Math 2300 **Integration Worksheet** Spring 2011

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**1.** (Use of tables.) Use formula 21, part IV of your integral tables to show that

$$\int (2\sec^3\theta - \sec\theta)\,d\theta = \sec\theta\tan\theta + C.$$

Some hints: (a) Consider $m = 3$. (b) Recall that $\sec\theta = \dfrac{1}{\cos\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

Using formula 21 in the table of integrals we obtain:

$$\begin{aligned}\int (2\sec^3\theta - \sec\theta)\,d\theta &= 2\int \frac{1}{\cos^3\theta}d\theta - \int \frac{1}{\cos\theta}d\theta \\ &= 2\left[\left(\frac{1}{2}\right)\frac{\sin\theta}{\cos^2\theta} + \frac{1}{2}\int \frac{1}{\cos\theta}d\theta\right] - \int \frac{1}{\cos\theta}d\theta \\ &= \frac{\sin\theta}{\cos^2\theta} + C \\ &= \sec\theta\tan\theta + C.\end{aligned}$$

**2.** (Trig substitutions.) Show that

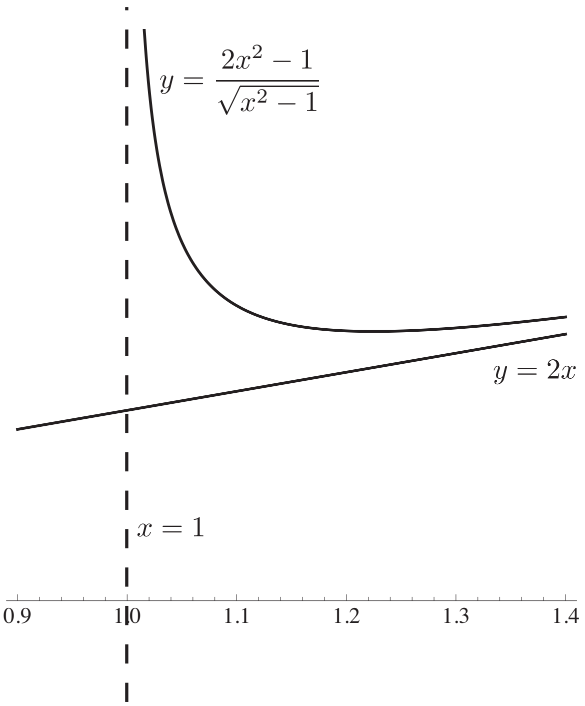
$$\int \frac{2x^2 - 1}{\sqrt{x^2 - 1}}\,dx = x\sqrt{x^2 - 1} + C.$$

Hint: use the trig substitution $x = \sec\theta$; simplify using the fact that $\sec^2\theta - 1 = \tan^2\theta$, and then use problem **1**.

If $x = \sec\theta$, then $dx = \frac{\sin\theta}{\cos^2\theta}d\theta$, so

$$\begin{aligned}\int \frac{2x^2 - 1}{\sqrt{x^2 - 1}}dx &= \int \left(\frac{2\sec^2\theta - 1}{\sqrt{\sec^2\theta - 1}}\right)\left(\frac{\sin\theta}{\cos^2\theta}\right)d\theta \\ &= \int \left(\frac{\frac{2}{\cos^2\theta} - 1}{\sqrt{\tan^2\theta}}\right)\left(\frac{\sin\theta}{\cos^2\theta}\right)d\theta \\ &= \int \left(\frac{2 - \cos^2\theta}{\cos^2\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{\sin\theta}{\cos^2\theta}\right)d\theta \\ &= \int \frac{2 - \cos^2\theta}{\cos^3\theta}d\theta \\ &= \int (2\sec^3\theta - \sec\theta)d\theta \\ &= \sec\theta\tan\theta + C \\ &= x\sqrt{x^2 - 1} + C.\end{aligned}$$

**3.** (Improper integrals.) Consider the unbounded region between the line $x = 1$, the line $y = 2x$, and the curve $y = \dfrac{2x^2 - 1}{\sqrt{x^2 - 1}}$. See the picture below. (Note that, for purposes of clarity, the vertical axis has been left out.)



Explain why the area of this region is given by an *improper integral.* Carefully express this improper integral as a limit *one or more* proper integrals.

