Math 2300 – Calculus II – University of Colorado

Fall 2010 – Midterm III

SOLUTIONS IN RED

1. **(15 points)** Consider the surface given by $z = f(x, y)$, where $f(x, y) = \sqrt{32 - x^2 - y^2}$.

   (a) What region of the xy plane is the domain of $f(x, y)$? Describe this region geometrically.

   The function is defined on the region where $32 - x^2 - y^2 \geq 0$, that is $x^2 + y^2 \leq 32$. Geometrically, this represents the disc (filled circle) in the xy-plane centered at the origin and of radius $\sqrt{32}$.

   (b) What is the equation for the cross-section of the above surface with the vertical plane $y = -4$? Describe this cross-section geometrically.

   The cross-section with $y = -4$ is the curve in the xz-plane given by $z = \sqrt{32 - x^2 - (-4)^2}$, that is $z = \sqrt{16 - x^2}$. Geometrically, this corresponds to the upper half of the circle centered at the origin of the xz-plane and of radius 4.

   (c) What is the equation for the cross-section of the above surface with the horizontal plane $z = 4$? Describe this cross-section geometrically.

   The cross-section with the plane $z = 4$ is the curve in the xy-plane given by $\sqrt{32 - x^2 - y^2} = 4$, i.e. $x^2 + y^2 = 16$. Geometrically, this is the circle centered at the origin and of radius 4.

   (d) What is the distance between the point where the above surface intersects the z axis and the points where it intersects the x axis?

   The surface intersects the z-axis at the point where $x = y = 0$ and $z = \sqrt{32 - 0^2 - 0^2} \Rightarrow z = \sqrt{32}$. It intersects the x-axis at the point where $y = z = 0$ and $0 = \sqrt{32 - x^2 - 0^2} \Rightarrow x = \pm\sqrt{32}$. The distance between $A(0, 0, \sqrt{32})$ and $B_1(\sqrt{32}, 0, 0)$ is, using the distance formula, $d = \sqrt{32 + 32} = 8$. The distance between $A(0, 0, \sqrt{32})$ and $B_2(\sqrt{32}, 0, 0)$ is the same (the two x-intercepts are equidistant from the z-intercept).

   Note: The surface is the upper hemisphere ($z \geq 0$) of the sphere centered at $(0, 0, 0)$ and of radius $\sqrt{32}$.

2. **(20 points)** Imagine that Boulder is located at the point $(0, 0)$ on a map, with distances measured in miles, North pointing in the direction of the positive y-axis, and East pointing in the direction of the positive x-axis. Suppose that the temperature, T, at location (x, y) is given by

$$T = f(x, y) = f(x, y) = 60e^{(12x - x^2 + 6y - 2y^2)/100}.$$

   (For the purposes of this problem, assume the temperature does not vary with time.)

(a) What is the temperature in Boulder?

In Boulder: $f(0,0) = 60e^{0/100} = 60$.

(b) What is the temperature in Lafayette, 11 miles east of Boulder?

In Lafayette: $f(11,0) = 60e^{(12\cdot 11 - 11^2)/100} = 60 \cdot e^{.11} \sim 67$.

(c) Find $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$.

$\frac{\partial}{\partial x} f(x,y) = 0.6(12 - 2x)e^{(12x - x^2 + 6y - 2y^2)/100}$
$\frac{\partial}{\partial y} f(x,y) = 0.6(6 - 4y)e^{(12x - x^2 + 6y - 2y^2)/100}$

(d) Suppose you start in Boulder and head due East, until you get to Lafayette. Near the start of your journey, are temperatures getting warmer or cooler? What about near the end of your journey?

The "East-West" partial derivative $\frac{\partial}{\partial y} f(x,y)$ is positive for $x < 6$ and negative for $x > 6$. So, heading East out of Boulder, the temperatures will increase up to 6 miles away from Boulder; after that, the temperatures will decrease going East until Lafayette (at $x = 11$).

