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$, which makes the test inconclusive. Similarly, comparison with $\sum_{k=1}^{\infty} \frac{1}{k}$ is not helpful, since $\frac{\sqrt{k+1}}{k^2+1} \leq \frac{1}{k^{3/2}}$, and our series will have terms smaller than the terms of the convergent series $\sum_{k=1}^{\infty} \frac{1}{k}$, which is again inconclusive. Finally, if we try direct comparison with $\sum_{k=1}^{\infty} \frac{\sqrt{2k}}{k^2}$, we find that $\frac{\sqrt{k+1}}{k^2+1} \leq \frac{\sqrt{2k}}{k^2}$ for k large enough. Since the series $\sum_{k=1}^{\infty} \frac{\sqrt{2k}}{k^2}$ is convergent, our original series will also converge (by direct comparison).

7. The limit comparison test can be used to determine whether the series converges. Decide which series to compare with.

$$\sum_{n=1}^{\infty} \frac{200n^2 - 100n - 1}{n^3 + n^2 + n + 1}$$

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n}$

We can use Limit Comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (harmonic series), our series also diverges.

8. Which of the following defines a convergent sequence of partial sums?

(a) Each term in the sequence is closer to the last term than any two prior consecutive terms.

(b) Assume that the sequence of partial sums converges to a number, L. Regardless of how small a number you give me, say ϵ, one can find a value N such that the N^{th} term of the sequence is within ϵ of L.

(c) Assume that the sequence of partial sums converges to a number, L. One can find a value N such that all the terms in the sequence, past the N^{th} term, are less than L.

(d) Assume that the sequence of partial sums converges to a number, L. Regardless of how small a number you give me, say ϵ, one can find a value N such that all the terms in the sequence, past the N^{th} term, are within ϵ of L.

9. **True or False?** A sequence of partial sums that is bounded and always increasing is a convergent sequence.

(a) True

(b) False

TRUE. This theorem is true for any sequence, in particular for any sequence of partial sums.

10. **True or False?** If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

(a) True

(b) False

FALSE. a_n may not decrease to zero fast enough to warrantee convergence of the series. For example, the harmonic series $\sum \frac{1}{n}$ diverges, although $\lim_{n\to\infty} \frac{1}{n} = 0$.

11. **True or False?** If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges.

(a) True

(b) False

Chapter 10 – Taylor series

1. True/False: If $\sum a_n$ is convergent, then the power series $\sum a_n x^n$ has convergence radius at least $R = 1$.

   The fact that $\sum a_n$ converges means that the power series around zero $\sum a_n x^n$ is convergent for $x = 1$. In other words, $x = 1$ is in the convergence interval for the series. That implies that $R \geq 1$.

2. Consider the Taylor series $f(x) = \sum a_n x^n$, and $g(x) = \sum b_n x^n$. True/False: Is the Taylor series of the function $(f + g)(x) = \sum(a_n + b_n)x^n$? Is the Taylor series of the function $(fg)(x) = \sum(a_n \cdot b_n)x^n$?

   The first part is TRUE. The second part is FALSE.

3. True/False: Is the convergence radius for the Taylor series of f the same as the convergence radius for the Taylor series of the derivative f'? How about their convergence intervals?

   The first part is TRUE (the convergence radii for f and f' are the same). The second part is FALSE (the convergence intervals might differ in whether the endpoints are included or not).

4. True/False: Is the function $f(x) = \sum_{0}^{\infty} \frac{x^n}{n!}$ a solution to the differential equation $\frac{df}{dx} = f$, with initial condition $f(0) = 1$?

   TRUE. The power series represents the Taylor series for $f(x) = e^x$, which satisfies the equation $\left(\frac{d}{dx} e^x = e^x\right)$ and the initial condition $(f(0) = e^0 = 1)$.

5. If one uses the Taylor polynomial $P_3(x)$ of degree $n = 3$ to approximate $\sin x = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ at $x = 0.1$, would one get an overestimate or an underestimate?

   $\sin 0.1 = \sum_{0}^{\infty} \frac{(-1)^n 0.1^{2n+1}}{(2n+1)!}$ is an alternating series. If cut off after its term of degree $n = 3$ (which is negative), the approximation $P_3(0.1)$ would be an underestimate.

6. If one uses the Taylor polynomial $P_5(x)$ of degree $n = 5$ to approximate $e^x = \sum_{0}^{\infty} \frac{x^n}{n!}$ at $x = -0.2$, would one get an overestimate or an underestimate?

$e^{-0.2} = \sum_0^\infty \frac{(-0.2)^n}{n!}$ is an alternating series. If cut off after its term of degree $n = 5$ (which is negative), the approximation $P_5(-0.2)$ would be an underestimate.

