Goal. To use integration by parts to study an interesting "interpolation" of the factorial function $n!$.

For this tutorial project, we are going to investigate properties of the integral

$$I(p) = \int_0^1 (-\ln x)^p \, dx.$$

Note that $I(p)$ defines a *function* of p: different values of p will yield different values of the integral $I(p)$.

1. Let's begin by looking at $I(1)$:

$$I(1) = \int_0^1 (-\ln x)^1 \, dx = -\int_0^1 \ln x \, dx.$$

(a) Explain why $I(1)$ is an *improper* integral, and write $I(1)$ as a limit of proper integrals.

Solution. The integral is improper because the integrand $-\ln x$ has a vertical asymptote at $x = 0$ which is between the limits of integration (in fact, it is one of the limits of integration). So when we write $I(1)$ we mean

$$I(1) = \int_0^1 (-\ln x) \, dx = \lim_{a \to 0^+} \int_a^1 (-\ln x) \, dx.$$

(b) Show that $I(1) = 1$. (*Hint.* Integrate by parts by letting $u = \ln x$ and $dv = dx$.)

Solution. Who am I to disregard a hint? If $u = -\ln x$ then $du = -\frac{1}{x} \, dx$, and if $dv = dx$ then $v = x$. Integrating by parts gives the following:

$$\begin{aligned} I(1) = \lim_{a \to 0^+} \int_a^1 (-\ln x) \, dx &= \lim_{a \to 0^+} \left(x(-\ln x)\Big|_a^1 - \int_a^1 x \left(-\frac{1}{x} \right) dx \right) \\ &= \lim_{a \to 0^+} \left((1(-\ln 1) - a(-\ln a)) + \int_a^1 dx \right) \\ &= \lim_{a \to 0^+} \left((a(-\ln a)) + x\big|_a^1 \right) = \lim_{a \to 0^+} \left((a(-\ln a)) + (1 - a) \right). \end{aligned}$$

(I used $\ln 1 = 0$ to knock out a term there.) Surely as a goes to zero, $1 - a$ goes to 1. But what about $a(-\ln a)$? Blithely plugging in zero gives $0 \ln 0 = 0 \cdot \infty$ which will require a more careful study. This is a job for L'Hôpital's rule, which applies after a standard one-over-one-over trick:

$$\begin{aligned} \lim_{a \to 0^+} a(-\ln a) &= \lim_{a \to 0^+} \frac{-\ln a}{1/a} = \frac{\infty}{\infty}, \text{ so L'Hôpital's Rule applies:} \\ &= \lim_{a \to 0^+} \frac{-1/a}{-1/a^2} = \lim_{a \to 0^+} \frac{-1}{a} \cdot \frac{a^2}{-1} = \lim_{a \to 0^+} a = 0. \end{aligned}$$

Putting all the work together shows that $I(1) = 1$ after all.

2. Now let's consider $I(2)$:

$$I(2) = \int_0^1 (-\ln x)^2\, dx.$$

(a) Explain why $I(2)$ is also an *improper* integral, and write $I(2)$ as a limit of proper integrals.

Solution. $I(2)$ has the same problem that $I(1)$ had: the natural logarithm approaches an asymptote as x goes down to zero. (Squaring it isn't going to fix anything!) So when we write $I(2)$ we mean

$$I(2) = \int_0^1 (-\ln x)^2\, dx = \lim_{a\to 0^+} \int_a^1 (-\ln x)^2\, dx.$$

(b) Show that $I(2) = 2$. (*Hint.* Integrate by parts. When you do this, you should end up with (among other things) an integral that looks like $I(1)$. You then should use your answer to problem #1b above.)

Solution. Showing a bit of a lack of creativity we'll pick u and v just like in #1b. Let $u = (-\ln x)^2$, so that $du = 2(-\ln x)\frac{-1}{x}$ by the chain rule. All that's left in the integrand is dx, so we'll let that be dv, i.e., $dv = dx$, so $v = x$. Now we integrate by parts:

$$\begin{aligned} I(2) = \lim_{a\to 0^+} \int_a^1 (-\ln x)^2\, dx &= \lim_{a\to 0^+} \left(x(-\ln x)^2\Big|_a^1 - \int_a^1 x \left(2(-\ln x)\frac{-1}{x} \right) dx \right) \\ &= \lim_{a\to 0^+} \left(\left(1(-\ln 1)^2 - a(-\ln a)^2\right) + 2\int_a^1 (-\ln x)\, dx \right) \\ &= \lim_{a\to 0^+} \left(-a(-\ln a)^2 + 2\int_a^1 (-\ln x)\, dx \right) \end{aligned}$$

That last integral is familiar: $\lim_{a\to 0^+} \int_a^1 (-\ln x)\, dx = \int_0^1 (-\ln x)\, dx = I(1)$.

