MATH 2300 – CALCULUS II – UNIVERSITY OF COLORADO

Fall 2010 – Midterm I

SOLUTIONS IN RED

1. (26 points) Consider the integral $\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$

   (a) (6 points) Draw a picture of the area that this integral represents. Evaluate this integral any way you like.

   The graph of the function $f(x)=\sqrt{2-x^2}$ for x in $[0,\sqrt{2}]$ represents the upper right quarter (first quadrant) of the circle centered at the origin and of radius $r=\sqrt{2}$. We now want to compute the integral:

   Solution 1: Since the integrand f is positive on $[0,\sqrt{2}]$, the integral $\int_0^{\sqrt{2}}\sqrt{2-x^2}\,dx$ represents the area under the graph of f and above the x-axis; this is exactly the area of a quarter of the disc, which is $\frac{1}{4}\pi r^2=\frac{2\pi}{4}=\frac{\pi}{2}$.

   Solution 2: We make the trig substitution, and write x in $[-\sqrt{2},\sqrt{2}]$ as $x=\sqrt{2}\sin t$, with t in $[\frac{-\pi}{2},\frac{\pi}{2}]$. Then the radical becomes: $\sqrt{2-x^2}=\sqrt{2-2\sin^2 t}=\sqrt{2\cos^2 t}=\sqrt{2}\cos t$. The integration limits for t are $t=\sin^{-1}(0)=0$ and $t=\sin^{-1}(1)=\frac{\pi}{2}$, and $dx=\sqrt{2}\cos t\,dt$. So:

$$\begin{aligned}\int_0^{\sqrt{2}}\sqrt{2-x^2}\,dx &= \int_0^{\pi/2}\sqrt{2}\cos t\sqrt{2}\cos t\,dt=\int_0^{\pi/2}2\cos^2 t\,dt=\int_0^{\pi/2}[1+\cos(2t)]\,dt\\ &= \left(t+\frac{\sin(2t)}{2}\right)\Big|_0^{\pi/2}=\left(\frac{\pi}{2}+\frac{\sin(\pi)}{2}\right)-\left(0+\frac{\sin(0)}{2}\right)=\frac{\pi}{2}\simeq 1.57\end{aligned}$$

   (b) (6 points) Compute LEFT(2) and MID(2). Show your work. An answer that appears to have come from a calculator will receive 0 points.

$$\text{LEFT}(2)=\Delta x\,[f(x_0)+f(x_1)]=\frac{\sqrt{2}}{2}\left[\sqrt{2-0^2}+\sqrt{2-(\sqrt{2}/2)^2}\right]=\frac{\sqrt{2}}{2}\left[\sqrt{2}+\sqrt{3/2}\right]\simeq 1.86$$

$$\text{MID}(2)=\Delta x\left[f(\tfrac{x_0+x_1}{2})+f(\tfrac{x_1+x_2}{2})\right]=\frac{\sqrt{2}}{2}\left[\sqrt{2-(\sqrt{2}/4)^2}+\sqrt{2-(3\sqrt{2}/4)^2}\right]\simeq 1.63$$

   (c) (4 points) The formula for error in an approximation is given by

$$\text{ERROR} = \text{ACTUAL VALUE} - \text{APPROXIMATION}.$$

   Compute the errors in LEFT(2) and MID(2).

$\text{Error Left}(2) \simeq 1.57 - 1.86 = -0.29$
$\text{Error Mid}(2) \simeq 1.57 - 1.62 = -0.05$

(d) (6 points) Estimate the errors in LEFT(20) and MID(20) without actually computing these values. Again, an answer that appears to have come from a calculator will receive 0 points.

$\text{Error Left}(20) \simeq -0.29/10 = -0.029$
$\text{Error Mid}(20) \simeq -0.05/100 = -0.0005$

(e) (4 points) Order the following approximations from least to greatest for any integer $n \geq 2$.

$$\text{Left}(n), \qquad \text{Right}(n), \qquad \text{Mid}(n) \qquad \text{and} \qquad \text{Trap}(n)$$

The function f is decreasing, so $\text{Right}(n) < \int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx < \text{Left}(n)$.