7. If one uses the Taylor polynomial $P_4(x)$ of degree $n = 4$ to approximate $\frac{1}{1+x}$ at $x = -0.1$, would one get an overestimate or an underestimate? What is the actual error?

   $\frac{1}{1-x} = \sum_0^\infty x^n$, so $\frac{1}{1+0.1} = \sum_0^\infty 0.1^n$ is a series with positive terms. Any Taylor polynomial approximation $P_n(-0.1)$ would be an underestimate. More precisely, the approximate value $P_4(-0.1) = 1 - 0.1 + 0.01 - 0.001 + 0.0001 = 0.9091$, and the real value is $\frac{1}{1-0.1} = 1.111...$, so the error is ~ 0.202.

8. You want to estimate $\sin x$ using the first 3 nonzero terms in the Taylor series. What formula for the error bound would you use to get the best estimate for the error, without computing the error?

   In principle, we could use either the error bound for $|E_3(x)| \le \frac{M|x|^4}{4!}$, where $M = \max_{[0,x]} |f^{(4)}(y)| = \max_{[0,x]} |\sin y|$, or the error bound for $|E_4(x)| \le \frac{M|x|^5}{5!}$, where $M = \max_{[0,x]} |f^{(5)}(y)| = \max_{[0,x]} |\cos y|$. The second one is more convenient, since is gives a tighter bound.

2 Conceptual problems – older material

Chapter 7 – Integration

1. Suppose that $\int_{-1}^{1} h(z)\,dz = 7$ and that $h(z)$ is an even function. Calculate the following.

   (a) $\int_0^1 h(z)\,dz$

   Since h is even, we have $\int_{-1}^{1} h(z)\,dz = 2\int_0^1 h(z)\,dz$, so $\int_0^1 h(z)\,dz = 7/2$.

   (b) $\int_{-4}^{-2} 5h(z+3)\,dz$

   We make the substitution $w = z + 3$, so that $dw = dz$. Then
   $\int_{-4}^{-2} 5h(z+3)\,dz = 5\int_{-1}^{1} h(w)\,dw = 5 \cdot 7 = 35$.

2. Use the fact that $\int_0^\infty e^{-x}\sin(x)dx = \frac{1}{2}$ to find $\int_0^\infty e^{-x}\cos(x)dx$

   We use integration by parts to express: $\int_0^b e^{-x}\sin x\; dx$. We take $u = \sin x$ and $dv = e^{-x}\,dx$, so that $du = \cos x\; dx$ and $v = -e^{-x}$. Then:

$$\int_0^b e^{-x}\sin x\; dx = \left(-e^{-x}\sin x\right)\Big|_0^b - \int_0^b -e^{-x}\cos x\; dx = e^{-b}\sin b + \int_0^b e^{-x}\cos x\; dx$$

   Since $\lim_{b\to\infty} \frac{\sin b}{e^b} = 0$ (by the Sandwich Theorem), it follows that:

$$\int_0^\infty e^{-x}\sin x\; dx = \int_0^\infty e^{-x}\cos x\; dx = \frac{1}{2}$$

Chapter 8 – Polar integration and other

1. Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$.

   We calculate the arc length using the direct formula:

   $$L = \int_a^b \sqrt{(dy/dx)^2 + 1}\, dx = \int_1^2 \sqrt{\left[\frac{3}{2}x^{1/2}\right]^2 + 1}\, dx = \int_1^2 \sqrt{\frac{9}{4}x + 1}\, dx$$

   With a substitution $w = 9/4x + 1$, $dw = 9/4dx$, the integral becomes:

   $$L = \frac{4}{9}\int_{13/4}^{22/4} \sqrt{w}\, dw = \frac{4}{9} \cdot \frac{2}{3}\, w^{3/2}\Big|_{13/4}^{22/4} = \frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

2. Find the arc length of the curve $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$.

   $$y = (x^6 + 8)/(16x^2) = \frac{1}{16}\left(x^4 + \frac{8}{x^2}\right)$$

   $$\frac{dy}{dx} = \frac{1}{16}\left(4x^3 - \frac{16}{x^3}\right) = \frac{1}{4}\left(x^3 - \frac{4}{x^3}\right)$$

