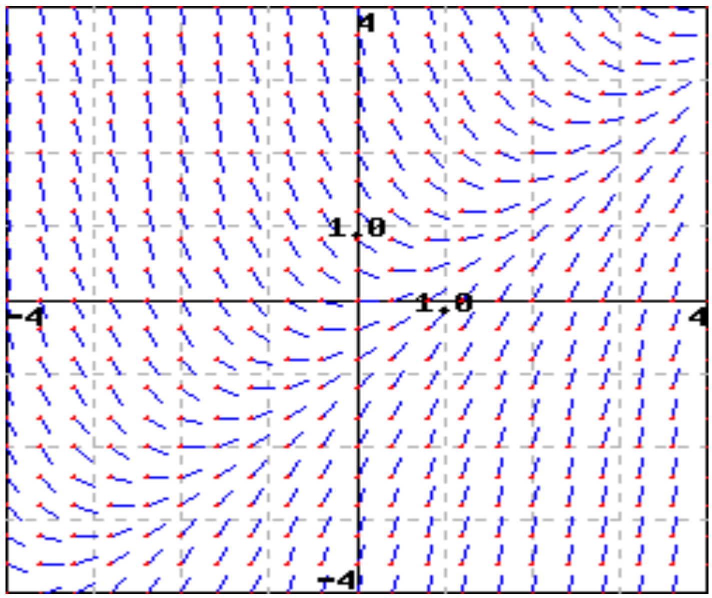
MATH 2300 – Review problems for Midterm #3

1. The slope field for the equation $\frac{dy}{dx} = x - y$ is shown below:



   (a) Sketch the solutions that pass through the points $(0,0)$, $(3,1)$ and $(1,0)$.

   (b) From your sketch, what is the equation of the solution to the differential equation that passes through $(1,0)$? Justify your answer.

   (c) Verify that your solution is correct by substituting it into the differential equation.

2. Suppose that a corpse was discovered in a motel room at midnight and its temperature was $80°F$. The temperature of the room is kept constant at $H_r = 60°F$. Two hours later the temperature of the corpse dropped to $75°F$. Find the time of death, if the temperature H of the corpse satisfies Newton's Law of Cooling:
$$\frac{dH}{dt} = k(H - H_r)$$

3. Consider the equation: $x\dfrac{dy}{dx} + y = x^2 + 1$

   (a) Is the equation separable?

   (b) Show that $y = \frac{x^2}{3} + \frac{2}{3x}$ is a solution of the equation, satisfying the initial condition $y(1) = 1$.

   (c) What is the domain for the solutions going through $(1, y_0)$? How about the ones going through $(-1, y_0)$?

   (d) Find the general solution. (Hint: without separating, notice that the left side is equal to $\frac{d}{dx}(xy)$, and integrate to solve for y.)

4. Consider f a function with $f(0) = 1$ and $f(1) = -2$. Consider the solution of the differential equation $f(x)\dfrac{dy}{dx} - f'(x) = 0$, which satisfies the initial condition $y(0) = 1$. What is the value of this solution at $x = 1$?

5. Consider the function of two variables: $f(x,y) = \sqrt{1-x-y}$.

   (a) What region of the xy plane is the domain of this function?
   (b) What is the cross-section with the vertical plane $y=-1$? In a separate xz coordinate system, plot the cross-sectional curve.
   (c) What is the cross-section with the xy horizontal plane?
   (d) Where does the surface which is the graph of the function cross the coordinate axes?

6. (a) Calculate the mutual distances between the points $A(1,0,0)$, $B(0,\sqrt(2),0)$ and $C(1,0,1)$.
   (b) How large (in radians) are the angles of the triangle ABC?