The function f is concave down, so $\text{Trap}(n) < \int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx < \text{Mid}(n)$.

Since MID(n) and TRAP(n) are generally closer estimates than LEFT(n) and RIGHT(n), we can order:

$$\text{Right}(n) < \text{Trap}(n) < \int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx < \text{Mid}(n) < \text{Left}(n)$$

2. (12 points)

(a) (8 points) Prove that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\,dx$$

Start by rewriting:

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^{n-1} x \sin x\,dx$$

Use an integration by parts with $u = \sin^{n-1} x$ and $dv = \sin x\,dx$. Then $du = (n-1)\sin^{n-2} x \cos x\,dx$ and $v = -\cos x$, so:

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^{n-1} x \sin x\,dx = \left[-\sin^{n-1} x \cos x\right]\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x\,dx \\ &= 0 + (n-1)\int_0^{\pi/2} \sin^{n-2} x(1-\sin^2 x)\,dx \qquad \text{(multiply out)} \\ &= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,dx - (n-1)\int_0^{\pi/2} \sin^n x\,dx \end{aligned}$$

Bring $-(n-1)\int_0^{\pi/2}\sin^n x\,dx$ to the left side and get:

$$\int_0^{\pi/2}\sin^n x\,dx-(n-1)\int_0^{\pi/2}\sin^n x\,dx=(n-1)\int_0^{\pi/2}\sin^{n-2}x\,dx$$

Adding the terms together on the left:

$$n\int_0^{\pi/2}\sin^n x\,dx=(n-1)\int_0^{\pi/2}\sin^{n-2}x\,dx$$

Dividing both sides by the coefficient n, it follows immediately that:

$$\int_0^{\pi/2}\sin^n x\,dx=\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$$

(b) (4 points) Use the formula in (a) to evaluate $\displaystyle\int_0^{\pi/2}\sin^3 x\,dx$.

We apply the formula at (a) for $n=3$:

$$\int_0^{\pi/2}\sin^3 x\,dx=\frac{2}{3}\int_0^{\pi/2}\sin x\,dx=\frac{2}{3}[-\cos x]|_0^{\pi/2}=\frac{2}{3}[0-(-1)]=\frac{2}{3}$$

3. (12 points)

(a) (6 points) Show that

$$\int_0^{\infty}x^2e^{-x^2}\,dx=\frac{1}{2}\int_0^{\infty}e^{-x^2}\,dx$$

We start by setting up the limit for the improper integral:

$$\int_0^{\infty}x^2e^{-x^2}\,dx=\lim_{b\to\infty}\int_0^{b}x^2e^{-x^2}\,dx$$

We use integration by parts, with $u=x$ and $dv=xe^{-x^2}dx$. Then $du=dx$ and (using a substitution $w=x^2$) $v=\int xe^{-x^2}=-\frac{1}{2}e^{-x^2}$.

$$\begin{aligned}\lim_{b\to\infty}\int_0^{b}x^2e^{-x^2}\,dx &= \lim_{b\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]\Bigg|_0^b+\frac{1}{2}\lim_{b\to\infty}\int_0^{b}e^{-x^2}\,dx\\ &= \lim_{b\to\infty}\left[-\frac{1}{2}be^{-b^2}\right]+\frac{1}{2}\lim_{b\to\infty}\int_0^{b}e^{-x^2}\,dx\end{aligned}$$

But

$$\lim_{b\to\infty}\left[-\frac{1}{2}be^{-b^2}\right]=-\frac{1}{2}\lim_{b\to\infty}\left[\frac{b}{e^{b^2}}\right]$$