(e) Suppose that, instead of the above formula for T, we knew only the numerical values of $f(0,0)$ and $\frac{\partial f}{\partial x}(0,0)$ given by that formula. Use these numbers to estimate the temperature in Lafayette. Is this approximation close to the actual value you found in part (b) of this problem? What's the lesson to be learned here?

Going East out of Boulder, we keep $y = 0$ constant. Using a degree one Taylor approximation, we can write that, for values of x close to zero: $f(x,0) \sim f(0,0) + \dfrac{\partial f}{\partial x}(0,0)x = 60 + 7.2x$. Using this estimate for $x = 11$ (corresponding to the position of Lafayette), we get: $f(11,0) \sim 60 + 7.2 \cdot 11 = 139.2$. This is very coarse overestimate of the actual temperature in Lafayette, calculated at (b). The large error is not surprising, since we approximated a function with exponential behavior by a linear polynomial, and used this approximation at a value of the variable far from the base point of the linear expansion.

3. (a) Define the function $S(x)$ by the following Taylor series:

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

What is the radius of convergence for $S(x)$?

We use the Ratio Test:

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!} x^2 = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 < 1,$$ for all real values of x. So the Ratio Test is conclusive and proves convergence of the power series for all values of x. It follows that the radius of convergence of $S(x)$ is $R = \infty$.

(b) Define the function $C(x)$ by the following Taylor series:

$$C(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

What is the radius of convergence for $C(x)$?

Through a calculation very similar to the one above for $S(x)$, it follows that the radius of convergence of $C(x)$ is also $R = \infty$.

(c) Show that $C'(x) = S(x)$.

$$C'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right) = \sum_{n=1}^{\infty} \frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = S(x)$$

(d) Show that that $S'(x) = C(x)$.

$$S'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = C(x)$$

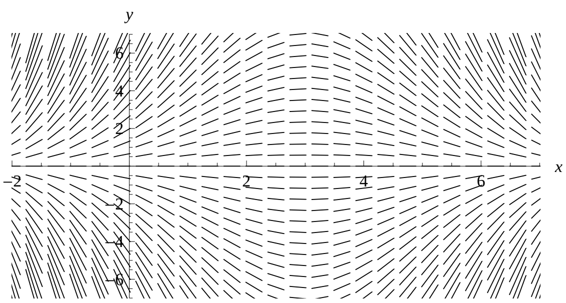
Note: you might have recognized the series for the hyperbolic trig functions $S(x) = \sinh(x)$ and $C(x) = \cosh(x)$. Then, after first verifying that these are indeed their respective Taylor series, you could use the algebraic form (rather than Taylor series form) for $S(x)$ and $C(x)$ to prove the properties in parts (c) and (d).

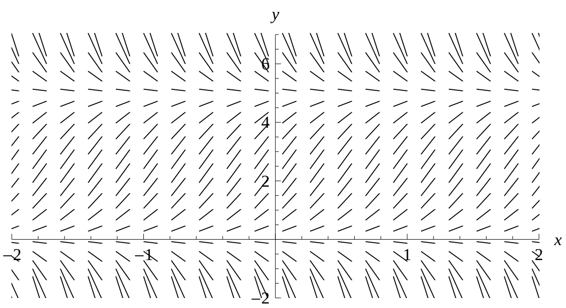
4. **(15 points)**

(a) Which of the slope fields (i)–(iv) below could be the slope field for the "logistic" differential equation