7. Consider the series

$$J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

   Show that $J(x)$ is a solution to the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -x^2 y$$

8. Imagine that Boulder is located at the point $(0,0)$ on a map with distances measured in miles and North pointing in the direction of the positive y-axis and East in the direction of the positive x-axis. Suppose that the temperature, T, at location (x,y) is given by

$$T = f(x,y) = 60e^{-2x+3y}.$$

   (a) Find and interpret $\frac{\partial}{\partial x}f(x,y)$.
   (b) Find and interpret $\frac{\partial}{\partial y}f(x,y)$.
   (c) Suppose you are in Boulder. What is the temperature? Use the partial derivatives to estimate the temperature if you went one mile South and one mile East.

9. Let $f(x,y) = 3\cos(4x)\sin(2y)$.

   (a) Find $\frac{\partial}{\partial x}f(x,y)$. This is a new function of the variables x,y. Call it $g(x,y)$.
   (b) Find $\frac{\partial}{\partial y}f(x,y)$. This is a new function of the variables x,y. Call it $h(x,y)$.
   (c) Find $\frac{\partial}{\partial x}g(x,y)$ and $\frac{\partial}{\partial y}g(x,y)$.
   (d) Find $\frac{\partial}{\partial x}h(x,y)$ and $\frac{\partial}{\partial y}h(x,y)$.
   (e) What do you notice about $\frac{\partial}{\partial y}g(x,y)$ and $\frac{\partial}{\partial x}g(x,y)$. Interpret this in terms of the original function $f(x,y)$.

10. Define

$$f_n(x) = \frac{\sin(nx)}{n^2}.$$

   (a) Show that $\sum_{n=1}^{\infty} f_n(x)$ converges for all x.

(b) Show that $\sum_{n=1}^{\infty} f_n'(x)$ diverges for $x = 2n\pi$ where n is an integer.

11. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence of f and f'.

12. How many terms of the Taylor series for $\ln(1 + x)$ centered at $x = 0$ do you need to estimate the value of $\ln(1.4)$ to within 0.001?

13. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second degree Taylor polynomial, estimate how far the car moves in the next second.

14. (Simple, but I like this one.) Find the integral and express the answer as an infinite series.

$$\int \frac{e^x}{x}\,dx$$

15. Using series, evaluate the limit

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}.$$

16. (Word this differently, but you get the idea.) Use the Lagrange Error Bound for $P_n(x)$ to find a reasonable bound for the error in approximating the quantity $e^{0.60}$ with a third degree Taylor polynomial for e^x centered at $x = 0$.

17. Consider the error in using the approximation $\sin\theta \approx \theta - \theta^3/3!$ on the interval $[-1, 1]$. Where is the approximation an overestimate? Where is it an underestimate? What is the magnitude of the largest possible error?

18. This problem will present another derivation of the Taylor series for e^x centered at $x = 0$.

    (a) Use the tangent line to e^x at $x = 0$ to show that $e^x \geq 1 + x$.

    (b) Given the fact that if $f(x) \geq g(x)$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ for any a, b (which isn't that surprising if you think about definite integrals representing area under a curve), use

$$e^x = 1 + \int_0^x e^t dt$$

    to show that $e^x \geq 1 + x + x^2/2$.

    (c) Using part (b), show that $e^x \geq 1 + x + x^2/2 + x^3/6$.

    (d) Thinking about the Taylor series for e^x, why is it not surprising that e^x is greater than any of the finite polynomial approximations when $x \geq 0$? What happens to the higher power terms of the approximation as the degree increases? How would you use this to show that in the limit the polynomial approaches e^x?

19. (a) Find the Taylor series around $x = 0$ for

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

(b) Find the Taylor series around $x = 0$ for

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

(c) One of the functions from part (a) and part (b) is even and the other is odd. Which is which? Justify your answer in terms of the Taylor series.

(d) Show that $\cosh x + \sinh x = e^x$.

20. Find the first three non-zero terms in the Taylor expansion for $\tan x$ around $x = 0$.

21. Find the radius of convergence for

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

with k a positive integer.