Since both top and bottom of the fraction go to ∞ when $b\to\infty$, we can use L'Hôpital's rule. Calculate the limit for the ratio of the derivatives:

$$\lim_{b\to\infty}\left[\frac{1}{2be^{b^2}}\right] = 0$$

Hence the original limit is zero, which means that:

$$\lim_{b\to\infty}\int_0^b x^2 e^{-x^2}\,dx = \frac{1}{2}\lim_{b\to\infty}\int_0^b e^{-x^2}\,dx$$

(b) (6 points) Show that

$$\int_0^2 e^{w^2}\,dw = \int_0^1 2e^{4x^2}\,dx$$

Start with the right side: make a substitution that will transform $4x^2$ into w^2. That is, $w = 2x$. Then $dw = 2dx$, and the new integration limits are $w(0) = 0$ and $w(1) = 2$.

$$\int_0^1 2e^{4x^2}\,dx = \int_0^2 2e^{w^2}\,\frac{dw}{2} = \int_0^2 e^{w^2}\,dw$$

4. (12 points)

(a) (6 points) Let $f(x)$ be a continuous function. Prove that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Start with the right side, for example, and make the substitution $w = a - x$. Then $dw = dx$ and the new integration limits are $w(0) = a$ and $w(a) = 0$:

$$\int_0^a f(a-x)\,dx = \int_a^0 f(w)\,(-dw) = -\int_a^0 f(w)\,dw = -\int_0^a f(w)\,dw = -\int_0^a f(x)\,dx$$

(b) (6 points) Let $f'(x)$ be a continuous function and let $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty f'(x)\,dx = -f(0)$$

$$\begin{aligned}\int_0^\infty f'(x)\,dx &= \lim_{b\to\infty}\int_0^b f'(x)\,dx = \lim_{b\to\infty}[f(x)]|_0^b \\ &= \lim_{b\to\infty}[f(b)-f(0)] = \lim_{b\to\infty} f(b) - f(0) = 0 - f(0) = -f(0)\end{aligned}$$

5. (12 points) Assume $\int_1^\infty f(x)\,dx$ converges and that a is a positive constant.

(a) (6 points) Does $\int_1^\infty f(a+x)\,dx$ converge? Explain in 3 sentences or less.

Solution 1: Make the substitution $w = a + x$. Then:

$$\int_1^\infty f(a+x)\,dx = \int_{a+1}^{a+\infty} f(w)\,dw = \int_{a+1}^\infty f(w)\,dw$$

So $\int_{a+1}^\infty f(w)\,dw$ converges, since it differs only by a finite value $\int_1^{a+1} f(w)\,dw$ from the original integral $\int_1^\infty f(w)\,dw$, which is known to be convergent.

Solution 2: For $a >$, the transformation from $f(x)$ to $f(a+x)$ is a shift to the left (notice that the interval $[1,\infty]$ is still contained in the domain of the new function). This horizontal shift does not affect the convergence of the integral, only the finite limit value.

(b) (6 points) Does $\int_1^\infty (a + f(x))\,dx$ converge as well? Explain in 3 sentences or less.

Solution 1:

$$\int_1^\infty (a+f(x))\,dx = \int_1^\infty a\,dx + \int_1^\infty f(x)\,dx$$

But $\int_1^\infty a\,dx = \lim_{b\to\infty} \int_1^b a\,dx = \lim_{b\to\infty} ab = \infty$. Since we known that $\int_1^\infty f(x)\,dx$ is convergent, their sum will be blow up to ∞, making $\int_1^\infty (a+f(x))\,dx$ divergent.

Solution 2: The function $a + f(x)$ is a shift up by a units. This will prevent the proper decay of $f(x)$, and thus introduces an additional positive infinite area between the x-axis and the graph of the integrand, preventing it from converging. So $\int_1^\infty (a+f(x))\,dx$ diverges.

6. (12 points) Some values of the continuous, differentiable function $g(x)$ are given in the table below.

x	1	5/4	3/2	7/4	2
$g(x)$	2	3	4	7	10

(a) (6 points) Estimate the integral $\int_1^2 \frac{g(t)}{t}\,dt$ using these data.