$$= \lim_{a\to 0^+} -a(-\ln a)^2 + 2I(1) = \lim_{a\to 0^+} -a(-\ln a)^2 + 2.$$

Now we just need to deal with this limit. Just like the integral turned back into something we saw before, this limit will become something we saw before. Plugging in zero gives $0\cdot\infty$ which can be mitigated just like before:

$$\lim_{a\to 0^+} a(-\ln a)^2 = \lim_{a\to 0^+} \frac{(-\ln a)^2}{1/a} = \frac{\infty}{\infty}\text{ , so L'Hôpital applies:}$$

$$= \lim_{a\to 0^+} \frac{2(-\ln a)\frac{-1}{a}}{-1/a^2} = \lim_{a\to 0^+} \frac{-2(-\ln a)}{a}\cdot\frac{a^2}{-1} = \lim_{a\to 0^+} 2a(-\ln a).$$

But we already know that the limit of $a(-\ln a)$ as a goes to zero is 0, from the application of L'Hôpital's rule in the previous problem. All that survived in our calculation of $I(2)$ is 2, so $I(2) = 2$.

3. Perhaps you sense a pattern. Now we'll consider $I(3)$:

$$I(3) = \int_0^1 (-\ln x)^3\,dx.$$

(a) Explain why $I(3)$ is also an *improper* integral, and write $I(3)$ as a limit of proper integrals.

Solution. If squaring $\ln x$ in #2a didn't matter, cubing it won't matter either. Still an asymptote at zero. We write

$$I(3) = \int_0^1 (-\ln x)^3\,dx = \lim_{a\to 0^+}\int_a^1 (-\ln x)^3\,dx.$$

(b) Show that $I(3) = 6$. (*Hint.* Integrate by parts. When you do this, you should end up with (among other things) an integral that looks like $I(2)$. You then should use your answer to problem #2b above.)

This is going to look a lot like the previous one, so I'll essentially copy and paste. Integrate by parts by letting $u = (-\ln x)^3$, so $du = 3(-\ln x)^2\frac{-1}{x}\,dx$. Then let $dv = dx$, so that $v = x$. Now

$$\begin{aligned}
I(3) = \lim_{a\to 0^+}\int_a^1 (-\ln x)^3\,dx &= \lim_{a\to 0^+}\left(x(-\ln x)^3\Big|_a^1 - \int_a^1 x\left(3(-\ln x)^2\frac{-1}{x}\right)\,dx\right)\\
&= \lim_{a\to 0^+}\left(\left(1(-\ln 1)^3 - a(-\ln a)^3\right) + 3\int_a^1 (-\ln x)^2\,dx\right)\\
&= \lim_{a\to 0^+}\left(-a(-\ln a)^3 + 3\int_a^1 (-\ln x)^2\,dx\right)
\end{aligned}$$

That last integral is familiar: $\lim_{a\to 0^+}\int_a^1 (-\ln x)^2\,dx = \int_0^1 (-\ln x)^2\,dx = I(2)$.

$$= \lim_{a\to 0^+} -a(-\ln a)^3 + 3I(2) = \lim_{a\to 0^+} -a(-\ln a)^3 + 6.$$

Now we have to evaluate the limit $-a(-\ln a)^3$. One dose of L'Hôpital's rule will fix this right up:

$$\begin{aligned}
\lim_{a\to 0^+} a(-\ln a)^3 &= \lim_{a\to 0^+}\frac{(-\ln a)^3}{1/a} = \frac{\infty}{\infty}\text{ , so L'Hôpital applies:}\\
&= \lim_{a\to 0^+}\frac{3(-\ln a)^2\frac{-1}{a}}{-1/a^2} = \lim_{a\to 0^+}\frac{-3(-\ln a)^2}{a}\cdot\frac{a^2}{-1} = \lim_{a\to 0^+} 3a(-\ln a)^2.
\end{aligned}$$

Now from above the limit of $a(-\ln a)^2$ is 0, so our work shows that $I(3) = 6$.

4. Let's try to extrapolate from the cases we've seen so far. Using the same kind of techniques as you used in the above three problems, show that, for any positive integer n,

$$I(n+1) = (n+1)I(n).$$

Solution. I'll copy and paste and change 3 to n. First, so we know what we're working with,

$$I(n+1) = \int_0^1 (-\ln x)^{n+1}\,dx = \lim_{a\to 0^+} \int_a^1 (-\ln x)^{n+1}\,dx.$$

Integrate by parts with $u = (-\ln x)^{n+1}$, $du = (n+1)(-\ln x)^n \frac{-1}{x}\,dx$, $dv = dx$, and $v = x$. We get the following:

$$\begin{aligned} I(n+1) &= \lim_{a\to 0^+} \int_a^1 (-\ln x)^{n+1}\,dx \\ &= \lim_{a\to 0^+} \left(x(-\ln x)^{n+1}\Big|_a^1 - \int_a^1 x\left((n+1)(-\ln x)^n \frac{-1}{x}\right)\,dx \right) \\ &= \lim_{a\to 0^+} \left(\left(1(-\ln 1)^{n+1} - a(-\ln a)^{n+1}\right) + (n+1)\int_a^1 (-\ln x)^n\,dx \right) \\ &= \lim_{a\to 0^+} \left(-a(-\ln a)^{n+1} + (n+1)\int_a^1 (-\ln x)^n\,dx \right) \end{aligned}$$