22. Find the Taylor series and radius of convergence for

$$f(x) = (x+2)^2(x-1).$$

23. This is problem 2 from a different perspective.

(a) Define the function $C(x)$ by the following Taylor series:

$$C(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

What is the radius of convergence for $C(x)$? What is $C(0)$? Is $C(x)$ even or odd? What function can you think of that has these same properties? Why is this function not that function?

(b) Define the function $S(x)$ by the following Taylor series:

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

What is the radius of convergence for $S(x)$? What is $S(0)$? Is $S(x)$ even or odd? What function can you think of that has these same properties? Why is this function not that function?

(c) Show that $C'(x) = S(x)$ and that $S'(x) = C(x)$.

(d) Let $E(x) = C(x) + S(x)$. Show that $E'(x) = E(x)$ and that $E(0) = 1$.

(e) Solve the differential equation

$$\frac{dE}{dx} = E$$

with initial condition $E(0) = 1$.

(f) Using Taylor series, show that $C(x) + S(x) = e^x$.

24. A yam is put into an oven where its temperature, H, is modeled by $\dfrac{dH}{dt} = -k(H - H_0)$, where t is time and k and H_0 are positive constants. Below are four solutions to this differential equation. Which curve(s) correspond(s) to the

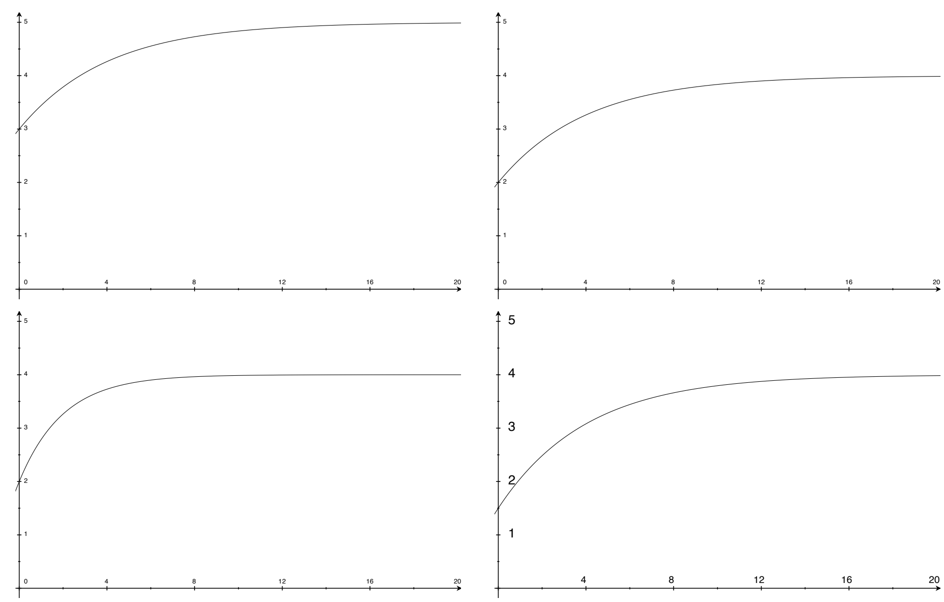
(a) Warmest oven?

(b) Lowest initial temperature?

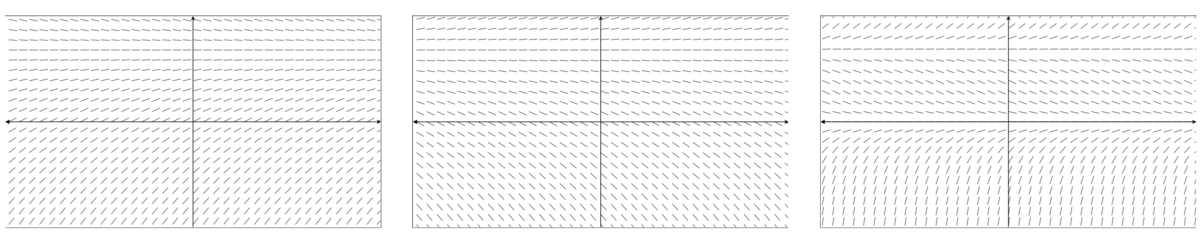
(c) Same initial temperature?

