Math 2300 – Calculus II – University of Colorado

Fall 2010 – Final exam problems

1 Short-answer type questions

Chapter 7 – Integration

1. Decide whether the following statements are true or false. Give a brief justification for your answer.

   (a) If f is continuous for all x and $\int_0^\infty f(x)dx$ converges, then so does $\int_a^\infty f(x)dx$ for all poitive a.

   (b) if $f(x)$ is continuous and positive for $x > 0$ and if $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)dx$ converges.

   (c) If $\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ both converge then $\int_0^\infty (f(x) + g(x))dx$ converges.

   (d) If $\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ both diverge then $\int_0^\infty (f(x) + g(x))dx$ diverges.

2. For the next four problems, let a be any positive number, and suppose $f(x)$ is continuous and $\int_0^\infty f(x)dx$ converges.

   (a) $\int_0^\infty af(x)dx$ converges.

   (b) $\int_0^\infty f(ax)dx$ converges.

   (c) $\int_0^\infty f(a + x)dx$ converges.

   (d) $\int_0^\infty (a + f(x))dx$ converges.

3. For the following problems, state which of the integration techniques you would use to evaluate the integral, but **DO NOT** evaluate the integrals. If your answer is **substitution**, also list w and dw; if your answer is **integration by parts**, also list u, dv, du and v; if your answer is **partial fractions**, set up the partial fraction decomposition, but do not solve for the numerators; if your answer is **trigonometric substitution**, write which substitution you would use.

   (a) $\int \tan x dx$

   (b) $\int \frac{dx}{x^2-9}$

   (c) $\int e^x \cos x dx$

   (d) $\int \frac{\sqrt{9-x^2}}{x^2} dx$

   (e) $\int \frac{\sin(\ln x)}{x} dx$

   (f) $\int x^{3/2} \ln x dx$

   (g) $\int \frac{1}{\sqrt{x^2+4}} dx$

   (h) $\int \frac{e^x+x}{e^x+1} dx$

4. True or False: For $x > 0$, $e^x = 1 + \displaystyle\int_0^x e^t\, dt$

Chapter 9 – Sequences and series

1. The sequence $(\frac{1}{2})^n$

   (a) Converges, but not monotonically

   (b) Converges, monotonically increasing

   (c) Converges, monotonically decreasing

   (d) Diverges

2. Determine the general term of the following series if the starting value is $n = 1$.

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{4} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{16} - \cdots$$

(a) $\frac{(-1)^n (x-1)^n}{2n}$
(b) $\frac{(-1)^{n+1} (x-1)^n}{2^n}$
(c) $\frac{(-1)^n (x-1)^n}{2^n}$
(d) $\frac{(-1)^{n+1} 2(x-1)^n}{2^n}$

3. The power series $\sum C_n x^n$ diverges at $x = 7$ and converges at $x = -3$. At $x = -9$, the series is

(a) Conditionally convergent
(b) Absolutely convergent
(c) Divergent
(d) Cannot be determined.

4. The power series $\sum C_n (x-5)^n$ converges at $x = -5$ and diverges at $x = -10$. At $x = 11$, the series is

(a) Conditionally convergent
(b) Absolutely convergent
(c) Divergent
(d) Cannot be determined.

5. The power series $\sum C_n x^n$ diverges at $x = 7$ and converges at $x = -3$. At $x = -4$, the series is

(a) Conditionally convergent
(b) Absolutely convergent
(c) Divergent
(d) Cannot be determined.

6. In order to determine if the series converges or diverges, the comparison test can be used. Decide which series provides the best comparison.

$$\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+1}$$

(a) $\sum_{k=1}^{\infty} \frac{1}{k}$
(b) $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$
(c) $\sum_{k=1}^{\infty} \frac{\sqrt{2k}}{k^2}$

7. The limit comparison test can be used to determine whether the series converges. Decide which series to compare with.

$$\sum_{n=1}^{\infty} \frac{200n^2 - 100n - 1}{n^3 + n^2 + n + 1}$$

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n}$

8. Which of the following defines a convergent sequence of partial sums?

   (a) Each term in the sequence is closer to the last term than any two prior consecutive terms.