   We use the direct formula for arc length:

   $$\begin{aligned} L &= \int_a^b \sqrt{(dy/dx)^2 + 1}\, dx = \int_2^3 \sqrt{\frac{1}{16}\left(x^3 - \frac{4}{x^3}\right)^2 + 1}\, dx \\ &= \int_2^3 \sqrt{\frac{1}{16}\left(x^6 - 8 + \frac{16}{x^6}\right) + 1}\, dx = \int_2^3 \sqrt{\frac{1}{16}\left(x^6 + 8 + \frac{16}{x^6}\right)}\, dx \\ &= \int_2^3 \sqrt{\frac{1}{16}\left(x^3 + \frac{4}{x^3}\right)^2}\, dx = \int_2^3 \frac{1}{4}\left(x^3 + \frac{4}{x^3}\right)\, dx = \frac{1}{4}\left(\frac{x^4}{4} - \frac{2}{x^2}\right)\Big|_2^3 \end{aligned}$$

Chapter 10 – Taylor series

1. Suppose that x is positive but very small. Arrange the following in **increasing** order:

   $$x,\ \ \sin x,\ \ \ln(1 + x),\ \ 1 - \cos(x).\ \ e^x - 1,\ \ x\sqrt{1 - x}.$$

   $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

   $$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$$

   $$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} \cdots$$

   $$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

   $x\sqrt{1 - x} = x - \dfrac{x^2}{2} + \dfrac{x^3}{4 \cdot 2!} \cdots$ (using the binomial series for $(1 - x)^{1/2}$, then multiplying by x).

   Using the fact that the values of x^n get smaller (more negligible) as we increase the power n, we can order the series based on comparing their first few terms:

$$1 - \cos x < x\sqrt{1-x} < \ln(1+x) < \sin x < e^x - 1$$

2. All the derivatives of some function f exist at 0, and the Taylor series for f about 0 is

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

Find $f'(0),\ f''(0),\ \text{and} f^{(10)}(0)$.

When differentiating the Taylor series n times, the first $n-1$ terms vanish. When we then plug in $x = 0$, the terms that still contain powers of x which are ≥ 1 will also become zero. So the only surviving part is the part obtained from the term $c_n x^n$, which after differentiating n times becomes $c_n n!$. In our case:

$f'(0) = 1$

$f''(0) = 2 \cdot \frac{1}{2}$

$f^{(10)}(0) = 10 \cdot 9 \cdots 1 \cdot \frac{1}{10} = 9!$

3. Solve exactly for the variable x:

$$1 + x + x^2 + x^3 + \cdots = 5$$

.

On the left side, we are looking at a geometric series: $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$. The equation is equivalent with $\dfrac{1}{1-x} = 5 \ \Leftrightarrow\ 1 - x = \dfrac{1}{5} \ \Leftrightarrow\ x = \dfrac{4}{5}$.

4. Solve exactly for the variable x:

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1$$

.

On the left side, we are looking at the Taylor series around zero for the function $e^x - 1$. The equation is equivalent with $e^x - 1 = 1 \ \Leftrightarrow\ e^x = 2 \ \Leftrightarrow\ x = \ln 2$.

5. (a) Find the Taylor series about 0 for $f(x) = x^2 e^{x^2}$

   We start with the Taylor series around zero for $e^y = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We substitute $y = x^2$ and get the series for $e^{x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. We multiply by x^2 and get the series for $x^2 e^{x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$

   (b) Is this function even or odd? Justify your answer.

   The function is even, since it only contains even powers in the Taylor series expansion around zero.

   (c) Find $f^{(3)}(0)$ and $f^{(6)}(0)$.

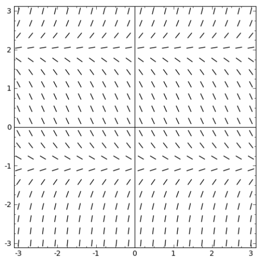
   Similarly as in problem 2, $f^{(n)}(0)$ is the value of the term of degree n under differentiation n times:

$f^{(3)}(0)$ should come from the term of power $n = 3$, which is not represented in the series. So $f^{(3)}(0) = 0$.

$f^{(6)}(0)$ comes from $\dfrac{x^6}{2!}$, and will be $f^{(3)}(0) = \dfrac{6!}{2!}$

