Note: This review worksheet does not cover *all* material that you are expected to know.

1. An object is dropped from a 600 foot tower. (The acceleration due to gravity is 32 ft/s^2.)

   (a) When does it hit the ground? **Solution:** The height $s(t)$, in feet, of the object above the ground after t seconds is

   $$s(t) = -16t^2 + 600.$$

   The object hits the ground when $s(t) = 0$, meaning

   $$0 = -16t^2 + 600.$$

   Solving for t gives $t = \pm\sqrt{600/16} \approx \pm 6.12372$. We discard the negative answer because it makes no sense. So: the ball hits the ground after about 6.12372 seconds.

   (b) How fast is it going on impact? **Solution:** We have $s'(t) = 32t$, so $s'(6.12372) = 195.96$. The ball is moving at about 195.96 ft/sec when it hits the ground.

2. Find the average value of the function $f(x) = \sqrt{x}$ from 0 to 4. **Solution:** The average value is

   $$\frac{1}{4-0}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\cdot\frac{2}{3}x^{3/2}\Big|_0^4 = \frac{4^{3/2}}{6} = \frac{8}{6} = \frac{4}{3}.$$

3. Find $\dfrac{dy}{dx}$ if $\cos(2y) = xy$.

Solution:

$$\begin{aligned}\frac{d}{dx}[\cos(2y)] &= \frac{d}{dx}[xy]\\ 2\frac{dy}{dx}\cdot(-\sin(2y)) &= x\frac{dy}{dx} + y\\ (-2\sin(2y) - x)\frac{dy}{dx} &= y\\ \frac{dy}{dx} &= -\frac{y}{2\sin(2y) - x}.\end{aligned}$$

4. Let $f(x) = 3 + \cos(2x + \frac{\pi}{2})$. Find the local linearization of $f(x)$ near the point $x = 0$, and use your linearization to find an approximation of $f(0.4)$.

Solution: We have $f'(x) = -2\sin(2x + \frac{\pi}{2})$. So the local linearization of $f(x)$ near $x = 0$ is

$$f(0) + f'(0)(x - 0) = 3 + \cos(2\cdot 0 + \frac{\pi}{2}) + [-2\sin(2\cdot 0 + \frac{\pi}{2})]x = 3 - 2x.$$

so

$$f(0.4) \approx 3 - 2(0.4) = 2.2.$$

5. Let $f(x) = (x-1)^2 e^{-x}$.

   (a) State the domain of $f(x)$. **Solution:** The domain of $f(x)$ consists of all real numbers

   (b) What is the behavior of $f(x)$ as $x \to +\infty$ and $x \to -\infty$? **Solution:** Since both $(x-1)^2$ and e^{-x} go to $+\infty$ as $x \to -\infty$, we have

$$\lim_{x\to-\infty} (x-1)^2 e^{-x} = +\infty.$$

   On the other hand, since $\lim_{x\to+\infty}(x-1)^2 = +\infty$ and $\lim_{x\to+\infty} e^x = +\infty$, we can apply l'Hôpital's rule as follows:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x-1)^2}{e^x} = \lim_{x\to+\infty} \frac{2(x-1)}{e^x} = \lim_{x\to+\infty} \frac{2}{e^x} = 0$$

   (which is to say: e^{-x} *decays*, as $x \to +\infty$, more quickly than $(x-1)^2$ *grows* as $x \to \infty$.

   (c) Find the intervals of increase and decrease of $f(x)$, and all local maxima and minima of $f(x)$. **Solution:**

$$\begin{aligned} f'(x) &= (x-1)^2(-e^{-x}) + 2(x-1)e^{-x} = e^{-x}[-(x-1)^2 + 2(x-1)] \\ &= e^{-x}[-x^2+4x-3] = -e^{-x}[(x-1)(x-3)]. \end{aligned}$$

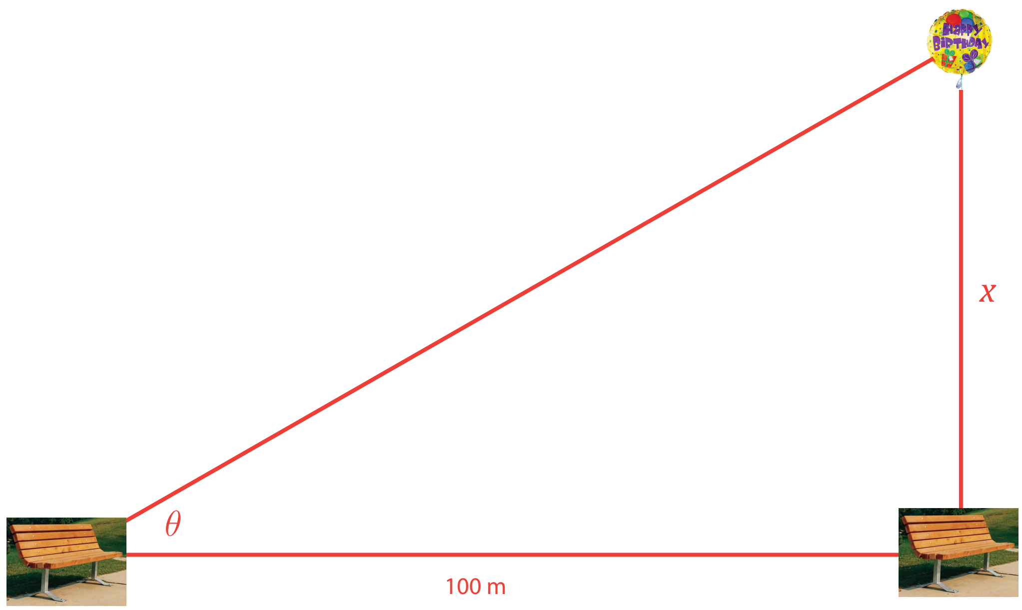
   Since e^{-x} is never zero, we find that $f'(x) = 0$ when $x = 1$ or $x = 3$. So there are three intervals to consider: $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$. We check that $f'(0)$ is negative, $f'(2)$ is positive, and $f'(4)$ is negative. Conclusion: $x = 1$ is a local minimum and $x = 3$ a local maximum; f is increasing on $(1, 3)$ and decreasing on $(-\infty, 1)$ and $(3, \infty)$.

