Math 2300 – Calculus II – University of Colorado

Fall 2010 – Midterm II

SOLUTIONS IN RED

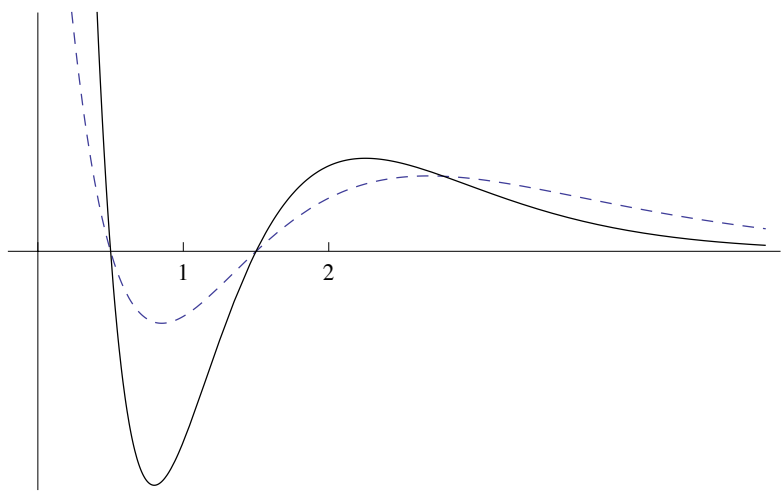
1. Consider the graphs of $f(x)$ (the solid curve) and $g(x)$ (the dashed curve) in the figure below. Suppose that

$$\int_0^1 f(x)\ dx \quad \text{converges}$$

and that

$$\int_2^\infty f(x)\ dx \quad \text{diverges}$$

Assume that the behavior suggested as $x \to 0$ and $x \to \infty$ continues for that part of the domain $(0, \infty)$ not shown in the graph.



For each of the following, indicate what this graph suggests about the convergence of the indicated integral. Circle one answer only for each part. No explanation is necessary.

(a) $\int_0^1 g(x)\ dx$

converges diverges convergence cannot be determined

$\displaystyle\int_0^1 f(x)\ dx$ converges $\Rightarrow \displaystyle\int_0^{\frac{1}{2}} f(x)\ dx$ converges, because $\displaystyle\int_{\frac{1}{2}}^1 f(x)\ dx$ is finite. Since $0 < g(x) < f(x)$ on the interval $(0, 1/2)$, it follows (by direct comparison) that $\displaystyle\int_0^{\frac{1}{2}} g(x)\ dx$ converges. Since $\displaystyle\int_{\frac{1}{2}}^1 g(x)\ dx$ is finite, it follows that $\displaystyle\int_0^1 g(x)\ dx$ converges.

(b) $\int_1^\infty g(x)\ dx$

converges diverges convergence cannot be determined

$\displaystyle\int_2^\infty f(x)\ dx$ diverges $\Rightarrow \displaystyle\int_3^\infty f(x)\ dx$ diverges, because $\displaystyle\int_2^3 f(x)\ dx$ is finite. Since $0 < f(x) < g(x)$ on the interval $(3, \infty)$, it follows (by direct comparison) that $\displaystyle\int_3^\infty g(x)\ dx$ diverges. Since $\displaystyle\int_2^3 g(x)\ dx$ is finite, it follows that $\displaystyle\int_2^\infty g(x)\ dx$ diverges.

(c) $\int_0^\infty g(x)\ dx$

converges diverges convergence cannot be determined

$\displaystyle\int_0^\infty g(x)\ dx = \int_0^1 g(x)\ dx + \int_1^2 g(x)\ dx + \int_2^\infty g(x)\ dx$. The first integral converges by (a), the second is finite (not an improper integral), and the third diverges by (b). Altogether, $\displaystyle\int_0^\infty g(x)\ dx$ is divergent.

(d) $\int_0^\infty (f(x) + g(x))\ dx$

converges diverges convergence cannot be determined

$\displaystyle\int_0^\infty (f(x)+g(x))\ dx = \int_0^\infty f(x)\ dx + \int_0^\infty g(x)\ dx$. Both parts diverge ($+\infty$), so the sum diverges as well.

(e) $\int_0^\infty (f(x) - g(x))\ dx$

converges diverges convergence cannot be determined

$\displaystyle\int_0^\infty (f(x)-g(x))\ dx = \int_0^\infty f(x)\ dx - \int_0^\infty g(x)\ dx$. Both parts diverge ($+\infty$), but nothing can be determined about the difference, without further information (the values of $f(x) - g(x)$ for $x \to 0$ and $x \to \infty$ could be small enough to make the integral convergent, but also large enough to make it divergent).

2. Given $\displaystyle\sum_{n=1}^\infty a_n = 0.72$, $b_n = n^2$, $c_n = n^3$ determine whether or not the following statements are True or False. To receive full credit, you must appropriately justify all your statements – with calculations, a sentence or two explaining your decisions, and/or a relevant picture that illustrates your thinking.

