MATH 2300 – review problems for Exam 2 **CORRECTED ANSWERS**

1. A pyramid of constant density δ gm/cm^3 has a square base of sidelength 40 cm, and a height of 10 cm.

   (a) Find the mass of the pyramid.

$$\text{Mass=volume} \times \text{density} = \delta \times \int_0^{10} (4z)^2\,dz = \frac{\delta\, 16z^3}{3}\bigg|_0^{10} = \frac{16000\,\delta}{3}\,\text{gm}.$$

   (b) Find the center of mass of the pyramid.

   Put the apex of the pyramid at the origin, with the base of the pyramid perpendicular to the z axis. Then by symmetry, the center of mass has x-coordinate $\overline{x} = 0$ and y-coordinate $\overline{y} = 0$. Furthermore,

$$\overline{z} = \frac{\delta \int_0^{10} z(4z)^2\,dz}{\frac{16000\,\delta}{3}} = \frac{3}{16000} \cdot \frac{16z^4}{4}\bigg|_0^{10} = \frac{30}{4} = \frac{15}{2}\,\text{cm}.$$

2. (corrected) A metal plate of constant density 5 gm/cm^2 has a shape bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 1$.

   (a) Find the mass of the plate.

$$\text{Mass=area} \times \text{density} = 5 \times \int_0^1 \sqrt{x}\,dx = \frac{10}{3}\,\text{gm}.$$

   (b) Find the center of mass of the plate.

$$\overline{x} = \frac{5\int_0^1 x\sqrt{x}\,dx}{\frac{10}{3}} = \frac{3}{5}\,\text{cm}, \quad \overline{y} = \frac{5\int_0^1 y(1-y^2)\,dy}{\frac{10}{3}} = \frac{3}{8}\,\text{cm}.$$

3. Find the mass of the plate in the previous problem if, instead of constant density, the plate has density:

   (a) $\delta(x) = 1 + x$,

$$\text{Mass } = \int_0^1 (1+x)\sqrt{x}\,dx = \frac{16}{15}\,\text{gm}.$$

   (b) $\delta(y) = 1 + y$.

$$\text{Mass } = \int_0^1 (1+y)(1-y^2)\,dy = \frac{11}{12}\,\text{gm}.$$

4. Find the area of the region bounded by the curve $r = \sqrt{\theta}$, for $0 \le \theta \le \pi$. $\int_0^\pi [(\sqrt{\theta})^2/2]\,d\theta = \pi^2/4$.

5. Find the area of the region that lies within the limaçon $r = 3 + 2\cos(\theta)$ and outside the circle $r = 4$.

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} [(3 + 2\cos\theta)^2 - 4^2]\, d\theta = \frac{13\sqrt{3}}{2} - \frac{5\pi}{3}.$$

6. Find the area of the region common to the circles $r = \cos(\theta)$ and $r = \sqrt{3}\sin(\theta)$. (Hint: $\tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$.)

$$\frac{1}{2}\int_0^{\pi/6} (\sqrt{3}\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (\cos\theta)^2\, d\theta = \frac{1}{24}\left(5\pi - 6\sqrt{3}\right) \approx 0.221486.$$

7. Find the exact length of the polar curve $r = e^\theta$, for $0 \le \theta \le \pi/2$.

$$\int_0^{\pi/2} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta = \sqrt{2}\int_0^{\pi/2} e^\theta\, d\theta = \sqrt{2}(e^{\pi/2} - 1).$$

8. (corrected) Let $\{f_n\}$ be the sequence defined recursively by $f_1 = 5$ and $f_n = f_{n-1} + 2n + 4$.

   (a) Check that the sequence g_n whose n-th term is $g_n = n^2 + 5n - 1$ satisfies this recurrence relation, and that $g_1 = 5$. (This tells us $g_n = f_n$ for all n.) We check:

   $$g_{n-1} + 2n + 4 = ((n-1)^2 + 5(n-1) - 1) + 2n + 4 = n^2 + 5n - 1 = g_n$$

   and plainly $g_1 = 5$.

   (b) Use the result of part (a) to find f_{20} quickly, with the aid of a calculator. $f_{20} = 20^2 + 5 \cdot 20 - 1 = 499$.

9. Decide whether each of the following sequences converges. If a series converges, what does it converge to? If not, why not?