Chapter 11 – Differential equations and slope fields

1. For each differential equation, find the corresponding slope field. (*Not all slope fields will be used.*)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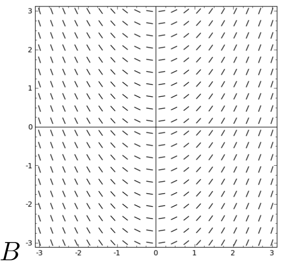
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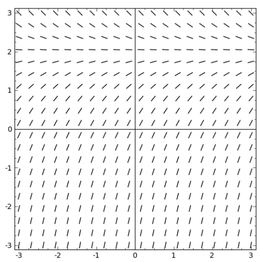
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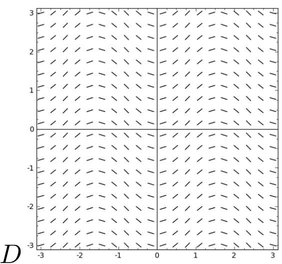
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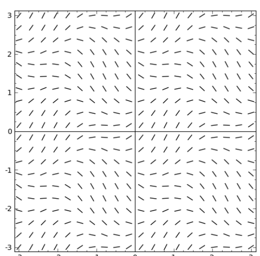
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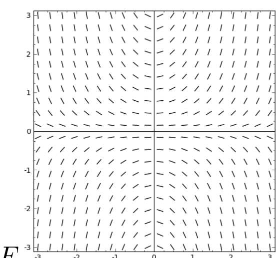
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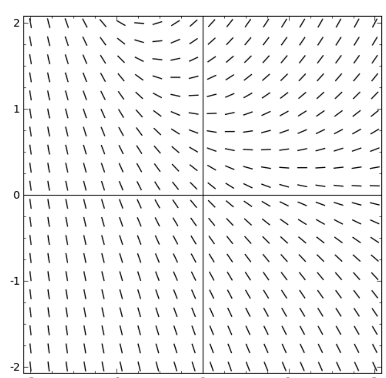
Equation	**Slope Field**
$\dfrac{dy}{dx} = x$	B
$y' = xy$	F
$y' = \sin(2x)$	D
$\dfrac{dy}{dx} = 2 - y$	C

2. A slope field for the differential equation $y' = y - e^{-x}$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions. Make sure to label each sketched graph.

(a) $y(0) = 0$ (b) $y(0) = 1$ (c) $y(0) = -1$



Chapters 9&11 – Word problems on series and differential equations

1. Suppose that ibuprofen is taken in 200mg does every six hours, and that all 200mg are delivered to the patient's body immediately when the pill is taken. After six hours, $r = 12.5\%$ of the ibuprofen remains. Find expressions for the amount of ibuprofen in the patient immediately before and after the n^{th} pill taken. Include work; without work, you may receive no credit.

   The amount left in the body after 6 hours after taking a full dose of 200mg is $200r = 200 * 0.125$mg. Every 6 hours that elapse will diminish this by another factor r. Adding up all the residue from all the past doses left in to body before taking the n-th dose, we get:

   $$M_{\text{before}}(n) = 200 * r + 200 * r^2 + \cdots + 200 * r^{n-1} = 200\left(\frac{1-r^n}{1-r} - 1\right)$$

   Taking the n-th dose boosts this amount by another 200mg; so, after taking the n-th dose:

   $$M_{\text{after}}(n) = 200 * r + 200 * r^2 + \cdots + 200 * r^{n-1} = 200\frac{1-r^n}{1-r}$$

2. A population of rabbits lives on an island where they have no predators, evolving only based on their interactions: an increase in population facilitates further reproduction, but also introduces competition over resources. For this particular species, the reproduction rate is $a > 0$ and the competition rate is $b > 0$, so that the equation describing the population rate is: $\frac{dP}{dt} = aP - bP^2$

   (a) What are the equilibria that the population can reach (in terms of a and b)?

   The equilibria are the constant values of P for which the slope field $\frac{dP}{dt}$ is horizontal. We obtain them by setting the slope field equal to zero: $aP - bP^2 = 0$, and solving for P: $P(a - bP) = 0 \Leftrightarrow$ $P = 0$ and $P = a/b$ are the equilibria for the population P.

   (b) Sketch the slope field corresponding to this equation.

   The slope field is constant along horizontal lines. It is zero along $P = 0$ and $P = a/b$, positive for $0 < P < a/b$ and negative outside of this range. This tells us that the equilibrium $P = 0$ is unstable, and that the equilibrium $P = a/b$ is stable.

   (c) What is the highest rate that the population can reach (in terms of a and b)?

   The highest positive rate is reached at the peak of the parabola $P \to P(a - bP)$, that is when $P = \frac{a}{2b}$.

   (d) What happens in the long term if the population starts at $P(0) = \frac{2a}{b}$? How about $P(0) = \frac{a}{2b}$?

   If started at $P(0) = \frac{2a}{b}$, the population will increase, eventually tapering off towards a saturation equilibrium value of $P = a/b$. If started at $P(0) = \frac{a}{2b}$ will decay, eventually stabilizing towards the same stable equilibrium $P = a/b$.

3. When an object is removed from a furnace and placed in an environment with a constant temperature of 80F, its temperature is 1500F. One hour after it is removed, the temperature is 1120F.

   (a) What is the temperature 5 hours after the object is removed from the furnace?

   (b) How long will it take the object to get to 120F, when it can be picked up by hand?