(a) $\displaystyle\lim_{n\to\infty} a_n = 0.72$

FALSE. Since the series $\displaystyle\sum_{n=1}^\infty a_n$ converges, it follows immediately that the underlying sequence $\lim_{n\to\infty} a_n = 0 \neq 0.72$.

(b) $\dfrac{b_n}{c_n}$ converges.

TRUE. $\displaystyle\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$ (because when n gets arbitrarily large, $\frac{1}{n}$ gets arbitrarily close to zero.

(c) $\displaystyle\lim_{n\to\infty} s_n = 0.72$ where $s_n = a_1 + a_2 + \cdots + a_n$

TRUE. $\displaystyle\lim_{n\to\infty} s_n = 0.72$ is the same thing as $\displaystyle\sum_{n=1}^\infty a_n = 0.72$, by definition.

(d) $\displaystyle\sum_{n=1}^\infty \frac{b_n}{c_n}$ converges.

FALSE. $\displaystyle\sum_{n=1}^\infty \frac{b_n}{c_n} = \sum_{n=1}^\infty \frac{1}{n}$, which is divergent (the harmonic series).

(e) $\sum_{n=1}^{\infty}(-1)^n \frac{b_n}{c_n}$ converges.

TRUE. $\sum_{n=1}^{\infty}(-1)^n \frac{b_n}{c_n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$, which is convergent by the Alternating Series Test (indeed, the series is alternating, and the sequence of the positive parts of each term $\frac{1}{n}$ decreases to 0 as $n \to \infty$).

3. A ball is dropped from a height 20 ft and bounces with no air resistance. Each bounce is 4/5 of the height of the bounce before.

(a) Find an expression for the height to which the ball rises after it hits the floor for the first, second and third times.

The ball bounces back each time to 4/5 of the previous height. Its initial height is $h_0 = 20$. So: after the first bounce, it reaches the height $h_1 = \frac{4}{5}h_0 = 16$, after the second bounce it reaches $h_2 = \frac{4}{5}h_1 = 12.8$, and after the third it reaches $h_3 = \frac{4}{5}h_2 = 10.24$. In general, after the n-th bounce it raises back to $h_n = h_0\left(\frac{4}{5}\right)^n = 20\left(\frac{4}{5}\right)^n$.

(b) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the n-th time. Express your answer in closed form.

By the time it hits the floor the n-the time, the ball has traveled the distance:
$d_n = h_0 + 2(h_1 + \ldots + h_{n-1}) = 2(h_0 + \ldots h_{n-1}) - h_0 = 2h_0\left[1 + (4/5) + \ldots + (4/5)^{n-1}\right] - h_0 = 2h_0\frac{1-(4/5)^n}{1-4/5} - h_0 = 200[1-(4/5)^n] - 20$

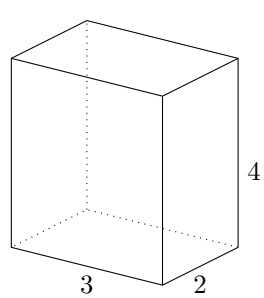
(c) What is the distance that the ball would travel if allowed to bounce infinitely?

If allowed to bounce infinitely, the ball accumulates the distance given by the above formula, when $n \to \infty$, that is $d_\infty = 200 - 20 = 180$.

4. A closed container measuring $2 \times 3 \times 4\,\mathrm{m}^3$ holds a compressible liquid with density function given by

$$D(h) = 20(6-h)\,\frac{\mathrm{kg}}{\mathrm{m}^3}$$

where h is measured in meters from the bottom of the container. The container is placed in the orientation shown below. How many kilograms can the container hold?



The mass of the container is given by $M = \int_0^4 A(h)D(h)\ dh$, where $A(h)$ is the area of the horizontal cross-section made through the container at height h. The container has identical rectangular cross-sections at any height h, of length $L = 3$ m and width $W = 2$ m, so $A(h) = 3 \cdot 2 = 6$ m^2, for all $0 \le h \le 4$. Then the integral for the mass becomes:

$$M = \int_0^4 6D(h)\ dh = \int_0^4 6 \cdot 20(6-h)\ dh = 120 \int_0^4 (6-h)\ dh = 120(24-8) = 1920 \text{ kg}$$

5. (a) Apply each of the following tests, if possible, to determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges. If a given test is conclusive, what does it tell you about convergence of the series? If a given test is not conclusive, explain why not.

   (i) The Ratio Test.

   To apply the Ratio Test, we calculate the limit: $\lim_{n\to\infty} \frac{\ln(n+1)}{(n+1)^2} \cdot \frac{n^2}{\ln(n)}$ (the absolute values were ignored, since all factors are positive). Since $\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$, we are left with calculating: $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)}$. Since both the numerator and denominator go to ∞ when $n \to \infty$, we can apply L'Hopital's Rule and calculate $\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$. So, by L'Hopital's Rule, we also get that $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} = 1$, so the original limit: $\lim_{n\to\infty} \frac{\ln(n+1)}{(n+1)^2} \cdot \frac{n^2}{\ln(n)} = 1$. This is the only limit value for which the Ratio Test is inconclusive. So the Ratio Test cannot be used to determine the convergence of this series.