(d) Largest value of k?

You must explain your reasoning to receive credit.



25. Show below are the slope fields of three differential equations, "A", "B", and "C". For each slope field, the axes intersect at the origin.



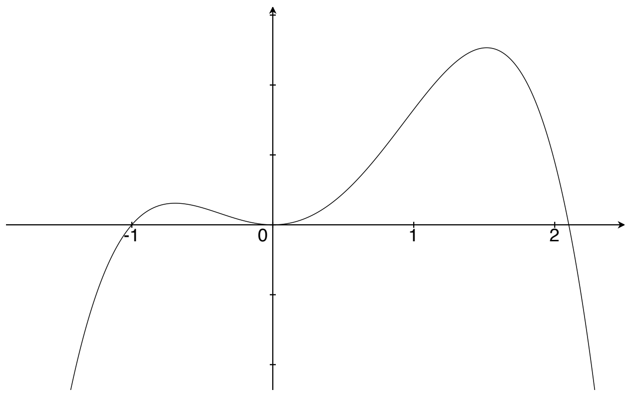
For each of the following functions, indicate which, if any, of the differential equations, "A", "B", and "C" it could be the solution of. Note that any of the functions could be a solution to zero, one, or more than one of the differential equations. If a function is a solution to none of the differential equations clearly write "None" as your asnwer.

(a) $y = 0$

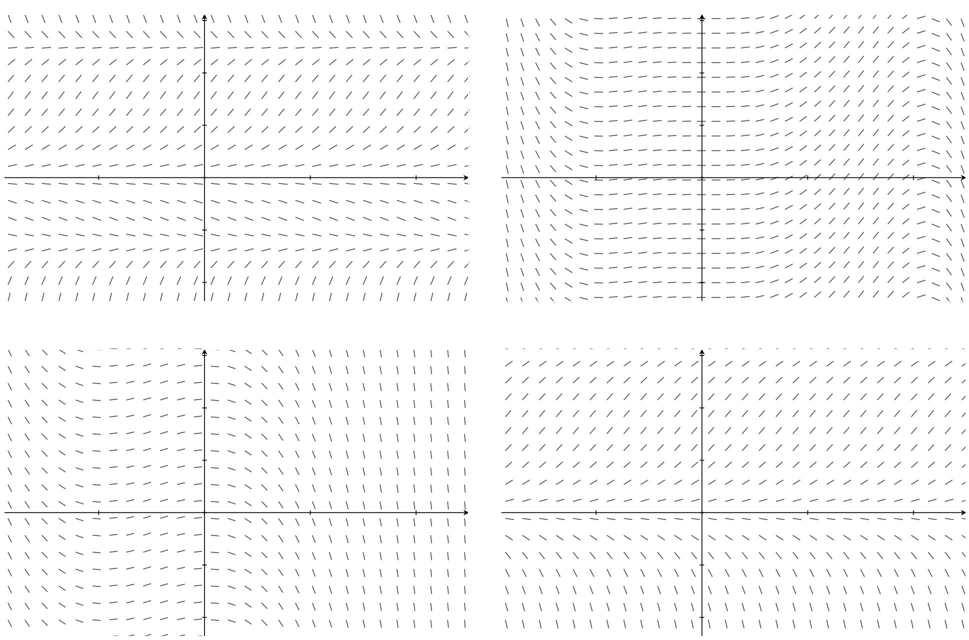
(b) $y = 1$

(c) $y = 1 + ke^x$

26. Suppose that $\frac{dy}{dx} = f(x)$, where $f(x)$ is shown in the graph below.



(a) Which of the slope fields below (which have ticks with unit spacing) could be the slope field of this differential equation? Explain briefly.



(b) Are there any equilibrium solutions to this differential equation? If so, what are they? If not, why not?