   (c) How long will it take the object to reach 80F?

Chapter 12

1. Find an equation for a sphere if one of its diameters has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.

   The center of the sphere is at the midpoint of the diameter: $\left(\dfrac{2+4}{2}, \dfrac{1+3}{2}, \dfrac{4+10}{2}\right) = (3, 2, 7)$. The radius is half of the diameter length: $\dfrac{1}{2}\sqrt{(4-2)^2 + (3-1)^2 + (10-4)^2} = \sqrt{44}/2 = \sqrt{11}$. So the equation of the sphere is:

   $$(x-3)^2 + (y-2)^2 + (x-7)^2 = 11$$

2. Find the equation of the largest sphere with center $(5, 4, 9)$ contained in the first octant.

   The center of the sphere itself is clearly in the first octant, at a distance of $x = 5$ from yz-plane, at a distance $y = 4$ from the xz-plane, and at a distance $z = 9$ from the xy-plane. For the sphere to be contained in the first octant, we have to make sure that the radius is small enough so that the spherical surface is at most tangent to the coordinate planes. Since the xz-plane is the one that is closest to the center, it is enough to ask for the radius to be at most 4. I other words, the largest sphere will have radius $R = 4$ and equation:

   $$(x-5)^2 + (y-4)^2 + (z-9)^2 = 16$$

   One can easily check that, indeed, this sphere does not intersect the xy and yz-planes, and that it barely touches the xz-plane at one point.

3. Find the equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

   Since the origin is on the surface of the sphere, the radius will be the distance from the center to the origin: $r = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. So the equation of the sphere is: $(x-1)^2 + (y-2)^2 + (x-3)^2 = 14$.

4. A cube is located such that its top four corners have the coordinates $(-1, -2, 2)$, $(-1, 3, 2)$, $(4, -2, 2)$, and $(4, 3, 2)$. Give the coordinate of the center of the cube.

Chapter 14 – Partial derivatives

1. On your Calc II final, you were asked to calculate the partial derivatives for a function f that is too crazy to remember. When you get back to your dorm and check your answers with your room-mates, they tell you that they got

   $g(x, y) = \dfrac{\partial f}{\partial x}(x, y) = ye^{x^2+y^2}$ and $h(x, y) = \dfrac{\partial f}{\partial y}(x, y) = xe^{x^2+y^2}$

   You show them your result (which is different), and tell them that they are all wrong. Going through the following steps, show them why they are wrong:

   (a) Calculate $\dfrac{\partial g}{\partial y}(x, y)$ for their function $g(x, y)$.

   $$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(ye^{x^2+y^2}\right) = e^{x^2+y^2} + 2y^2e^{x^2+y^2}$$

   (b) Calculate $\dfrac{\partial h}{\partial x}(x, y)$ for their function $h(x, y)$.

   $$\frac{\partial h}{\partial x} = \frac{\partial}{\partial y}\left(xe^{x^2+y^2}\right) = e^{x^2+y^2} + 2x^2e^{x^2+y^2}$$

(c) Using the answers from (a) and (b), what would you say to convince them that their result can't be right?

If the answer were indeed showing the correct derivatives, than they should have satisfied:

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial h}{\partial x}$$

which they don't (as shown by our calculations at (a) and (b)).

2. The function T below describes the temperature evolution along the course of 24 hours (starting and ending at midnight) at each point on the water surface in the harbor: $T(x, y, t) = ye^{-x^2 \sin(\pi t/24)}$ ($-200 < x < 200$ is the span of the harbor along the beach, and $y > 0$ is the distance from the shore into the water, both in meters).

(a) At the points $(x, 20)$ (20m along the shore line), is the temperature raising or dropping at 6am?

At the points 20m along the shore: $T(x, 20, t) = 20e^{-x^2 \sin(\pi t/24)}$. Then

$T_t(x, 20, t) = 20e^{-x^2 \sin(\pi t/24)} \cdot (-x^2 \cos(\pi t/24)) \cdot \dfrac{\pi}{24}$

$T_t(x, 20, 6) = 20e^{-x^2 \sin(\pi/4)} \cdot (-x^2 \cos(\pi/4)) \cdot \dfrac{\pi}{24} < 0$, for all x

The temperature is dropping at all points $(x, 20)$.

(b) If a person swims at mid-day in a straight line, parallel to the shore, 20m from the shore. Where will he feel that the water is getting colder, and where will he feel it getting warmer? (Assume that he is swimming fast enough that the time can be considered insignificant for the temperature change).

$$T_x(x, 20, 12) = \frac{\partial}{\partial x}\left(20e^{-x^2 \sin(\pi/2)}\right) = \frac{\partial}{\partial x}\left(20e^{-x^2}\right) = -40xe^{-x^2}$$

The water is warming up for $-200 < x < 0$, reaches a max temperature at $x = 0$, then is steadily cooling off for $0 < x < 200$.

