WORKSHEET: Integration of trigonometric functions **MATH 2300**

Evaluate the following integrals:

1. $\int \sin^3 x\, dx$ (Hint: Use the identity $\sin^2 x + \cos^2 x = 1$. Then make a substitution.)

Solution

$$\begin{aligned}\int \sin^3 x\, dx &= \int \sin^2 x \sin x\, dx = \int (1-\cos^2 x)\sin x\, dx \\ &= \int \sin x\, dx - \int \cos^2 x \sin x\, dx = -\cos x + \int w^2\, dw \\ &= -\cos x + \frac{1}{3}\cos^3 x + C\end{aligned}$$

(we let $w = \cos x$ in the second integral).

2. $\int \sin^5 x \cos^2 x\, dx$ (Hint: Write $\sin^5 x$ as $(\sin^2 x)^2 \sin x$.)

Solution

$$\begin{aligned}\int \sin^5 x\cos^2 x\, dx &= \int (\sin^2 x)^2 \sin x \cos^2 x\, dx = \int (1-\cos^2 x)^2 \cos^2 x \sin x\, dx \\ &= -\int (1-w^2)^2 w^2\, dw = -\int (1-2w^2+w^4)w^2\, dw \\ &= -\int w^2\, dw + 2\int w^4\, dw - \int w^6\, dw = -\frac{w^3}{3} + \frac{2w^5}{5} - \frac{w^7}{7} + C \\ &= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} + -\frac{\cos^7 x}{7} + C\end{aligned}$$

(we let $w = \cos x$).

3. $\int \sin^7 x \cos^5 x\, dx$

Solution

$$\begin{aligned}\int \sin^7 x\cos^5 x\, dx &= \int (\sin^2 x)^3 \sin x \cos^5 x\, dx = \int (1-\cos^2 x)^3 \cos^5 x \sin x\, dx \\ &= -\int (1-w^2)^3 w^5\, dw = -\int (1-3w^2+3w^4-w^6)w^5\, dw \\ &= -\int w^5\, dw + 3\int w^7\, dw - 3\int w^9\, dw + \int w^{11}\, dw = -\frac{w^6}{6} + \frac{3w^8}{8} - \frac{3w^{10}}{10} + \frac{w^{12}}{12} + C \\ &= -\frac{\cos^6 x}{6} + \frac{3\cos^8 x}{8} - \frac{3\cos^{10} x}{10} + \frac{\cos^{12} x}{12} + C\end{aligned}$$

(again we let $w = \cos x$).

4. In general, how would you go about trying to find $\int \sin^m x \cos^n x \, dx$, where m is odd? (Hint: consider the previous three problems.)

Solution If m is odd, we can write $m = 2k+1$ where k is an integer. Then

$$\begin{aligned}\int \sin^m x \cos^n x \, dx &= \int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx \\ &= \int (1-\cos^2 x)^k \cos^n x \sin x \, dx = -\int (1-w^2)^k w^n \, dw\end{aligned}$$

(at the last step, we put $w = \cos x$). Now multiply everything out and integrate. Finally, plug back in $w = \cos x$ to get your answer in terms of x.

5. Find $\int \sin^2 x \, dx$, in each of the following two ways:

(a) Use the identity $\sin^2 x = \frac{1}{2}(1-\cos(2x))$.

Solution

$$\int \sin^2 x \, dx = \frac{1}{2}\int (1-\cos(2x)) \, dx = \frac{1}{2}\Big(x - \frac{\sin(2x)}{2}\Big) + C = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

(b) Integrate by parts, with $u = \sin x$ and $dv = \sin x \, dx$.

Solution

$$\begin{aligned}\int \sin^2 x \, dx &= \sin x(-\cos x) - \int (-\cos x)\cos x \, dx = -\sin x \cos x + \int \cos^2 x \, dx \\ &= -\sin x \cos x + \int \cos^2 x \, dx = -\sin x \cos x + \int (1-\sin^2 x) \, dx \\ &= -\sin x \cos x + \int dx - \int \sin^2 x \, dx = -\sin x \cos x + x - \int \sin^2 x \, dx.\end{aligned}$$

If we add $\int \sin^2 x \, dx$ to both sides, we get

$$2\int \sin^2 x \, dx = -\sin x \cos x + x,$$

or

$$\int \sin^2 x \, dx = -\frac{\sin x \cos x}{2} + \frac{x}{2} + C,$$

(c) Show that your answers to parts (a) and (b) above are the same. Hint: $\sin(2x) = 2\cos x \sin x$.

Solution The hint gives the answer immediately.

6. Do the integral in problem (1), above, again, but this time by parts, using $u = \sin^2 x$ and $dv = \sin x\, dx$. (After this, you'll probably need to do a substitution.) **Solution** We have $du = 2 \sin x \cos x\, dx$ and $v = -\cos x$, so

$$\int \sin^3 x\, dx = \sin^2 x(-\cos x) + 2 \int \cos^2 x \sin x\, dx = -\cos x \sin^2 x - 2 \int u^2\, du$$
$$= -\cos x \sin^2 x - 2\frac{u^3}{3} + C = -\cos x \sin^2 x - \frac{2\cos^3 x}{3} + C$$

(we used the substitution $u = \cos x$).

7. EXTRA CREDIT: Can you show your answers to problems (1) and (6) above are the same? **Solution** We subtract the answer to problem 6 from the answer to problem 1:

$$-\cos x + \frac{1}{3}\cos^3 x - \left(-\cos x \sin^2 x - \frac{2\cos^3 x}{3}\right) = \cos^3 x - \cos x + \cos x \sin^2 x$$
$$= \cos^3 x - \cos x(1 - \sin^2 x) = \cos^3 x - \cos x(\cos^2 x) = \cos^3 x - \cos^3 x = 0,$$

which shows that the two answers are the same.