   (b) Assume that the sequence of partial sums converges to a number, L. Regardless of how small a number you give me, say ϵ, one can find a value N such that the N^{th} term of the sequence is within ϵ of L.

   (c) Assume that the sequence of partial sums converges to a number, L. One can find a value N such that all the terms in the sequence, past the N^{th} term, are less than L.

   (d) Assume that the sequence of partial sums converges to a number, L. Regardless of how small a number you give me, say ϵ, one can find a value N such that all the terms in the sequence, past the N^{th} term, are within ϵ of L.

9. **True or False?** A sequence of partial sums that is bounded and always increasing is a convergent sequence.

   (a) True

   (b) False

10. **True or False?** If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

   (a) True

   (b) False

11. **True or False?** If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges.

   (a) True

   (b) False

Chapter 10 – Taylor series

1. True/False: If $\sum a_n$ is convergent, then the power series $\sum a_n x^n$ has convergence radius at least $R = 1$.

2. Consider the Taylor series $f(x) = \sum a_n x^n$, and $g(x) = \sum b_n x^n$. True/False: Is the Taylor series of the function $(f+g)(x) = \sum (a_n + b_n)x^n$? Is the Taylor series of the function $(fg)(x) = \sum (a_n \cdot b_n)x^n$?

3. True/False: Is the convergence radius for the Taylor series of f the same as the convergence radius for the Taylor series of the derivative f'? How about their convergence intervals?

4. True/False: Is the function $f(x) = \sum_{0}^{\infty} \frac{x^n}{n!}$ a solution to the differential equation $\frac{df}{dx} = f$, with initial condition $f(0) = 1$?

5. If one uses the Taylor polynomial $P_3(x)$ of degree $n = 3$ to approximate $\sin x = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ at $x = .1$, would one get an overestimate or an underestimate?

6. If one uses the Taylor polynomial $P_5(x)$ of degree $n = 5$ to approximate $e^x = \sum_0^\infty \frac{x^n}{n!}$ at $x = -.2$, would one get an overestimate or an underestimate?

7. If one uses the Taylor polynomial $P_4(x)$ of degree $n = 4$ to approximate $\frac{1}{1+x}$ at $x = -.1$, would one get an overestimate or an underestimate? What is the actual error?

8. You want to estimate $\sin x$ using the first 3 nonzero terms in the Taylor series. What formula for the error bound would you use to get the best estimate for the error, without computing the error?

2 Conceptual problems – older material

Chapter 7 – Integration

1. Suppose that $\int_{-1}^{1} h(z)\,dz = 7$ and that $h(z)$ is an even function. Calculate the following.

   (a) $\int_0^1 h(z)\,dz$

   (b) $\int_{-4}^{-2} 5h(z+3)\,dz$

2. Use the fact that $\int_0^\infty e^{-x}\sin(x)dx = \frac{1}{2}$ to find $\int_0^\infty e^{-x}\cos(x)dx$

Chapter 8 – Polar integration and other

1. Find the arc length of the curve $y = x^{3/2}$ from $(1,1)$ to $(2, 2\sqrt{2})$.

2. Find the arc length of the curve $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$.

Chapter 10 – Taylor series

1. Suppose that x is positive but very small. Arrange the following in **increasing** order:

$$x,\ \ \sin x,\ \ \ln(1+x),\ \ 1-\cos(x).\ \ e^x - 1,\ \ x\sqrt{1-x}.$$

2. All the derivatives of some function f exist at 0, and the Taylor series for f about 0 is

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

   Find $f'(0),\ f''(0),\ \text{and} f^{(10)}(0)$.

3. Solve exactly for the variable x:

$$1 + x + x^2 + x^3 + \cdots = 5$$

.