The area of this region is given by an improper integral because near $x = 1$, the curve $y = \frac{2x^2-1}{\sqrt{x^2-1}}$ is unbounded and because the upper limit of integration is infinite. We must address both of these cases when we write the solution.

The area of this region is given by:

$$\begin{aligned}
\int_1^\infty \frac{2x^2 - 1}{\sqrt{x^2 - 1}} dx - \int_1^\infty 2x\, dx &= \lim_{b \to 1^+} \int_b^\infty \frac{2x^2 - 1}{\sqrt{x^2 - 1}} dx - \int_1^\infty 2x\, dx \\
&= \lim_{b \to 1^+} \int_b^2 \frac{2x^2 - 1}{\sqrt{x^2 - 1}} dx + \int_2^\infty \frac{2x^2 - 1}{\sqrt{x^2 - 1}} dx - \int_1^\infty 2x\, dx \\
&= \lim_{b \to 1^+} \int_b^2 \frac{2x^2 - 1}{\sqrt{x^2 - 1}} dx + \lim_{c \to \infty} \int_2^c \frac{2x^2 - 1}{\sqrt{x^2 - 1}} dx - \lim_{d \to \infty} \int_1^d 2x\, dx.
\end{aligned}$$

**4.** (More on improper integrals; l'Hôpital's rule.) Using your answers to problems **2** and **3** above, show that the above unbounded region has area equal to $\frac{1}{2}$.

Continuing where we left off in problem 3, we see that

$$\begin{aligned}
\cdots &= \lim_{b\to 1^+}\int_b^2 \frac{2x^2-1}{\sqrt{x^2-1}}dx + \lim_{c\to\infty}\int_2^c \frac{2x^2-1}{\sqrt{x^2-1}}dx - \lim_{d\to\infty}\int_1^d 2x dx \\
&= \lim_{b\to 1^+}\left(x\sqrt{x^2-1}\right)\Big|_b^2 + \lim_{c\to\infty}\left(x\sqrt{x^2-1}\right)\Big|_2^c - \lim_{d\to\infty} x^2\Big|_1^d \\
&= 2\sqrt{3} - \lim_{b\to 1^+}\left(b\sqrt{b^2-1}\right) + \lim_{c\to\infty}\left(c\sqrt{c^2-1}\right) - 2\sqrt{3} - \lim_{d\to\infty} d^2 + 1 \\
&= \lim_{a\to\infty}\left(a\sqrt{a^2-1} - a^2\right) + 1.
\end{aligned}$$

We're now left with a limit of the form "$\infty - \infty$". We can rewrite it and apply L'Hopital's rule. To rewrite it, consider the following.

$$\begin{aligned}
a\sqrt{a^2-1} - a^2 &= \left(a\sqrt{a^2-1} - a^2\right)\left(\frac{a\sqrt{a^2-1}+a^2}{a\sqrt{a^2-1}+a^2}\right) \\
&= \frac{a^2(a^2-1) - a^4}{a\sqrt{a^2-1}+a^2} \\
&= \frac{-a^2}{a\sqrt{a^2-1}+a^2} \\
&= \frac{-a}{\sqrt{a^2-1}+a}.
\end{aligned}$$

Now we have a limit in the form "$\frac{\infty}{\infty}$" so we can apply L'Hopital's rule by finding the derivatives with respect to $a$ of the numerator and denominator of the fraction. Thus we obtain

$$\begin{aligned}
\lim_{a\to\infty}\left(a\sqrt{a^2-1} - a^2\right) - 1 &= \lim_{a\to\infty}\left(\frac{-a}{\sqrt{a^2-1}+a}\right) + 1 \\
&= \lim_{a\to\infty}\left(\frac{-1}{a(a^2-1)^{-1/2}+1}\right) + 1.
\end{aligned}$$

Lastly observe that

$$\begin{aligned}
\lim_{a\to\infty}\left(\frac{a}{\sqrt{a^2-1}}\right) &= \lim_{a\to\infty}\left(\sqrt{\frac{a^2}{a^2-1}}\right) \\
&= \lim_{a\to\infty}\left(\sqrt{\frac{1}{1-\frac{1}{a^2}}}\right) \\
&= 1.
\end{aligned}$$

Thus

$$\lim_{a\to\infty}\left(a\sqrt{a^2-1}-a^2\right)-1=-\frac{1}{2}+1=\frac{1}{2}.$$