   (a) The sequence whose n-th term is $a_n = 1 - \frac{1}{n}$. Converges to 1. (The $\frac{1}{n}$ part goes to zero.)

   (b) The sequence whose n-th term is $b_n = \sqrt{n+1} - \sqrt{n}$. Converges to 0:

   $$\sqrt{n+1} - \sqrt{n} = \sqrt{n+1} - \sqrt{n}\left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right) = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

   and the denominator of this expression grows as n gets large, so it approaches zero.

   (c) The sequence whose n-th term is $c_n = \cos(\pi n)$. Diverges. The sequence is $\{-1, 1, -1, 1, -1, \cdots\}$ which oscillates.

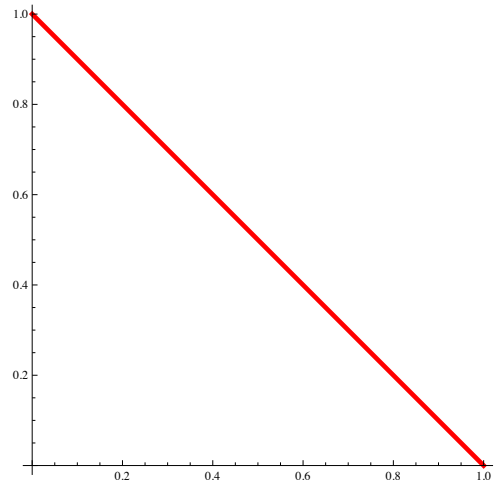
   (d) The sequence $\{d_n\}$, where $d_1 = 2$ and

   $$d_n = 2d_{n-1} \quad \text{for } n > 1.$$

   Diverges. The sequence is $\{2, 4, 8, 16, \cdots\}$ and the terms go to infinity.

10. Consider the region defined in polar coordinates by $0 \le \theta \le \pi/2$ and $0 \le r \le \dfrac{1}{\sin(\theta) + \cos(\theta)}$.

(a) Sketch a graph of this region.



(b) Use your graph to find the area. (The integral in polar coordinates is hard!) The area is $\frac{1}{2}\times$ base $\times$ height $= 1/2$.

11. Find the arc length of the part of the cardioid $r = 1 + \cos(\theta)$ where $0 \le \theta \le \pi/2$.

$$\int_0^{\pi/2} \sqrt{2(1+\cos(\theta))}\, d\theta = 2\sqrt{2}.$$

12. The following integrals represent the area of some region in the xy plane. Draw a graph of the area, labeling the axes and giving the equation(s) of the function(s).

(a) $\displaystyle\int_{-2}^{0} (-4x)\, dx$ The triangle bounded by $y = -4x$, $x = -2$, and the x-axis.

(b) $\displaystyle\int_{-3}^{3} (-\sqrt{9-x^2})\, dx$ The semicircle bounded by $y = -\sqrt{9-x^2}$ and the x-axis.

(c) $\displaystyle\int_{1}^{2} 3y\, dy$ The region bounded by $y = x/3$, $x = 3$, $x = 6$, and the x-axis.

(d) $\displaystyle\int_{0}^{1} (\sqrt{y} - y)\, dy$ The region bounded by $y = x^2$ and $y = x$.

13. Using slices parallel to the base, write a definite integral representing the volume of a cone with a height of 10 cm and a base of diameter 6 cm.

$$\int_0^{10} \pi \left(\frac{3}{10}x\right)^2 dx \quad \text{or} \quad \int_0^{10} \pi \left(3 - \frac{3}{10}x\right)^2 dx$$

14. Find the sum of the series. For what values of the variable does the series converge to this sum?

(a) $1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{8} + \cdots = \dfrac{2}{2-x}$, for $|x| < 2$

(b) $y - y^2 + y^3 - y^4 + \cdots = \dfrac{y}{y+1}$, for $|y| < 1$

(c) $4 + z + \dfrac{z^2}{3} + \dfrac{z^3}{9} + \cdots = \dfrac{z-12}{z-3}$, for $|z| < 3$

15. (corrected) A ball is dropped from a height of 10 feet and bounces. Assume that there is no air resistance. Each bounce is $\frac{3}{4}$ of the height of the bounce before.

(1) Find an expression for the height to which the ball rises after it hits the floor for the nth time. $H(n) = 10(\frac{3}{4})^n$

(2) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the nth time. $D(n) = 10 + (2 \cdot 10 \cdot (3/4))\dfrac{(1-(3/4)^{n-1}}{1-(3/4)}$.