4. Solve exactly for the variable x:

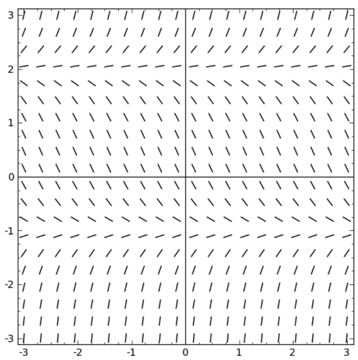
$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1$$

.

5. (a) Find the Taylor series about 0 for $f(x) = x^2 e^{x^2}$
   (b) Is this function even or odd? Justify your answer.
   (c) Find $f^{(3)}(0)$ and $f^{(6)}(0)$.

Chapter 11 – Differential equations and slope fields

1. For each differential equation, find the corresponding slope field. (*Not all slope fields will be used.*)



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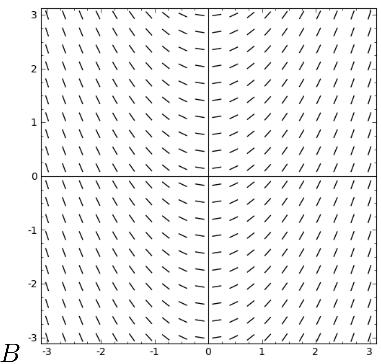
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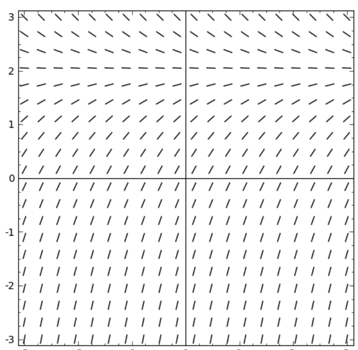
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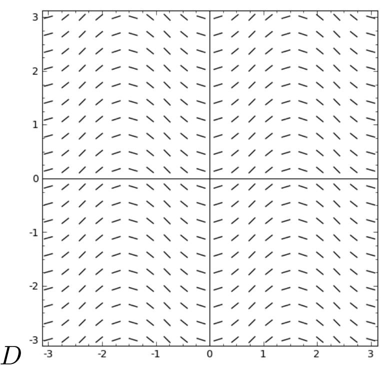
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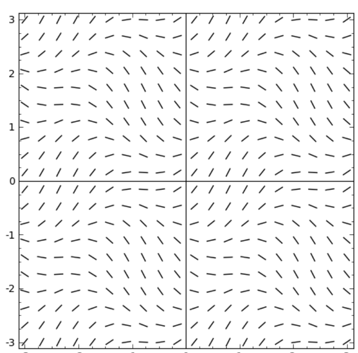
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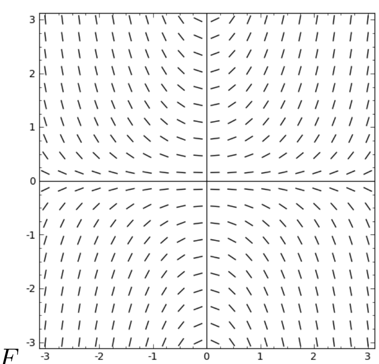
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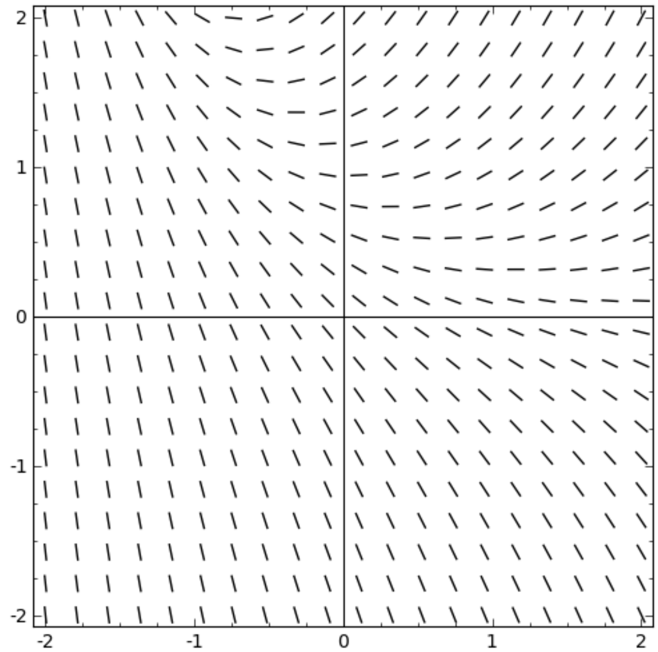
Equation	**Slope Field**
$\frac{dy}{dx} = x$	_____
$y' = xy$	_____
$y' = \sin(2x)$	_____
$\frac{dy}{dx} = 2 - y$	_____