   (d) Find the inflection points, and the intervals where the function is concave up/down. Show your work clearly. **Solution:**

$$f''(x) = \frac{d}{dx}(e^{-x}[-x^2+4x-3]) = e^{-x}\left(x^2 - 6x + 7\right).$$

   Since e^{-x} is never zero, we find that $f''(x) = 0$ when $x^2 - 6x + 7 = 0$ or, by the quadratic formula, $x = 3 \pm \sqrt{2}$. So there are three intervals to consider: $(-\infty, 3-\sqrt{2})$, $(3-\sqrt{2}, 3+\sqrt{2})$, and $(3+\sqrt{2}, \infty)$. We check that $f''(0)$ is positive, $f''(3)$ is negative, and $f''(6)$ is positive. Conclusion: $x = 3-\sqrt{2}$ and $x = 3+\sqrt{2}$ are inflection points; f is concave down on $(3-\sqrt{2}, 3+\sqrt{2})$ and concave up on $(-\infty, 3-\sqrt{2})$ and $(3+\sqrt{2}, \infty)$.

6. A person is sitting on a park bench and watching a balloon, which lifts off from another park bench 100 m (horizontally) away from him, rise up in the air. The balloon is rising up at a constant speed of 5 m/s. The person tilts his head upwards in order to keep the balloon in sight. How fast is the person tilting his head when the balloon is at the height of 50 m (above the take-off point)?



Solution: We have

$$\tan(\theta) = \frac{x}{100},$$

so

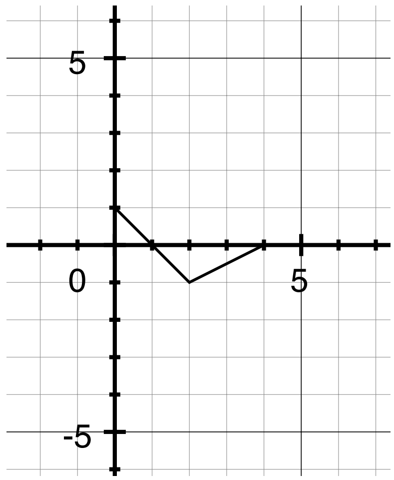
$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{100}\frac{dx}{dt} = \frac{5}{100}$$

since $dx/dt = 5$. When the balloon is 50 m above the ground, we have $\tan(\theta) = 50/100 = 1/2$, so $\sec^2(\theta) = 1 + \tan^2\theta = 1 + (1/2)^2 = 1.5$, so

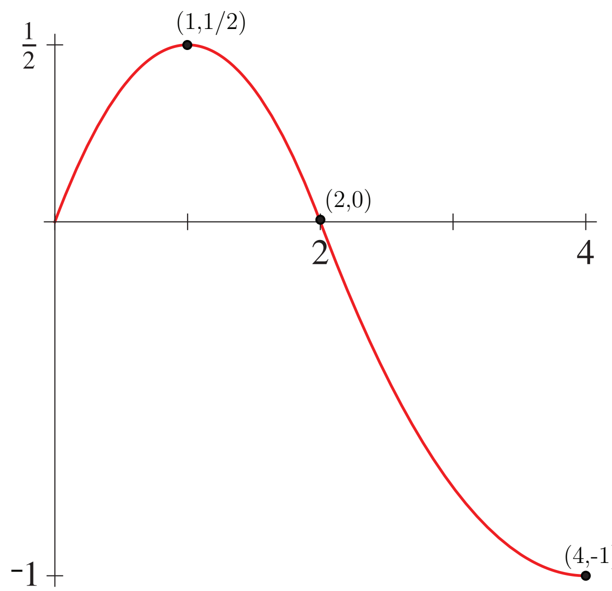
$$\frac{d\theta}{dt} = \frac{5}{100\sec^2(\theta)} = \frac{20}{300} = \frac{1}{15}.$$

That is: when the balloon is 50 m off the ground, the person on the bench is tilting his head upwards at 1/25 radians/sec.

7. The graph of $f'(x)$ is given below.



a) Sketch the graph of $f(x)$, from $x = 0$ to $x = 4$, assuming that $f(0) = 1$.
b) Clearly label the x and y-coordinates of any relative maxima, relative minima, and inflection points of $f(x)$.



Solution:

global maximum at $(1, 1/2)$, global minimum at $(4, -1)$, inflection point at $(2, 0)$