3. Find $f_x(1, 0)$ for $f(x, y) = \dfrac{xe^{\sin(x^2 y)}}{(x^2 + y^2)^{3/2}}$.

$$f(x, 0) = \frac{xe^0}{(x^2)^3/2} = \frac{x}{x^3} = \frac{1}{x^2}$$

$$f_x(x, 0) = \frac{-2}{x^3} \Leftrightarrow f_x(1, 0) = \frac{-2}{1} = -2$$

New material

Section 16.2 – Double integrals

1. Evaluate the double integral (using the most convenient method):

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\, dx\, dy$$

$\displaystyle\int_{e^y}^e \frac{x}{\ln x}\, dx$ seems quite difficult to solve directly. We chose to approach the double integral by changing the order of integration. The region of integration is described by: $0 \le y \le 1$ and $e^y \le x \le e$. An equivalent description is $1 \le x \le e$ and $0 \le y \le \ln x$. Changing the order of the variables, the integral can be rewritten as:

$$\int_1^e \int_0^{\ln x} \frac{x}{\ln x}\, dy\, dx = \int_1^e \left. \frac{x}{\ln x} y \right|_{y=0}^{y=\ln x} dx = \int_1^e x\, dx = \left. \frac{x^2}{2} \right|_1^e = \frac{e^2-1}{2}$$

$$\int_0^1 \int_x^1 \frac{y^2}{1+y^4} dydx$$

$\int_x^1 \frac{y^2}{1+y^4} dy$ is hard to solve. Again, we chose to change the order of integration. The region of integration is described by: $0 \le x \le 1$ and $x \le y \le 1$. An equivalent description is $0 \le y \le 1$ and $0 \le x \le y$. Changing the order of the variables, the integral can be rewritten as:

$$\int_0^1 \int_0^y \frac{y^2}{1+y^4}\, dx\, dy = \int_0^1 \left. \frac{y^2}{1+y^4} x \right|_{x=0}^{x=y} dy = \int_0^1 \frac{y^3}{1+y^4}\, dy = \left. \frac{1}{4} \ln(1+y^4) \right|_0^1 = \frac{\ln 2}{4}$$

2. The following sum of double integrals describes the mass of a thin plate $\mathcal{R}$ in the xy-plane, of density $\delta(x,y) = x+y$:

$$\text{mass} = \int_{-4}^0 \int_0^{2x+8} \delta(x,y)\, dy\, dx + \int_0^4 \int_0^{-2x+8} \delta(x,y)\, dy\, dx$$

   (a) Describe the thin plate (shape, intersections with the coordinate axes).
   (b) Write an expression for the mass of the plate as only one double integral.
   (c) Calculate the area and the mass of the plate.
   (d) Calculate the average density of the plate.

3. Consider the solid region $\mathcal{W}$ situated above the region $0 \le x \le 2$, $0 \le y \le x$, and bounded above by the surface $z = e^{x^2}$.

   An equivalent description for the region $0 \le x \le 2$, $0 \le y \le x$ is: $0 \le y \le 2$, $y \le x \le 2$.

   (a) Write an integral that evaluates the area of each cross-section of $\mathcal{W}$ with vertical planes $y = a$, where a is a constant in $[0,2]$. Can you evaluate this integral (as an expression depending on a)?

   In each cross-sectional plane $y = a$, the interval for x is given by $a \le x \le 2$, so the area of the cross-sections is $\int_a^2 f(x,a)\, dx = \int_a^2 e^{x^2} dx$, which can't be evaluated directly.

   (b) Write an integral that evaluates the area of each cross-section of $\mathcal{W}$ with vertical planes $x = b$, where b is a constant in $[0,2]$. Can you evaluate this integral (as an expression depending on b)?
   In each cross-sectional plane $x = b$, the interval for y is given by $0 \le y \le b$, so the area of the cross-sections is $\int_0^b f(b,y)\, dy = \int_0^b e^{b^2}\, dy = be^{b^2}$.