(3) Using without proof the fact that a ball dropped from a height of h feet reaches the ground in $\sqrt{h}/4$ seconds: Will the ball bounce forever? If not, how long it will take for the ball to come to rest?

The ball will not bounce forever. The total time it bounces is given by

$(\sqrt{10}/4) + a/(1-r)$, with $a = (1/2) \cdot \sqrt{10 \cdot (3/4)}$ and $r = \sqrt{3/4}$.

16. In theory, drugs that decay exponentially always leave a residue in the body. However in practice, once the drug has been in the body for 5 half-lives, it is regarded as being eliminated. If a patient takes a tablet of the same drug every 5 half-lives forever, what is the upper limit to the amount of drug that can be in the body?

Let P_n represent the percentage of the drug in the body after the nth tablet. Then

$$P_1 = 1 \text{ (100 percent)}$$

$$P_2 = 1(.5 * .5 * .5 * .5 * .5) + 1 = 1.03125$$

$$P_3 = 1.03125(.03125) + 1 = 1.03223$$

So,

$$P_n = \frac{1-(.03125)^n}{1-.03125}.$$

As $n \to \infty$, $P_n \to 1.0323$, so this is the maximum amount of the drug in the body.

17. Derive the volume formulas for the following shapes by using an appropriate integral:

(a) a circular cylinder of height H whose radius is R,
$\int_0^H \pi R^2 \, dh$

(b) a circular cone of height H whose radius at the base is R, $\int_0^H \pi \left(R - \frac{h}{H}\right) 2 \, dh$

(c) a square pyramid of height H whose base has side length S, $\int_0^H \left(S - \frac{h}{H}\right)^2 \, dh$

(d) a triangular pyramid of height H whose base is an equilateral triangle with side length S,
$\int_0^H \frac{\sqrt{3}}{4} \left(S - \frac{h}{H}\right)^2 \, dh$

(e) a sphere of radius R. $\int_{-R}^R \pi(R^2 - h^2) \, dh$

18. Suppose that $0 \le f(x) \le g(x)$ for $x \ge a$. If $\displaystyle\int_a^\infty f(x)\mathrm{d}x$ converges and $\displaystyle\int_a^\infty g(x)\mathrm{d}x$ diverges, then is the area between the curves $f(x)$ and $g(x)$ for $x \ge a$ finite or infinite? Infinite: this area is $\displaystyle\int_a^\infty g(x)\mathrm{d}x - \int_a^\infty f(x)\mathrm{d}x$, which is an infinite number minus a finite one.

19. Determine whether the following integrals converge or diverge:

(a) $\displaystyle\int_1^\infty \frac{5-2\sin(e^x)}{x^2}\mathrm{d}x$ converges

(b) $\displaystyle\int_1^\infty \frac{1}{x+\ln x}\mathrm{d}x$ diverges

(c) $\displaystyle\int_1^\infty \frac{x^2+x+1}{x^5+3x^2+1}\mathrm{d}x$ converges

(d) $\displaystyle\int_1^\infty \frac{1}{e^x-x}\mathrm{d}x$ converges

(e) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\mathrm{d}x$ converges

20. Consider the region bounded by $y=\sqrt{x}$, $y=0$, $x=1$.

(a) Sketch the solid obtained by rotating the above region around the x-axis.

(b) Using the sketch, write a Riemann sum approximating the volume of the solid.

$$\sum \pi(\sqrt{x})^2\triangle x = \sum \pi x\triangle x$$

(c) Convert your sum into an integral and find the volume.

$$\int_0^1 \pi x\,dx = \frac{\pi}{2}$$

(d) Repeat parts (a)-(c) with the same region rotated around the y-axis.

$$\sum \pi(y^2)^2\triangle y = \sum \pi y^4\triangle y$$
$$\int_0^1 \pi y^4\,dy = \frac{\pi}{5}$$

21. Using the table below, estimate the length of the curve given by $y=f(x)$ from $(3,4)$ to $(6,0.7)$.

x	3	3.5	4	4.5	5	5.5	6
$f(x)$	4	3.6	2.4	-1	-0.5	0	0.7
$f'(x)$	-0.8	-2.4	-6.8	1	1	1.4	-0.4

The appropriate arc length formula is

$$L=\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

So, we can find a good estimate of the total length by using the table

$$\begin{aligned} L \approx\ & \sqrt{1+(-0.8)^2}(.5)+\sqrt{1+(-2.4)^2}(.5)+\sqrt{1+(-6.8)^2}(.5) \\ & +\sqrt{1+(1)^2}(.5)+\sqrt{1+(1)^2}(.5)+\sqrt{1+(1.4)^2}(.5) \end{aligned}$$

22. Find the volume of the solid whose base is the region in the xy-plane bounded by the curves $y = x$ and $y = x^2$ and whose cross sections perpendicular to the x-axis are squares with one side in the xy-plane.

