MATH 2300: CALCULUS 2

April 6, 2011

TEST 3: SOLUTIONS

1. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

You must justify your answer to receive credit.

$$\lim_{n\to\infty} \left| \frac{((n+1)!)^2}{(2(n+1))!} \frac{(2n)!}{(n!)^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| = \frac{1}{4},$$

which is less than 1, so the series converges by the ratio test.

2. Suppose $a_n \geq 0$ and $b_n \geq 0$ for all n. Also suppose that the series $\sum a_n$ converges, and that $\lim_{n\to\infty} \frac{a_n}{b_n} = 2$. Which of the following are true, and which are false? Provide a brief explanation for each of your answers.

(a) The sequence a_n converges. True by the nth term test (if a series converges, then the nth term goes to zero).

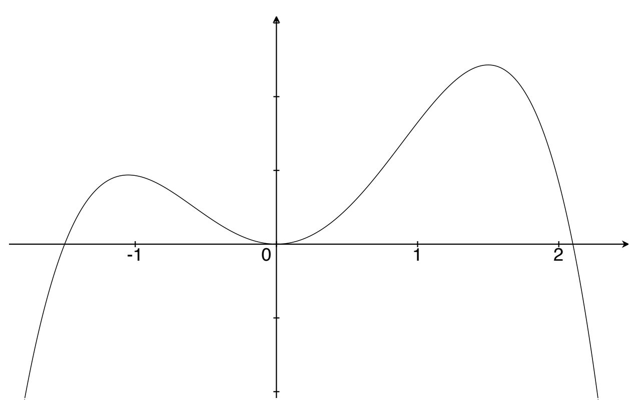
(b) $\lim_{n\to\infty} a_n = 1$. False: as noted above, the nth term goes to *zero*, not one.

(c) The series $\sum b_n$ is convergent. True by the limit comparison test.

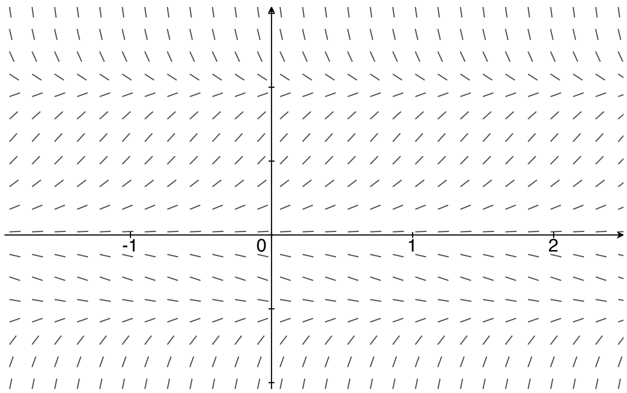
(d) The sequence b_n converges. True by the nth term test.

(e) The series $\sum \frac{b_n}{a_n}$ is convergent. False: for a series to converge, its nth term must go to zero, but b_n/a_n goes to $1/2$, since a_n/b_n goes to 2.

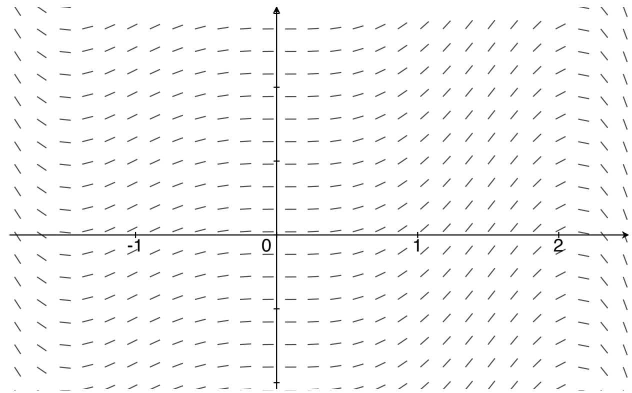
3. Suppose that $\frac{dy}{dx} = f(x)$, where $f(x)$ is shown in the graph below.