   (c) Write a convenient double integral that evaluates the volume of the region $\mathcal{W}$, and evaluate it.

$$\int\int_{\mathcal{R}} z dA = \int_0^2 \int_0^x e^{x^2}\, dy\, dx = \int_0^2 \int_y^2 e^{x^2}\, dx\, dy$$

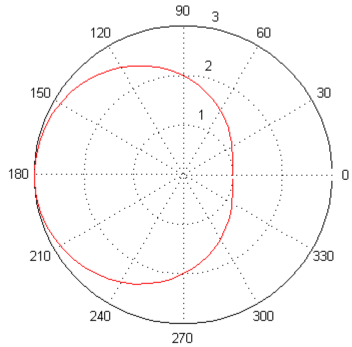
   The previous remarks indicate that the first option is the only convenient one, so the volume of the solid region can be calculated as:

$$\int_0^2 \int_0^x e^{x^2}\, dy\, dx = \int_0^2 \left. ye^{x^2} \right|_{y=0}^{y=x} dx = \int_0^2 xe^{x^2}\, dx = \left. \frac{e^{x^2}}{2} \right|_0^2 = \frac{e^4-1}{2}$$

Section 16.4 – Double integrals in polar coordinates

1. The plot below depicts the curve whose equation in polar coordinates is

$$r = 2 - cos(\theta):$$



   (a) Write an iterated double integral in polar coordinates whose numerical value equals the area enclosed by the curve.

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2-\cos(\theta)} r\ dr\ d\theta$$

Note: If we wanted to avoid using a double integral, we could use directly the area formula from Chapter 8:

$$\int_{\theta=0}^{\theta=2\pi} \frac{r^2(\theta)}{2}\ d\theta = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}(2-\cos\theta)\ d\theta$$

   (b) Evaluate your answer to part (a).

$$\begin{aligned} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2-\cos(\theta)} r\ dr\ d\theta &= \int_{\theta=0}^{\theta=2\pi} \frac{r^2}{2}\Big|_{r=0}^{r=2-\cos\theta}\ d\theta = \int_{\theta=0}^{\theta=2\pi} \frac{(2-\cos\theta)^2}{2}\ d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} 2 - 2\cos\theta + \frac{\cos^2\theta}{2}\ d\theta = \int_{\theta=0}^{\theta=2\pi} 2 - 2\cos\theta + \frac{\cos(2\theta)+1}{4}\ d\theta \\ &= \left(\frac{9\theta}{4} - 2\sin\theta + \frac{\sin(2\theta)}{8}\right)\Big|_0^{2\pi} = \frac{9\pi}{2} \end{aligned}$$

2. Evaluate the following integral where R is the region above the x-axis within $x^2 + y^2 = 9$:

$$\int_R \cos(x^2+y^2)dA$$

.

The interior of the circle $x^2 + y^2 = 9$ can be described in polar coordinates as: $0 \le \theta \le 2\pi$, $0 \le r \le 3$. In polar form, the integral becomes:

$$\int_R \cos(x^2+y^2)dA = \int_0^{2\pi} \int_0^3 \cos(r^2) r\ dr\ d\theta = 2\pi \left.\frac{\sin(r^2)}{2}\right|_0^3 = \pi\sin(9)$$

3. Evaluate the following integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\ dy\ dx$$

.

We use polar coordinates. The region described by $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$ is the first quadrant quarter of the disc centered at the origin, of radius 1. In polar coordinates, this can be described as: $0 \le \theta \le \pi/2$, $0 \le r \le 1$, so that the integral becomes:

$$\int_0^{\pi/2} \int_0^1 e^{r^2} r\ dr\ d\theta = \frac{\pi}{2} \frac{e^4 - 1}{2}$$

4. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^1 \int_{\sqrt{1-x^2}}^x xy\ dy\ dx + \int_1^{\sqrt{2}} \int_0^x xy\ dy\ dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy\ dy\ dx$$

into **one** double integral, then evaluate the double integral.

If we put together all three regions, we obtain the portion $0 \le \theta \le \pi/4$ of the filled annulus with radii $r = 1$ and $r = 2$. Using polar coordinates, we can rewrite the sum as the integral:

$$\begin{aligned} \int_0^{\pi/4} \int_1^2 r\cos\theta\ r\sin\theta\ r\ dr\ d\theta &= \int_1^2 r^3\ dr \int_0^{\pi/4} \sin\theta\ \cos\theta\ d\theta = \left.\frac{r^4}{4}\right|_1^2 \cdot \int_0^{\pi/4} \frac{\sin(2\theta)}{2}\ d\theta \\ &= \frac{15}{4} \left.\frac{-\cos(2\theta)}{2}\right|_0^{\pi/4} = \frac{15}{8} \end{aligned}$$