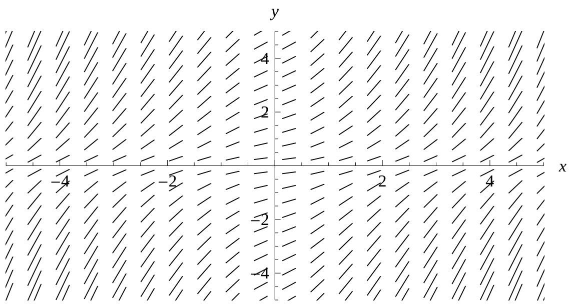
$$\frac{dy}{dx} = y(5-y) \ ?$$

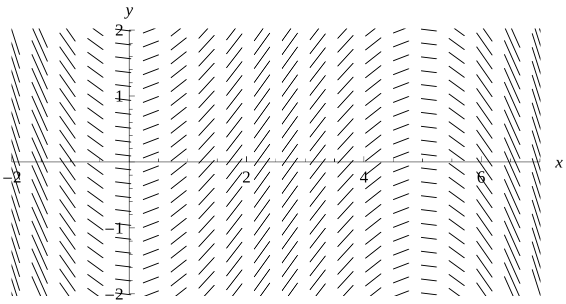
You must explain your answer to receive credit.





(i) (ii)





(iii) (iv)

(b) Are there any equilibrium solutions to this differential equation? If so, what are they? If not, why not?

The equilibrium solutions are the constant solutions $y = ct$ obtained by setting the slope field $\frac{dy}{dx} = 0$. In this case, $y(5-y) = 0$ gives two equilibrium solutions: $y = 0$ and $y = 5$.

(c) Is the above logistic differential equation separable? If not, why not? If so, separate variables, but *do not* solve. (You don't need to go any further than getting all y terms on one side and all x terms on the other.)

The equation is separable. It can be separated as: $\frac{dy}{y(5-y)} = dx$

(d) On the slope field you chose for part (a) of this problem, sketch in the solution curve for the above logistic differential equation that has initial condition $y(0) = 1$.

5. (a) Write down the second degree Taylor polynomial $P_2(x)$ approximating

$$f(x) = \ln\big(1 + x(1-x)\big)$$

near $x = 0$.

Solution 1: One can calculate directly the coefficients of $P_2(x)$, as follows:

$$f(0) = \ln(1) = 0 \Rightarrow c_0 = 0$$

$$f'(x) = \frac{1-2x}{1+x(1-x)} \Rightarrow c_1 = f'(0) = 1$$

$$f''(x) = \frac{-2[1+x(1-x)] - (1-2x)^2}{[1+x(1-x)]^2} \Rightarrow c_2 = \frac{f''(0)}{2!} = \frac{-3}{2}$$

So $P_2(x) = x - \frac{3x^2}{2}$.

Solution 2: One can use the Taylor series of $\ln(1+y)$ around $y = 0$, then substitute $y = x(1-x)$:

$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} \ldots$

$\ln(1+x(1-x)) = x(1-x) - \frac{[x(1-x)]^2}{2} +$ terms of order larger than 3 $=$ $x - x^2 - \frac{x^2}{2} +$ terms of order larger than 3

So $P_2(x) = x - x^2 - \frac{x^2}{2} = x - \frac{3x^2}{2}$

(b) Use your result from part (a) to approximate $\ln(1.09)$. Hint: $\frac{1}{10} \cdot \frac{9}{10} = 0.09$.

To approximate $\ln(1.09)$, we can use either $x = 0.1$ or $x = 0.9$. We chose to use $x = 0.1$ since it is closer to the base point $x = 0$, and we expect to get with it a better estimate than with $x = 0.9$ (it is easy to verify this by calculating both). $\ln(1.09) = f(0.1) \sim P_2(0.1) = 0.1 - \frac{3 \cdot 0.1^2}{2} = 0.985$.

6. Let

$$g(x,y) = \cos(x^2 y).$$

Show that

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} g(x,y)\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} g(x,y)\right].$$

$\frac{\partial}{\partial x} g(x,y) = -2xy\sin(x^2 y)$

$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} g(x,y)\right] = -2x\sin(x^2 y) - 2xy \cdot \cos(x^2 y) \cdot x^2$

$\frac{\partial}{\partial y} g(x,y) = -x^2 \sin(x^2 y)$

$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} g(x,y)\right] = -2x\sin(x^2 y) - x^2 \cdot \cos(x^2 y) \cdot 2xy$

It is clear that

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} g(x,y)\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} g(x,y)\right] = -2x\sin(x^2 y) - 2x^3 y\cos(x^2 y)$$