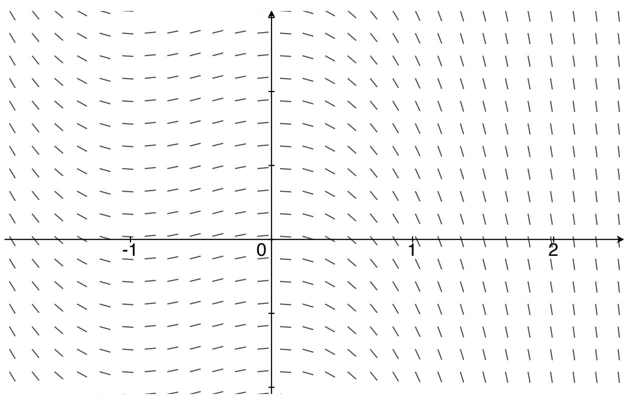
Which one of the slope fields (i), (ii), (iii), (iv) below could be the slope field of this differential equation? Explain briefly.



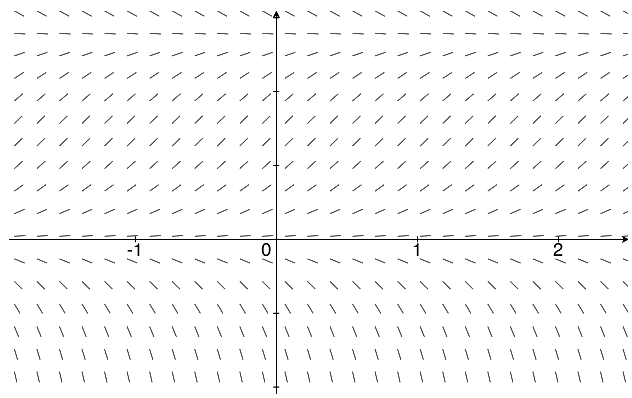
(i)



(ii)



(iii)



(iv)

$f(x) = 0$ when x is about -1.5, 0, and 2.1, so we look for a slope field with slopes equal to zero (that is, with horizontal slope segments) at such values of x. Thus, the correct answer is (ii).

4. (a) Write down the second degree Taylor polynomial $P_2(x)$ approximating

$$f(x) = e^{\sin(x)}$$

near $x = 0$.

$$\begin{aligned} f(x) &= e^{\sin(x)} & f(0) &= e^{\sin(0)} = 1 \\ f'(x) &= \cos(x)e^{\sin(x)} & f'(0) &= \cos(0)e^{\sin(0)} = 1 \\ f''(x) &= \cos^2(x)e^{\sin(x)} - \sin(x)e^{\sin(x)} & f''(0) &= \cos^2(0)e^{\sin(0)} - \sin(0)e^{\sin(0)} = 1 \end{aligned}$$

So

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2}.$$

(b) Use your result from part (a) to approximate $e^{\sin(0.1)}$.

$$e^{\sin(0.1)} \approx P_2(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} = 1.105.$$

5. Suppose $f(x)$ is a function such that, for any positive integer n,

$$|f^{(n+1)}(x)| \leq n!$$

for all x.

(a) Use the Lagrange Error Bound Formula to show that the error term $E_n(x) = f(x) - P_n(x)$, where $P_n(x)$ is the Taylor polynomial of degree n for $f(x)$ near $x = 0$, satisfies

$$|E_n(x)| \leq \frac{|x|^{n+1}}{n+1}.$$

We know that

$$|E_n(x)| \leq \frac{M|x|^{n+1}}{(n+1)!}$$

where M is an upper bound for $|f^{(n+1)}(u)|$ on $[0, x]$. But $|f^{(n+1)}(x)| \leq n!$ for all x, so

$$|E_n(x)| \leq \frac{n!|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{n+1}$$

for all x.

