Individual Written Homework Assignment 2 Solutions

January 24, 2011

Assignment: pp 22-24, problems 15, 17, 20, and 21

Problem 15 - The Measles Epidemic

We consider once again the specific rate equations:

$$S' = (-0.00001)SI \tag{1}$$

$$I' = 0.00001SI - \frac{I}{14} \tag{2}$$

$$R' = \frac{I}{14} \tag{3}$$

As discussed in the text on pages 1114, we saw that at time t = 1:

$$S = 44446.6 \tag{4}$$

$$I = 2903.4 \tag{5}$$

$$R = 2650.0. \tag{6}$$

Calculate the current rates of change S', I', and R' when $t = 1$, and then use these values to determine S, I, and R one day later.

Solution: First we need to find the current rates of change S', I' and R' when $t = 1$. To do this, we use equations (1), (2), and (3) along with our initial values listed at (4), (5), and (6).

$$\begin{aligned} S' &= (-0.00001)SI && (7)\\ &= (-0.00001)(44446.6)(2903.4) && (8)\\ &= -1290.5 && (9)\\ I' &= 0.00001SI - \frac{I}{14} && (10)\\ &= (0.00001)(44446.6)(2903.4) - (2903.4/14) && (11)\\ &= 1083.1 && (12)\\ R' &= \frac{I}{14} && (13)\\ &= 2903.4/14 && (14)\\ &= 207.4 && (15) \end{aligned}$$

Now we want to use these values to determine S, I, and R one day later. To find the value of S one day later, or at $t = 2$, we want to use the following equation: $\mathcal{S}_{t=n+1} = \mathcal{S}_{t=n} + (\mathcal{S}'_{t=1})(\triangle t)$ with the value of S at line (4) and S' at line (9) to calculate $S_{t=2}$:

$$
\begin{aligned}
S_{t=2} &= S_{t=1} + (S'_{t=1})(\triangle t) && (16)\\
&= 44446.6 + (-1290.5)(1) && (17)\\
&= 43156.1 && (18)
\end{aligned}
$$

Similarly for $I_{t=2}$ and $R_{t=2}$:

$$
\begin{aligned}
I_{t=2} &= I_{t=1} + (I'_{t=1})(\triangle t) && (19)\\
&= 2903.4 + (1083.1)(1) && (20)\\
&= 3986.5 && (21)
\end{aligned}
$$

$$
\begin{aligned}
R_{t=2} &= R_{t=1} + (R'_{t=1})(\triangle t) && (22)\\
&= 2650.0 + (207.4)(1) && (23)\\
&= 2857.4 && (24)
\end{aligned}
$$

Now we have finished the problem.

Problem 17

Double the time step. Go back to the starting time $t = 0$ and to the initial values:

$$
\begin{aligned}
S &= 45400 && (25)\\
I &= 2100 && (26)\\
R &= 2500 && (27)
\end{aligned}
$$

Recalculate the values of S, I, and R at time $t = 2$ by using a time step of $\triangle t = 2$. You should perform only a single round of calculations, and use the rates S, I, and R that are current at time $t = 0$.

Solution: To do this, we first need to calculate S', I', R' at $t = 0$:

$$
\begin{aligned}
S'_{t=0} &= (-0.00001)SI && (28)\\
&= (-0.00001)(45400)(2100) && (29)\\
&= -953.4 && (30)\\
I'_{t=0} &= 0.00001SI - \frac{I}{14} && (31)\\
&= (0.00001)(45400)(2100) - (2100/14) && (32)\\
&= 803.4 && (33)\\
R'_{t=0} &= \frac{I}{14} && (34)\\
&= 2100/14 && (35)\\
&= 150 && (36)
\end{aligned}
$$

Next, we need to modify the equations at (16) so that we can jump from $S_{t=0}$ to $S_{t=2}$ by using a step of $\triangle t = 2$:

$$
\begin{aligned}
S_{t=2} &= S_{t=0} + (S'_{t=0})(\triangle t) && (37)\\
&= 45400 + (-953.4)(2) && (38)\\
&= 43494.2 && (39)
\end{aligned}
$$

Similarly to get $I_{t=2}$ and $R_{t=2}$ by using a step of $\triangle t = 2$ we can modify equations (19) and (22):

$$
\begin{aligned}
I_{t=2} &= I_{t=1} + (I'_{t=0})(\triangle t) && (40)\\
&= 2100 + (803.4)(2) && (41)\\
&= 3706.8 && (42)\\
R_{t=2} &= R_{t=1} + (R'_{t=0})(\triangle t) && (43)\\
&= 2500 + (150)(2) && (44)\\
&= 2800 && (45)
\end{aligned}
$$

Problem 20

Suppose the spread of an illness similar to measles is modeled by the following rate equations:

$$
\begin{aligned}
S' &= -0.00002SI && (46)\\
I' &= 0.00002SI - 0.08I && (47)\\
R &= 0.08I. && (48)
\end{aligned}
$$

Note: the initial values S = 45400, etc. that we used in the text do not apply here.

