MATH 2300 **SOLUTIONS** **Worksheet**

Local Linearity and a preview of partial derivatives

In Chapter 14, we are going to study the partial derivatives with respect to x and y of a function $z = f(x, y)$. These partial derivatives are denoted $f_x(x, y)$ and $f_y(x, y)$ respectively. The idea is simple: $f_x(x, y)$ is just what you get by treating y like a *constant*, and differentiating $f(x, y)$ with respect to x. Similarly, $f_y(x, y)$ is what you get by treating x like a constant, and differentiating $f(x, y)$ with respect to y.

1. Consider the plane
$$z = f(x, y) = c + mx + ny.$$
Evaluate $f(0, 0)$, $f_x(0, 0)$, and $f_y(0, 0)$. [Note. $f_x(0, 0)$ means: compute the partial derivative $f_x(x, y)$ in general, as described above; then plug in $(x, y) = (0, 0)$. Similarly for $f_y(0, 0)$] What does these numbers represent geometrically?
$$f(0, 0) = c; \quad f_x(0, 0) = m, \quad f_y(0, 0) = n.$$
The number $f(0, 0)$ tells us where the plane hits the z-axis; the number $f_x(0, 0)$ tells us the "tilt" of the plane in the x direction; the number $f_y(0, 0)$ tells us its "tilt" in the y direction.

2. Using the previous problem, write the same plane in terms of $f(0, 0)$, $f_x(0, 0)$, and $f_y(0, 0)$.
$$z = f(x, y) = f(0, 0) + f_x(0, 0)\,x + f_y(0, 0)\,y$$

3. Let's talk about a specific plane, say $z = 5 + 2x - 3y$. OK, let's. How about: "Wow, check out that specific plane $z = 5 + 2x - 3y$! Nice shoes!" Show that $(1, 3, -2)$ lies on this plane. We note that
$$-2 = 5 + 2(1) - 3(3),$$
so $(x, y, z) = (1, 3, -2)$ satisfies $z = 5 + 2x - 3y$, so $(1, 3, -2)$ does lie on this plane.

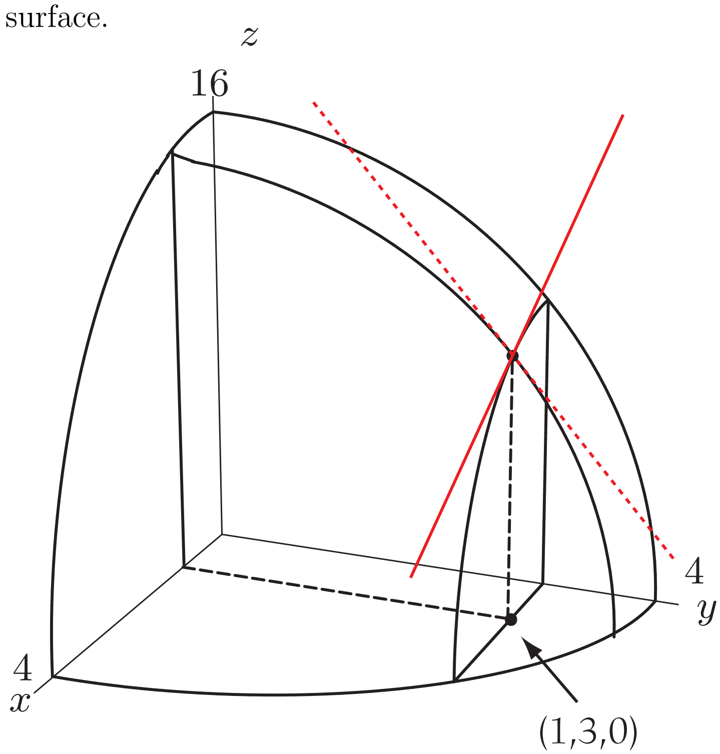
4. Show that the plane $z = 5 + 2x - 3y$ from problem 3 above may be rewritten in the form $z = -2 + 2(x - 1) - 3(y - 3)$.
$$-2 + 2(x - 1) - 3(y - 3) = -2 + 2x - 2 - 3y + 9 = 5 + 2x - 3y.$$

5. Evaluate $f(1, 3)$, $f_x(1, 3)$, and $f_y(1, 3)$, for $z = f(x, y)$ the plane of the previous two problems. These numbers should look familiar. Why did this happen?
$$f(1, 3) = -2; \quad f_x(1, 3) = 2, \quad f_y(1, 3) = -3.$$
If you write a plane $x = f(x, y)$ in the form
$$z = c + m_1(x - x_0) + m_2(y - y_0),$$
then $c = f(x_0, y_0)$, m_1 is the slope of the plane in the x direction $(= f_x(x, y))$, and m_2 is the slope of the plane in the y direction $(= f_y(x, y))$.