(b) Show that, for f as above, the Taylor series at $x = 0$ for $f(x)$ converges to $f(x)$ for $-1 \leq x \leq 1$. We only need to show that the error term $|E_n(x)| \to 0$ as $N \to \infty$. But by what we saw above,

$$|E_n(x)| \leq \frac{|x|^{n+1}}{n+1}.$$

For $|x| \leq 1$, the right hand side is $\leq 1/(n+1)$, which certainly goes to zero as $n \to \infty$, and we're done.

6. What's the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}?$$

Please show all of your work (and don't forget to check the endpoints).

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\cdot 3^{n+1}} \frac{n \cdot 3^n}{(x-2)^n} \right| = |x-2| \lim_{n\to\infty} \left| \frac{n}{(n+1)\cdot 3} \right| = \frac{|x-2|}{3}.$$

This converges for $|x-2|/3 < 1$, which means $|x-2| < 3$, which means $-3 < x-2 < 3$, which means $-1 < x < 5$.

We now check the endpoints: plugging in $x = -1$ gives us the series

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which convergesby the alternating series test. On the other hand, plugging in $x = 5$ gives

$$\sum_{n=1}^{\infty} \frac{(3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges by the p-series test with $p = 1$. So the interval of convergence is $[-1, 3)$.

7. Suppose the series

$$\sum_{n=5}^{\infty} C_n (x-3)^n$$

converges when $x = 0$ but diverges when $x = 9$. For each of the following values of x, determine whether the series converges or diverges there, or if there's not enough information to say. Explain each of your answers briefly.

(a) $x = 1$ Converges: since the series converges at $x = 0$, which is 3 units away from the center $a = 3$, it converges at least on the interval $[0, 6)$.

(b) $x = -5$ Diverges: since the series diverges at $x = 9$, which is 6 units away from the center $a = 3$, it diverges at least on the interval $(-\infty, -3)$ and on $[9, \infty)$.

(c) $x = -2$ Not enough information, since $x = -2$ is neither within 3 units of the center nor beyond 6 units from the center.

8. (a) Solve the initial value problem

$$\frac{dy}{dx} = \frac{\cos(x)}{e^y}, \quad y(0) = 0.$$

Please make sure you solve for y; that is, express your solution in the form

$$y = \text{ a function of } x.$$

$$\begin{aligned} e^y\, dy &= \cos(x)\, dx \\ \int e^y\, dy &= \int \cos(x)\, dx \\ e^y &= \sin(x) + C \\ y &= \ln(\sin(x) + C) \end{aligned}$$

Now plug in $y(0) = 0$:

$$\begin{aligned} 0 &= \ln(0 + C) = \ln(C) \\ e^0 &= C \\ C &= e^0 = 1 \end{aligned}$$