That last integral is what we want: $\lim_{a\to 0^+} \int_a^1 (-\ln x)^n\,dx = \int_0^1 (-\ln x)^n\,dx = I(n)$.

$$= \lim_{a\to 0^+} -a(-\ln a)^{n+1} + (n+1)I(n).$$

I told you I copied and pasted! Now for the limit of $a(-\ln a)^{n+1}$. Like what we saw above, we'll discover that if we start using L'Hôpital's rule, eventually we'll get down to a limit we've seen above.

$$\begin{aligned} \lim_{a\to 0^+} a(-\ln a)^{n+1} &= \lim_{a\to 0^+} \frac{(-\ln a)^{n+1}}{1/a} = \frac{\infty}{\infty}\text{, so L'Hôpital applies:} \\ &= \lim_{a\to 0^+} \frac{(n+1)(-\ln a)^n \frac{-1}{a}}{-1/a^2} \\ &= \lim_{a\to 0^+} \frac{-(n+1)(-\ln a)^n}{a} \cdot \frac{a^2}{-1} = \lim_{a\to 0^+} (n+1)a(-\ln a)^n \\ &= (n+1) \lim_{a\to 0^+} a(-\ln a)^n. \end{aligned}$$

Doing L'Hôpital's rule again will lower the exponent from n to $n-1$, then $n-2$, and so on, until eventually we get down to $a(-\ln a)^1$ whose limit we know is zero. So the limit is zero, and all that remains is $(n+1)I(n)$.

5. So far, we've seen that $I(1) = 1$, $I(2) = 2$, $I(3) = 6$, and $I(n+1) = (n+1)I(n)$ for any positive integer n. Do you see the pattern? That is, can you write down a nice, simple formula for $I(n)$, for any positive integer n? Try to explain, at least intuitively, how you deduced your formula.

   Solution. $I(n) = n!$ (read "n factorial", that is, $(n)(n-1)(n-2)\cdots(3)(2)(1)$, the product of all the positive integers less than or equal to n). One way to see this is to write down $I(n)$ and then repeatedly apply the rule $I(n) = nI(n-1)$. Doing so causes the following to happen:

$$\begin{aligned} I(n) &= nI(n-1) \\ &= (n)(n-1)I(n-2) \\ &= (n)(n-1)(n-2)I(n-3) \\ &\cdots \text{ some time passes} \cdots \\ &= (n)(n-1)(n-2)\cdots(3)(2)I(1) \\ &= (n)(n-1)(n-2)\cdots(3)(2)(1) = n!. \end{aligned}$$

6. The nice thing about our function $I(p)$ is that – unlike, say, the factorial function $n!$ – $I(p)$ make sense for *any* real number p. (Okay, technically, one can show that p must be greater than -1 for the integral defining $I(p)$ to make sense, but let's not worry about that now.) For example, $I(5.5)$ makes perfect sense:

$$I(5.5) = \int_0^1 (-\ln x)^{5.5}\,dx,$$

   even though $(5.5)!$ does not have any meaning. Now this doesn't mean that $I(5.5)$ is easy to *compute*. But, based an what you've seen in the above problems, can you name two integers (not necessarily consecutive integers) that $I(5.5)$ must sit in between? Explain. (If you have a calculator that does numerical integration, you can check your answer, but make a guess first.)

   Solution. It would certainly make sense that $I(5.5)$ should lie between $5!$ and $6!$. (This involves knowing that I is an increasing function, which we technically haven't shown yet, but it would certainly be a reasonable thing to hope that I satisfies. You might think about how you would go about taking the derivative of I with respect to p sometime.) In that case, we'd get $5! < I(5.5) < 6!$, or $120 < I(5.5) < 720$. WolframAlpha tells me $I(5.5) = 287.885\cdots$, in line with our guess.

7. Which of these graphs could be that of the function $I(p)$ we looked at today? Think about what properties you would expect a reasonable generalization of the factorial function to have. Also note that all of these functions pass through the points $(1, 1)$, $(2, 2)$, $(3, 6)$, $(4, 24)$, etc., so none of them are "wrong." But only one of them is actually $I(p)$. You might want to do a few numerical experiments like in #6 to try to rule some of them out. Do you think $I(p)$ is the "best" way to extend the factorial function? What shortcomings do the other graphs have?

   Solution. You were intended to think the top left graph is the "best," and the reason is because it is differentiable (smooth). The top right graph is not differentiable, since it has corners. The bottom left graph is not continuous since it has jumps. And the bottom right graph is just stupid, to be honest. It doesn't just increase! Because of that, an argument like $5! < I(5.5) < 6!$ wouldn't work, which would dissatisfying. The top left graph is actually the graph of $I(p)$.