6. Given a point $(a, b, f(a, b))$ on the plane $z = 5+2x-3y$ of the previous three problems, how can we write the equation for this plane in terms of $a, b, f(a, b), f_x(a, b)$, and $f_y(a, b)$?

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

7. Now things should get a little more interesting. Consider the surface obtained by graphing $z = f(x, y) = 16 - x^2 - y^2$. (See the figure below.) Verify that $(1, 3, 6)$ lies on this surface.



Graph of $z = 16 \quad x^2 \quad y^2$.

$$6 = 16 - 1^2 - 3^2,$$

so $(x, y, z) = (1, 3, 6)$ does lie on the surface in question.

8. By drawing on and labeling the graph above (and supplying some words, if appropriate, below), explain the geometric meaning of $f_x(1, 3)$ and $f_y(1, 3)$.

$f_x(1, 3)$ is the slope of the line that's parallel to the yz plane, and is tangent to the graph of $z = f(x, y)$ at $(x, y) = (1, 3)$. (See the red, dashed line above.)

Similarly, $f_y(1, 3)$ is the slope of the line that's parallel to the xz plane, and is tangent to the graph of $z = f(x, y)$ at $(x, y) = (1, 3)$. (See the red, solid line above.)

9. Write the equation of the plane passing through $(1, 3, 6)$ with the same partial derivatives as in the previous problem.

By problem 6 above, the equation of this plane is

$$z = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3) = 6 - 2(x-1) - 6(y-3).$$

(We have $f_x(x,y) = \frac{\partial}{\partial x}(16 - x^2 - y^2) = -2x$ and $f_y(x,y) = \frac{\partial}{\partial y}(16 - x^2 - y^2) = -2y$, so $f_x(1,3) = -2$ and $f_y(1,3) = -6$.)

10. What is the relationship between this plane and the tangent lines you have already drawn?

    It's the (unique) plane that contains both of these lines.

11. What is the relationship between this plane and the surface $z = 16 - x^2 - y^2$?

    This plane is the unique plane that's tangent to the graph of $z = 16 - x^2 - y^2$ at the point $(1,3,6)$.

12. Use this plane to approximate $f(1.1, 3.1)$. Compare this approximation with the actual value of $f(1.1, 3.1)$. Now use this plane to approximate $f(-1.1, -3.1)$, and compare this approximation to the actual value of $f(-1.1, -3.1)$.

    Since the plane "approximates" the surface $f(x,y)$, we have

    $$f(x,y) \approx 6 - 2(x-1) - 6(y-3).$$

    So

    $$f(1.1, 3.1) \approx 6 - 2(1.1-1) - 6(3.1-3) = 6 - 0.2 - 0.6 = 5.2$$

    and

    $$f(-1.1, -3.1) \approx 6 - 2(-1.1-1) - 6(-3.1-3) = 6 + 4.2 + 36.6 = 28.5.$$

    Compare these with the actual values

    $$f(1.1, 3.1) = 16 - (1.1)^2 - (3.1)^2 = 5.18$$

    and

    $$f(-1.1, -3.1) = 16 - (-1.1)^2 - (-3.1)^2 = 5.18$$

    We see that the tangent plane approximates the surface f much better at $(x,y) = (1.1, 3.1)$ than it does at $(x,y) = (-1.1, -3.1)$. It's pretty clear from the above picture why this is happening: $(1.1, 3.1, f(1.1, 3.1))$ is much closer to the point of tangency than is $(-1.1, -3.1, f(-1.1, -3.1))$. Since the surface f and the given tangent plane meet at $(1,3,6)$, they will approximate each other pretty well near than point, but they "bend away" from each other as we get far from that point.

13. This whole process should remind you of using tangent lines to approximate functions. Eventually you learned how to improve upon the approximations obtained from using a tangent line. How might we be able to improve upon our approximations using tangent planes?

For functions $f(x)$ of a single variable, we got better approximations by taking higher order Taylor polynomials. That is, we augmented the the tangent line $y = f(a) + f'(a)(x - a)$ with terms involving x^2, x^3, and so on. Maybe in the present situation, we could do a similar thing: add terms involving x^2, xy, y^2, x^3, x^2y, xy^2, y^3, and so on to the tangent plane approximation, in such a way that we get an approximating surface that conforms to the original surface even better than the tangent plane does.