2. A slope field for the differential equation $y' = y - e^{-x}$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions. Make sure to label each sketched graph.

(a) $y(0) = 0$ (b) $y(0) = 1$ (c) $y(0) = -1$



Chapters 9&11 – Word problems on series and differential equations

1. **A drug addiction problem for Mike:** Suppose that ibuprofen is taken in 200mg does every six hours, and that all 200mg are delivered to the patient's body immediately when the pill is taken. After six hours, 12.5% of the ibuprofen remains. Find expressions for the amount of ibuprofen in the patient immediately before and after the n^{th} pill taken. Include work; without work, you may receive no credit.

2. A population of rabbits lives on an island where they have no predators, evolving only based on their interactions: an increase in population facilitates further reproduction, but also introduces competition over resources. For this particular species, the reproduction rate is $a > 0$ and the competition rate is $b > 0$, so that the equation describing the population rate is: $\frac{dP}{dt} = aP - bP^2$

   (a) What are the equilibria that the population can reach (in terms of a and b)?
   (b) Sketch the slope field corresponding to this equation.
   (c) What is the highest rate that the population can reach (in terms of a and b)?
   (d) What happens in the long term if the population starts at $P(0) = 2b/a$? How about $P(0) = b/2a$?

3. When an object is removed from a furnace and placed in an environment with a constant temperature of 80F, its temperature is 1500F. One hour after it is removed, the temperature is 1120F.

   (a) What is the temperature 5 hours after the object is removed from the furnace?
   (b) How long will it take the object to get to 120F, when it can be picked up by hand?
   (c) How long will it take the object to reach 80F?

Chapter 12

1. Find an equation for a sphere if one of its diameters has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.

2. Find the equation of the largest sphere with center $(5, 4, 9)$ contained in the first octant.

3. Find the equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

4. A cube is located such that its top four corners have the coordinates $(-1,-2,2)$, $(-1,3,2)$, $(4,-2,2)$, and $(4,3,2)$. Give the coordinate of the center of the cube.

5. Evaluate the following integral where R is the region above the x-axis within $x^2+y^2=9$:

$$\int_R \cos(x^2+y^2)dA$$

.

Chapter 14 – Partial derivatives

1. On your Calc II final, you were asked to calculate the partial derivatives for a function f that is too crazy to remember. When you get back to your dorm and check your answers with your room-mates, they tell you that they got

   $g(x,y)=\dfrac{\partial f}{\partial x}(x,y)=ye^{x^2+y^2}$ and $h(x,y)=\dfrac{\partial f}{\partial y}(x,y)=xe^{x^2+y^2}$

   You show them your result (which is different), and tell them that they are all wrong. Going through the following steps, show them why they are wrong:

   (a) Calculate $\dfrac{\partial g}{\partial y}(x,y)$ for their function $g(x,y)$.

   (b) Calculate $\dfrac{\partial h}{\partial x}(x,y)$ for their function $h(x,y)$.

   (c) Using the answers from (a) and (b), what would you say to convince them that their result can't be right?