(a) Roughly how long does someone who catches this illness remain infected? Explain your reasoning.

Solution: From the equation $R = 0.08I$ we know that the recovery coefficient, b, is $0.08\frac{1}{days}$. Thus every day, 8% of the infected population recovers, so we know that someone who catches the disease remains infected for roughly $\frac{1}{(0.08\frac{1}{days})} = 12.5$ days.

(b) How large does the susceptible population have to be in order for the illness to take hold- that is, for the number of cases to increase? Explain your reasoning.

Solution: If the number of cases increases, then we know that I' positive. Thus we want to find when $I' = 0$ so we know when it changes from negative to positive.

$$
\begin{aligned}
I' &= 0.00002SI - 0.08I && (49)\\
&= I(0.00002S - 0.08) && (50)\\
&= 0 \text{ when } I = 0 \text{ and when } S = 4000 && (51)
\end{aligned}
$$

and

$$I' > 0 \text{ when } S > 4000 \text{ since } I \geq 0 \tag{52}$$

Thus I' begins to increase when $S > 4000$, or the susceptible population must be greater than 4000 for the illness to take hold.

(c) Suppose 100 people in the population are currently ill. According to the model, how many (of the 100 infected) will recover during the next 24 hours?

Solution: Since we know that 8% of the infection population will recover each day from the recovery coefficient, $b = 0.08$, and the infected population is currently 100, we can calculate the number of people that will recover in the next 24 hours:

$$100 * 0.08 \frac{\text{people}}{24 \text{ hours}} = 8 \text{ people} \tag{53}$$

Thus 8 people will recover in the next 24 hours.

(d) Suppose 30 new cases appear during the same 24 hours. What does that tell us about S'?

Solution: If 30 new cases appear in 24 hours, then we know that 30 people moved from the susceptible population to the infected population in one day. Thus $S' = -30 \frac{\text{people}}{\text{day}}$.

(e) Using the information in parts (c) and (d), can you determine how large the current susceptible population is?

Solution: We can use (c), (d) and the fact that $S' = -0.00002SI$ to figure out the current value of S.
From (c) we know that 8 people of 100 total infected have recovered in this 24 hours.
From (d) we know that we have 30 new cases.
Thus $I = 122$.
Also from (d) we know that $S' = -30 \frac{\text{people}}{\text{day}}$.
Now we can plug all of these into our equation for S' to get S in terms of S', I and b:

$$S' = -0.00002SI \tag{54}$$

$$-30 = -0.00002(S)(122) \tag{55}$$

Solving for S we get:

$$S = \frac{30}{(-0.00002)(122)} \tag{56}$$

$$= 12295.1 \tag{57}$$

Hence our current susceptible population is 12295.1 people.

Problem 21

(a) Construct the appropriate S-I-R model for a measles-like illness that lasts for 4 days. It is also known that a typical susceptible person meets only about 0.3% of infection population each day, and the infection is transmitted in only one contact out of six.

Solution: Since the illness lasts for 4 days, we know that our recover coefficient, b, should be $b = \frac{1}{4 \text{ days}} = 0.25\frac{1}{\text{days}}$. Since a typical susceptible person meets only about 0.3% of infection population each day, we know that $p = 0.3\% = .003$. Finally, since the infection is transmitted in only one contact out of 6, we know that $q = \frac{1}{6} = 0.167$.
Hence our S-I-R model for this measles-like disease looks like:

$$-qpSI = -(0.167)(.003)SI = S' = -0.00005SI \tag{58}$$
$$qpSI - bI = (0.167)(.003) - 0.25I = I' = 0.00005SI - 0.25I \tag{59}$$
$$bI = R' = 0.25I. \tag{60}$$

(b) How small does the susceptible population have to be for this illness to fade away without becoming an epidemic?

Solution: Recall problem 20(b). We will solve this problem in a very similar manner, thus we want to use equation (59), set $I' = 0$ and then solve for I and S:

$$I' = 0.00005SI - 0.25I \tag{61}$$
$$= I(0.00005S - 0.25) \tag{62}$$
$$= 0 \text{ when } I = 0 \text{ and when } S = 5000 \tag{63}$$

So if $S < 5000$, then $I' < 0$, and hence the illness will fade away without becoming an epidemic.