5. Find the volume of an ice cream cone bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

The two surfaces cross where: $z = \sqrt{8 - x^2 - y^2} = \sqrt{x^2 + y^2}$, that is along the circle $x^2 + y^2 = 4$ in the plane $z = 2$. The domain of integration $\mathcal{R}$ will then be the disc of radius 2: $x^2 + y^2 \le 4$, and the volume can be written as the double integral:

$$\int\int_{\mathcal{R}} \sqrt{8 - x^2 - y^2} - \sqrt{x^2 + y^2}\ dA$$

In polar coordinates, this becomes:

$$\begin{aligned} \int_0^{2\pi} \int_0^2 (\sqrt{8 - r^2} - r^2)\ r\ dr\ d\theta &= 2\pi \left(\int_0^2 (r\sqrt{8 - r^2}\ dr - \int_0^2 r^3\ dr \right) \\ &= \left.\left(-\frac{1}{3}(8 - r^2)^{3/2} - \frac{r^4}{4} \right)\right|_0^2 = \frac{16\sqrt{2} - 20}{3} \end{aligned}$$

Section 16.3 – Triple integrals

1. Consider the solid between the planes $z = 1 + x + y$ and $x + y + z = 1$ and above the triangle $x + y \leq 1$, $x \geq 0$, $y \geq 0$ in the xy-plane.

   (a) Set up (without evaluating it) a triple integral that would calculate the mass of this solid, if the density at each point (x, y, z) is given by $\delta(x, y, z)$.

   The xy region of integration can be described by: $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$. The solid it bounded by the plane surfaces: $z = 1 - x - y$ (below) and $z = 1 + x + y$ (above). The triple integral for the mass of the solid is:

$$\int_0^1 \int_0^{1-x} \int_{1-x-y}^{1+x+y} \delta(x, y, z)\, dz\, dy\, dx$$

   (b) Calculate the volume of the solid.

   The triple integral for the volume of the solid is:

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_{1-x-y}^{1+x+y} 1\, dz\, dy\, dx &= \int_0^1 \int_0^{1-x} [(1 + x + y) - (1 - x - y)]\, dy\, dx = \int_0^1 \int_0^{1-x} 2(x + y)\, dy\, dx \\
&= \int_0^1 \left(2yx + y^2\right) \Big|_{y=0}^{y=1-x} dx = \int_0^1 [2x(1 - x) + (1 - x)^2]\, dx \\
&= \int_0^1 (1 - x^2)\, dx = \left(x - \frac{x^3}{3}\right)\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}
\end{aligned}$$

2. Consider the plane $\mathcal{H}$ defined by $\dfrac{x}{3} + \dfrac{y}{2} + \dfrac{z}{6} = 1$.

   (a) What are the intersections of $\mathcal{H}$ with each of the three coordinate planes xy, yz and xz? Label each of them appropriately.

   The intersection with the xy-plane $(z = 0)$ is the line $\dfrac{x}{3} + \dfrac{y}{2} = 1$

   The intersection with the yz-plane $(x = 0)$ is the line $\dfrac{y}{2} + \dfrac{z}{6} = 1$

   The intersection with the xz-plane $(y = 0)$ is the line $\dfrac{x}{3} + \dfrac{z}{6} = 1$

   (b) What are the intersections of $\mathcal{H}$ with each of the three coordinate axes? Label each of them appropriately.

   The intersection with the x-axis $(y, z = 0)$ is the point: $(3, 0, 0)$.

   The intersection with the y-axis $(x, z = 0)$ is the point: $(0, 2, 0)$.

   The intersection with the z-axis $(x, y = 0)$ is the point: $(0, 0, 6)$.

   (c) Consider the solid region $\mathcal{W}$ bounded by the three coordinate planes and the plane $calH$. Set up (without calculating) a triple integral that corresponds to the mass of $\mathcal{W}$, if the density at each of its points is given by the function $\delta(x, y, z) = x + y$.

   The integration region in the xy-plane is (see part (a)): $0 \leq x \leq 3$, $0 \leq y \leq 2 - \dfrac{2x}{3}$. The limits for z are given by the xy-plane $z = 0$ (below) and by the given plane $z = 6 - 2x - 3y$ (above). The triple integral is:

$$\int_0^3 \int_0^{2-\frac{2x}{3}} \int_0^{6-2x-3y} (x+y)\ dz\ dy\ dx$$

(d) Calculate the integral in part (c).

3. The function f describes the density of birds around DIA, so that $w = f(x, y, z)$ represents the number of birds/m^3 at the point of coordinates (x, y, z) (the airport is situated at the origin). Write down a formula that would calculate the average bird density in half-spherical region of radius R around DIA.

$$f_{\text{average}} = \frac{1}{V} \int_{x=0}^{x=R} \int_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}} \int_{z=0}^{z=\sqrt{R^2-x^2-y^2}} f(x,y,z)\ dz\ dy\ dx \qquad \text{birds/m}^3$$

where V is the volume of the half-spherical region: $V = \dfrac{2\pi R^3}{3}$.