2. The function T below describes the temperature evolution along the course of 24 hours (starting and ending at midnight) at each point on the water surface in the harbor: $T(x,y,t)=ye^{-x^2\sin(\pi t/24)}$ ($-200<x<200$ is the span of the harbor along the beach, and $y>0$ is the distance from the shore into the water, both in meters).

   (a) At the points $(x,20)$ (20m along the shore line), is the temperature raising or dropping at mid-day ($t=12$)?

   (b) If a person swims at mid-day in a straight line,parallel to the shore, 20m from the shore. Where will he feel that the water is getting colder, and where will he feel it getting warmer? (Assume that he is swimming fast enough that the time can be considered insignificant for the temperature change).

3. Find $f_x(1,0)$ for $f(x,y)=\dfrac{xe^{\sin(x^2y)}}{(x^2+y^2)^{3/2}}$.

New material

Section 16.2 – Double integrals

1. Evaluate the double integral (using the most convenient method):

$$\int_0^1\int_{e^y}^{e}\frac{x}{\ln x}\,dx\,dy$$

$$\int_0^1\int_x^1\frac{y}{1+y^4}dydx$$

2. The following sum of double integrals describes the mass of a thin plate $\mathcal{R}$ in the xy-plane, of density $\delta(x,y) = x + y$:

$$\text{mass} = \int_{-4}^{0} \int_{0}^{2x+8} \delta(x,y)\ dy\ dx + \int_{0}^{4} \int_{0}^{-2x+8} \delta(x,y)\ dy\ dx$$

(a) Describe the thin plate (shape, intersections with the coordinate axes).

(b) Write an expression for the mass of the plate as only one double integral.

(c) Calculate the area and the mass of the plate.

(d) Calculate the average density of the plate.

3. Consider the solid region $\mathcal{W}$ situated above the region $0 \leq x \leq 2$, $0 \leq y \leq x$, and bounded above by the surface $z = e^{x^2}$.

(a) Write an integral that evaluates the area of each cross-section of $\mathcal{W}$ with vertical planes $y = a$, where a is a constant in $[0, 2]$. Can you evaluate this integral (as an expression depending on a)?

(b) Write an integral that evaluates the area of each cross-section of $\mathcal{W}$ with vertical planes $x = b$, where b is a constant in $[0, 2]$. Can you evaluate this integral (as an expression depending on b)?

(c) Write a convenient double integral that evaluates the volume of the region $\mathcal{W}$, and evaluate it.

Section 16.4 – Double integrals in polar coordinates

1. **Polar Integration: I need someone to generate the graph here.** The plot below depicts the curve whose equation in polar coordinates is

$$r = 2 - cos(\theta) :$$

(a) Write an iterated double integral in polar coordinates whose numerical value equals the area enclosed by the curve.

(b) Evaluate your answer to part (a).

2. Evaluate the following integral where R is the region above the x-axis within $x^2 + y^2 = 9$:

$$\int_R \cos(x^2 + y^2) dA$$

.

3. Evaluate the following integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2} dydx$$

.

4. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\; dydx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\; dydx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\; dydx$$

into **one** double integral, then evaluate the double integral.

5. Find the volume of an ice cream cone bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

Section 16.3 – Triple integrals

1. Consider the solid between the planes $z = 1 + x + y$ and $x + y + z = 1$ and above the triangle $x + y \leq 1$, $x \geq 0$, $y \geq 0$ in the xy-plane.

   (a) Set up (without evaluating it) a triple integral that would calculate the mass of this solid, if the density at each point (x, y, z) is given by $\delta(x, y, z)$.

   (b) Calculate the volume of the solid.

2. **Note:** This might work as a combination of 12.1 and 16.3. Part (d) can be included in the review sheet, but would be too long for the actual test.

   Consider the plane $\mathcal{H}$ defined by $\frac{x}{3} + \frac{y}{2} + \frac{z}{6} = 1$.

   (a) What are the intersections of $\mathcal{H}$ with each of the three coordinate planes xy, yz and xz? Label each of them appropriately.