We can use, for example, the trapezoidal sum for $n = 4$:

$$\begin{aligned} \text{TRAP}(4) &= \frac{1}{8}\left[\frac{g(1)}{1} + 2\frac{g(5/4)}{5/4} + 2\frac{g(3/2)}{3/2} + 2\frac{g(7/4)}{7/4} + \frac{g(2)}{2}\right] \\ &= \frac{1}{8}\left[2 + 2\frac{3}{5/4} + 2\frac{4}{3/2} + 2\frac{7}{7/4} + \frac{10}{2}\right] = \frac{1}{4}\left[2 + \frac{24}{5} + \frac{16}{3} + 8 + 5\right] \end{aligned}$$

(b) (6 points) Estimate the integral $\int_1^2 g(t)\,dt$ using these data.

$$\begin{aligned} \text{TRAP}(4) &= \frac{1}{8}[g(1)+2g(5/4)+2g(3/2)+2g(7/4)+g(2)] \\ &= \frac{1}{8}[2+2\cdot 3+2\cdot 4+2\cdot 7+10] \end{aligned}$$

7. (14 points) The Law of Mass Action tells us that the time, T, taken by a chemical to create a quantity p of the product (in molecules) is given by

$$T = \int_0^p \frac{k\,dx}{(a-x)(b-x)}$$

where a and b are initial quantities of the two ingredients used to make the product, and k is a positive constant. Suppose $0 < a < b$.

Let's first calculate $\int_0^p \frac{k\,dx}{(a-x)(b-x)} = \int_0^p \frac{k\,dx}{(x-a)(x-b)}$ using partial fractions:

$$\frac{k}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$$

We want to find expressions for A and B (that will depend on the constants k, a and b) so that:

$$A(x-b)+B(x-a) = k$$

Plug in $x = b$ and obtain $B(b-a) = k$, i.e. $B = \frac{k}{b-a}$.

Plug in $x = a$ and obtain $A(a-b) = k$, i.e. $A = \frac{k}{a-b} = -\frac{k}{b-a}$.

Then

$$\begin{aligned} \int \frac{k\,dx}{(x-a)(x-b)} &= -\frac{k}{b-a}\int\frac{dx}{x-a} + \frac{k}{b-a}\int\frac{dx}{x-b} = -\frac{k}{b-a}\ln|x-a| + \frac{k}{b-a}\ln|x-b| \\ &= \frac{k}{b-a}\big(\ln|x-b| - \ln|x-a|\big) = \frac{k}{b-a}\ln\left|\frac{x-b}{x-a}\right| \end{aligned}$$

$$\int \frac{k\,dx}{(x-a)(x-b)} = \frac{k}{b-a}\left[\ln\left|\frac{x-b}{x-a}\right|\right]\Bigg|_0^p = \frac{k}{b-a}\left(\ln\left|\frac{p-b}{p-a}\right| - \ln\left|\frac{b}{a}\right|\right) = \frac{k}{b-a}\ln\left(\frac{a}{b}\left|\frac{p-b}{p-a}\right|\right)$$

(a) (8 points) Find the time taken to make a quantity $p = a/2$ of the product.

$$T(a/2) = \frac{k}{b-a}\ln\left(\frac{a}{b}\left|\frac{a/2-b}{a/2-a}\right|\right) = \frac{k}{b-a}\ln\left(\frac{2b-a}{b}\right)$$

(b) (6 points) What happens to T as $p \to a$?

$$\lim_{p\to a} T(p) \quad = \quad \lim_{p\to a} \frac{k}{b-a} \ln\left(\frac{a}{b}\left|\frac{p-b}{p-a}\right|\right)$$

Since $\lim_{p\to a} \dfrac{p-b}{p-a} = \pm\infty$, it means that $\lim_{p\to a} \ln\left(\dfrac{a}{b}\left|\dfrac{p-b}{p-a}\right|\right) = \infty$. So: $\lim_{p\to a} T(p) = \infty$. In other words, a is the equilibrium quantity.