Since the cross sections are perpendicular to the x-axis we will have slices of thickness $\triangle x$. The volume of each slice will be given by $y\triangle x$ where $y = x - x^2$. The Riemann sum approximating the volume is then

$$\sum (x - x^2)^2 \triangle x$$

This leads us to computing the volume of the solid by the integral

$$\int_0^1 (x - x^2)^2 \, dx$$

23. Do the same thing as the previous problem except with semi-circle cross sections and then again with cross sections that are isosceles triangles of height 3.

For semi-circle cross sections we use the same method as above to get

$$\int_0^1 \frac{1}{2}\pi \left(\frac{x - x^2}{2}\right)^2 dx$$

as the volume of the solid. For the isosceles triangle cross-sections we get a volume of

$$\int_0^1 \frac{1}{2}(x - x^2)(3) \, dx$$

24. A steady wind blows a kite due east. The kite's height above ground from horizontal position $x = 0$ to $x = 80$ feet is given by

$$y = 150 - \frac{1}{40}(x - 50)^2$$

Find the distance traveled by the kite.

This distance traveled can be calculated by finding the arc length of the above curve from $x = 0$ to $x = 80$. Using the arc length formula we find the distance traveled to be

$$L = \int_0^{80} \sqrt{1 + (-\frac{1}{20}(x - 50))^2} \, dx$$

25. Supplemental Problems from Section 8.2: 17, 21

26. A certain bacteria is growing in a petri dish of volume 300 cm^3. Assume that each bacterium occupies 1×10^{-12} cm^3, and that the growth rate of the bacteria, starting at time $t = 1$ hour, is given by

$$r(t) = \frac{\sin(t) + 2}{t} \quad \frac{\text{bacteria}}{\text{hour}}.$$

Will the bacteria ever outgrow the petri dish? Explain your answer carefully.

Yes, the bacteria will outgrow the petri dish. We will show that total quantity of bacteria tends to ∞ as t goes to ∞, so the total volume of the bacteria will eventually exceed the volume of the petri dish, regardless of the size of the bacteria.

Note that the total quantity of bacteria after b hours is equal to (the amount of bacteria after 1 hour)+ $\int_1^b r(t)\,dt$. Thus, we are trying to show that

$$\lim_{b\to\infty}\int_1^b \frac{\sin(t)+2}{t}\,dt = \infty,$$

i.e. we want to show $\int_1^\infty \frac{\sin(t)+2}{t}\,dt = \infty$. We use a comparison. Notice that $1 \le \sin(t)+2$, so $\frac{1}{t} \le \frac{\sin(t)+2}{t}$. Thus, $\int_1^\infty \frac{1}{t}\,dt \le \int_1^\infty \frac{\sin(t)+2}{t}\,dt$. We know that $\int_1^\infty \frac{1}{t}\,dt$ diverges to ∞, so $\int_1^\infty \frac{\sin(t)+2}{t}\,dt$ must diverge to infinity as well.

27. Show that the volume contained in the solid obtained by rotating the curve $y = e^{-x}$, from $x = 1$ to ∞, about the x-axis is finite. To calculate the volume of the solid, we slice the solid perpendicular to the x-axis to get slices that look approximately like cylinders of height (or width, depending on your point of view) Δx. Now, we make the following calculations.