   (b) What are the intersections of $\mathcal{H}$ with each of the three coordinate axes? Label each of them appropriately.

   (c) Consider the solid region $\mathcal{W}$ bounded by the three coordinate planes and the plane $calH$. Set up (without calculating) a triple integral that corresponds to the mass of $\mathcal{W}$, if the density at each of its points is given by the function $\delta(x, y, z) = x + y$.

   (d) Calculate the integral in part (c).

3. The function f describes the density of birds around DIA, so that $w = f(x, y, z)$ represents the number of birds/m^3 at the point of coordinates (x, y, z) (the airport is situated at the origin). Write down a formula that would calculate the average bird density in half-spherical region of radius R around DIA.

Section 16.7 – Change of variables

1. If R is the triangle bounded by $x + y = 1,\ x = 0,$ and $y = 0$, evaluate the integral

$$\int\int_R \cos\left(\frac{x-y}{x+y}\right) dxdy$$

.

2. Find positive numbers a and b so that the change of variables $s = ax$, $t = by$ transforms the integral $\int\int_R dxdy$ into

$$\int\int_T \left|\frac{\partial(x,y)}{\partial(s,t)}\right| dsdt,$$

for R the rectangle $0 \leq x \leq 10, 0 \leq y \leq 1$ and T is the square $0 \leq s, t \leq 1$.

3. Use the change of variables $x = 2s + t$, $y = s - t$ to compute the integral $\int_R (x + y)\, dA$, where R is the parallelogram formed by $(0, 0)$, $(3, -3)$, $(5, -2)$, and $(2, 1)$.

4. Find a number a so that the change of variables $s = x + ay$, $t = y$ transforms the integral $\int \int_R dx\, dy$, where R is the parallelogram that has vertices $(0, 0)$, $(19, 0)$, $(-63, 7)$ and $(-44, 7)$ in the xy-place, into an integral

$$\int \int_T \left| \frac{\partial(x, y)}{\partial(s, t)} \right| ds\, dt$$

over the rectangle T in the st-plane.

3 Review Problems from Hughes-Hallett

3.1 Integration Techniques

1. Section 7.1 (Substitution) Exercises 11, 15, 59, 69, 79, 109
2. Section 7.2 (Integration by Parts) Exercises 13, 15, 31, 49, 59
3. Section 7.4 (Partial Fractions) Exercises 19, 25a, 37
4. Section 7.4 (Trigonometric Substitution) Exercises 23, 55, 59
5. Section 7.7 (Improper Integrals) Exercises 9, 15, 39, 49
6. Section 7.8 (Comparison of Improper Integrals) Exercises 7, 17, 29

3.2 Applications of Integration

1. Section 8.1 (Areas and Volumes) Exercises 3, 11, 25
2. Section 8.2 (Volumes by Revolution & Cross Sections) Exercises 9, 23, 35
3. Section 8.3 (Arc Length and Parametric Curves) Exercises 11, 15, 17, 19

3.3 Sequences and Series

1. Section 9.1 (Sequences) Exercises 3, 5, 11, 17, 19, 25, 41, 55
2. Section 9.2 (Geometric Series) Exercises 11, 15, 21, 23
3. Section 9.3 (Convergence of Series) Exercises 5, 11, 13, 21
4. Section 9.4 (Tests for Convergence) Exercises 3, 5, 11, 15, 25, 29

3.4 Taylor Polynomials

1. Section 10.1 (Taylor Polynomials) Exercises 1, 9, 29
2. Section 10.2 (Taylor Series) Exercises 15, 17, 23, 29, 43
3. Section 10.3 (New Taylor Series from Old) Exercises 3, 29

3.5 Differential Equations

1. Section 11.1 (What is a Differential Equation?) Exercises 3, 7, 17
2. Section 11.2 (Slope Fields) Exercises 5, 7
3. Section 11.4 (Separation of Variables) Exercises 3, 9, 13, 21, 41
4. Section 11.5 (Applications) Exercises 3, 9, 15