So

$$y = \ln(1 + \sin(x)).$$

(b) Show that, if y is the function you found in part (a) above, then

$$y'' = -e^{-y}.$$

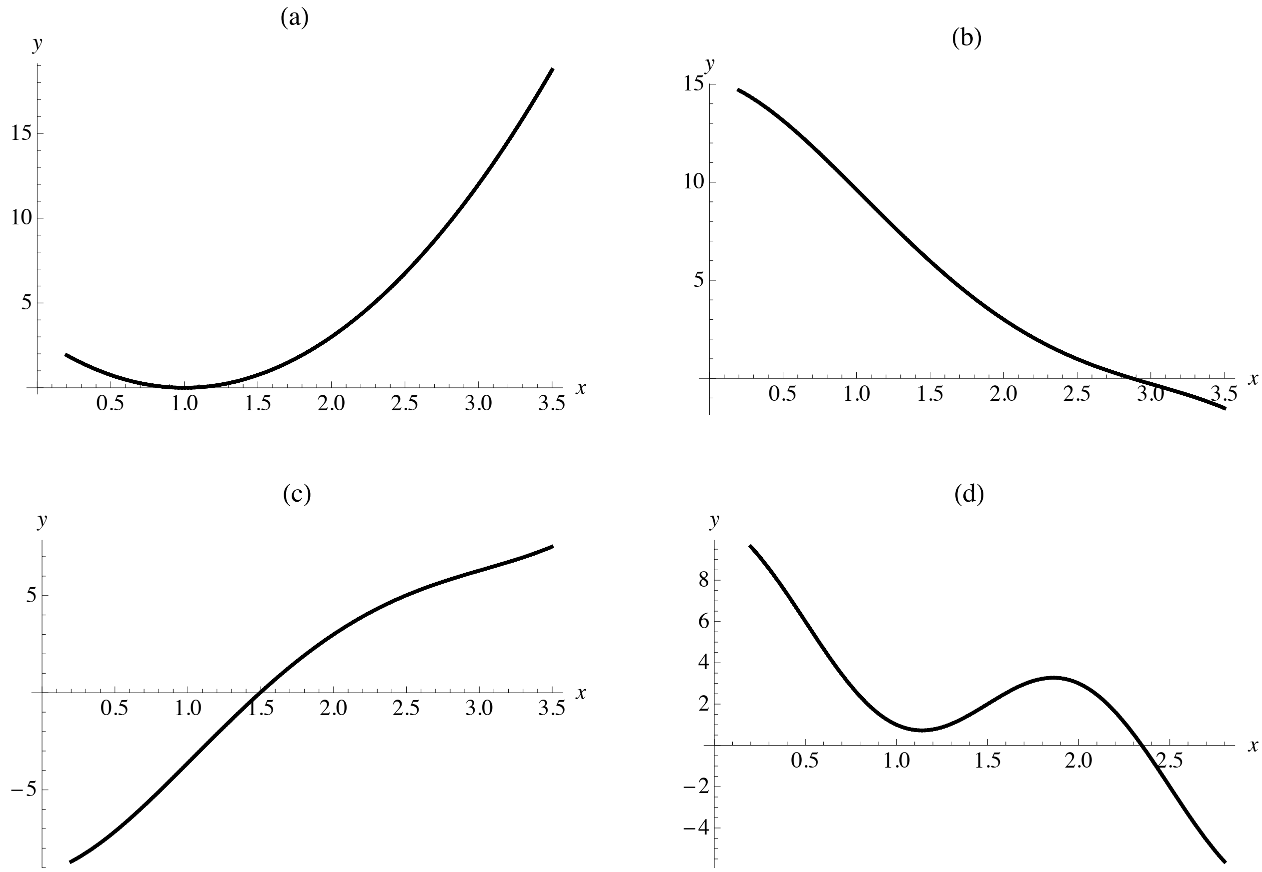
In computing y'', you may want to use the fact that $\cos^2 x + \sin^2 x = 1$ for any x.

$$\begin{aligned} y'' = (y')' &= \frac{d}{dx}\frac{d}{dx}[\ln(1 + \sin(x))] = \frac{d}{dx}\frac{\cos x}{1 + \sin(x)} \\ &= \frac{(1 + \sin(x))(-\sin(x)) - \cos(x)(\cos(x))}{(1 + \sin(x))^2} = \frac{-\sin(x) - \sin^2(x) - \cos^2(x)}{(1 + \sin(x))^2} \\ &= \frac{-1 - \sin(x)}{(1 + \sin(x))^2} = -\frac{1}{1 + \sin(x)} = -\frac{1}{e^{\ln(1+\sin(x))}} = -e^{-\ln(1+\sin(x))} = -e^{-y}. \end{aligned}$$

9. Which one of the functions (a), (b), (c), (d) sketched below could possibly have Taylor series

$$3 + 5(x-2) - 2(x-2)^2 + \cdots ?$$

Please explain.



Only (c), because it's the only function with $f'(2) > 0$ and $f''(2) < 0$.