    - The volume of a slice $\approx \pi r^2 h = \pi e^{-2x}\Delta x$
    - The volume of the solid $\approx \sum \pi e^{-2x}\Delta x$
    - The volume of the solid $= \lim_{\Delta x\to 0}\sum \pi e^{-2x}\Delta x = \int_1^\infty \pi e^{-2x}\,dx$

    Thus, we need to show that $\int_1^\infty \pi e^{-2x}\,dx$ is finite. We use a comparison. Using the facts that $f(y) = e^y$ is an increasing function of y and that $-2x$ is less than $-x$ for $x \ge 1$, we see that $0 \le \pi e^{-2x} \le \pi e^{-x}$ for $x \ge 1$. We conclude that $0 \le \int_1^\infty \pi e^{-2x}\,dx \le \int_1^\infty \pi e^{-x}\,dx$. Since $\int_1^\infty \pi e^{-x}\,dx$ is finite (this improper integral can be calculated directly), $\int_1^\infty \pi e^{-2x}\,dx$ must also be finite.

28. (corrected) The density of oil in a circular oil slick on the surface of the ocean at a distance r meters from the center of the slick is given by $\delta(r) = 50/(1+r)$ kg/m^2.

    (a) If the slick extends from $r = 0$ to $r = 10,000$ m, find a Riemann sum approximating the total mass of the oil in the slick.

    $$\sum 2\pi r_i \frac{50}{1+r_i}\Delta r$$

    (b) Find the exact value of the mass of oil in the slick by turning your sum into an integral, and evaluating it.

    $$2\pi\int_0^{10000} \frac{50r}{1+r}\,dr \approx 3.1387\cdot 10^6.$$

    (c) Within what distance r is half of the oil slick contained? about 5004 meters

29. Determine if the following statements are true or false. If the statement is true, clearly explain why. If the statement is false, give an example exhibiting why it is false.

(a) If f is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)\,dx$ converges. FALSE. $f(x)$ as given below is a counterexample.

$$f(x) = \begin{cases} 1, & x \le 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

(b) If f is continuous on $[0,\infty)$ and $\int_0^\infty f(x)\,dx$ diverges, then $\lim_{x\to\infty} f(x) \neq 0$. FALSE. This statement is logically equivalent to the previous one, so the above $f(x)$ is again a counterexample.

(c) If f' is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f'(x)\,dx = -f(0)$.
TRUE. Observe that

$$\begin{aligned} \int_0^\infty f'(x)\,dx &= \lim_{b\to\infty} \int_0^b f'(x)\,dx \\ &= \lim_{b\to\infty} [f(b) - f(0)] \\ &= -f(0). \end{aligned}$$

(d) If $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ both converge, then $\int_0^\infty (f(x)+g(x))\,dx$ converges. TRUE. Assume that $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ both converge. Then there is a number L_1 such that $\int_0^b f(x)\,dx$ is as close as we want to L_1 for large enough b (that is, for all b larger than some N_1). Similarly, there is a number L_2 such that $\int_0^b g(x)\,dx$ is as close as we want to L_2 for large enough b (that is, for all b larger than some N_2). This implies that $\int_0^b f(x)\,dx + \int_0^b g(x)\,dx = \int_0^b (f(x)+g(x))\,dx$ can be made as close as we want to $L_1 + L_2$ for large enough b. Thus, $\int_0^\infty (f(x)+g(x))\,dx$ converges (to $L_1 + L_2$).

(e) If $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ both diverge, then $\int_0^\infty (f(x)+g(x))\,dx$ diverges. FALSE. $\int_0^\infty \frac{1}{x}\,dx$ and $\int_0^\infty \frac{-1}{x}\,dx$ both diverge, but $\int_0^\infty \frac{1}{x} + \frac{-1}{x}\,dx = \int_0^\infty 0\,dx$ certainly converges.

30. Find the values of a for which the series converges/diverges:

(a) $\displaystyle\sum_{n=1}^\infty \left(\frac{1}{2a}\right)^n$ $|a| > 1/2$

(b) $\displaystyle\sum_{n=1}^\infty \frac{1}{a}\left(\frac{1}{2}\right)^n$ $a \neq 0$

(c) $\displaystyle\sum_{n=1}^\infty \left(\frac{2}{n}\right)^a$ $a > 1$

(d) $\displaystyle\sum_{n=1}^\infty (\ln a)^n$ $e^{-1} < a < e$

(e) $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^a}$ $a > 1$

(f) $\displaystyle\sum_{n=1}^\infty (1 + a^n)$ diverges for all a

(g) $\displaystyle\sum_{n=1}^\infty (1 + a)^n$ $-2 < a < 0$

(h) $\sum_{n=1}^{\infty} n^{\ln a}$ $0 < a < e^{-1}$

(i) $\sum_{n=1}^{\infty} a^{\ln n}$ $0 < a < e^{-1}$