   (ii) The Integral Test.

   The function corresponding to this series is $f(x) = \frac{\ln(x)}{x^2}$, which is continuous for $x > 0$, positive for $x \ge 1$, and increasing for x large enough. Indeed, taking the derivative of f (using the quotient rule) we get: $f'(x) = \frac{1 - 2\ln(x)}{x^3}$, which is negative for $x > e^{1/2}$. So the Integral Test can be applied, and the convergence of the series will be the same as the convergence of the improper integral $\int_1^{\infty} \frac{\ln(x)}{x^2}\ dx$. Next, we want to establish this convergence. We use integration by parts, with $u = \ln(x)$ and $dv = \frac{1}{x^2}\ dx$, and get: $\frac{\ln(x)}{x^2}\ dx = -\frac{\ln(x)}{x} - \frac{1}{x}$. Then $\int_1^{\infty} \frac{\ln(x)}{x^2}\ dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^2}\ dx = \lim_{b\to\infty} -\frac{\ln(b)}{b} - \frac{1}{b} = 0$ (that is, because $\lim_{b\to\infty} \frac{1}{b} = 0$, and $\lim_{b\to\infty} \frac{\ln(b)}{b} = 0$ by L'Hopital's Rule). This means that the integral $\int_1^{\infty} \frac{\ln(x)}{x^2}\ dx$ converges to zero. By the Integral Test, the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ also converges.

   (b) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

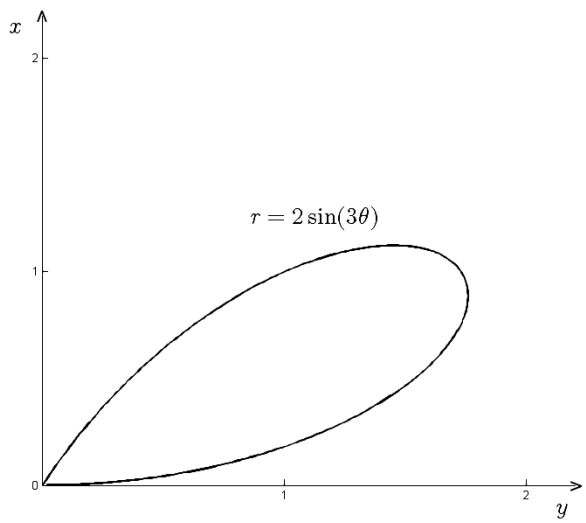
   You must justify your answer to receive credit.

   We use the Ratio Test to determine convergence:

$$\lim_{n\to\infty} \frac{3^{(n+1)}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n\to\infty} \frac{3^{(n+1)}}{3^n} \frac{(n+1)^2}{n^2} \frac{n!}{(n+1)!} = 3 \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

The Ratio Test is conclusive, and tells us that the series is convergent.

6. Consider one "petal" of the curve $r = 2\sin(3\theta)$, as shown in the following figure:



(a) What is the interval for the angle θ that corresponds to the piece of the curve shown in the figure?

Clearly, $r = 0$ at $\theta = 0$; the next zero for the radius occurs when the argument of sin is π, that is when $3\theta = \pi$, i.e. $\theta = \pi/3$. So the petal of the curve corresponds to the interval $[0, \pi/3]$ for θ.

(b) Write an expression for the perimeter of the petal shown in the figure. Simplify, but do not numerically evaluate your expression.

The petal can be written in parametric form, using cartesian coordinates:

$x(\theta) = r\cos(\theta) = 2\sin(3\theta)\cos(\theta)$
$y(\theta) = r\sin(\theta) = 2\sin(3\theta)\sin(\theta)$

Then

$$\frac{dx}{d\theta} = 6\cos(3\theta)\cos(\theta) - 2\sin(3\theta)\sin(\theta)$$
$$\frac{dy}{d\theta} = 6\cos(3\theta)\sin(\theta) + 2\sin(3\theta)\cos(\theta)$$

and

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= 36\cos^2(3\theta)[\cos^2(\theta) + \sin^2(\theta)] + 4\sin^2(3\theta)[\sin^2(\theta) + \cos^2(\theta)] \\ &= 36\cos^2(3\theta) + 4\sin^2(3\theta)\end{aligned}$$

The perimeter (arc length) of the petal is given by:

$$\int_0^{\pi/3} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/3} \left[36\cos^2(3\theta) + 4\sin^2(3\theta)\right] d\theta = 2\int_0^{\pi/3} \left[9\cos^2(3\theta) + \sin^2(3\theta)\right] d\theta$$