MATH 2300 – review problems for Exam 1 ANSWERS

1. Evaluate the integral $\int \sin x \cos x \, dx$ in each of the following ways: This one is self-explanatory; we leave it to you.

   (a) Integrate by parts, with $u = \sin x$ and $dv = \cos x \, dx$. The integral you get on the right should look much like the one you started with, so you can solve for this integral.

   (b) Integrate by parts, with $u = \cos x$ and $dv = \sin x \, dx$.

   (c) Substitute $w = \sin x$.

   (d) Substitute $w = \cos x$.

   (e) First use the fact that $\sin x \cos x = \frac{1}{2}\sin(2x)$, and then antidifferentiate directly.

   (f) Show that answers to parts (a)–(e) of this problem are all the same. It may help to use the identities $\cos^2 x + \sin^2 x = 1$ and $\cos(2x) = 1 - 2\sin^2 x$.

2. Let $f(x)$ be a continuous function on the set of all real numbers. Show that

$$\int_0^1 f(e^x)\, e^x \, dx = \int_1^e f(x)\, dx.$$

   We put $u = e^x$, so that $du = e^x \, dx$. Also, when $x = 0$, $u = e^0 = 1$; when $x = 1$, $u = e^1 = e$. So

$$\int_0^1 f(e^x)\, e^x \, dx = \int_1^e f(u)\, du,$$

   which is the same as $\int_1^e f(x)\, dx$.

3. (a) Explain why the integral

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2 - 4}}$$

   is improper. The integrand becomes infinite at $x = 2$.

   (b) Show that

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2 - 4}} = \sqrt{21}.$$

   We put $u = x^2 - 4$, to get

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2 - 4}} = \frac{1}{2}\int_0^{21} \frac{du}{\sqrt{u}} = \frac{1}{2}\lim_{a\to 0^+}\int_a^{21} \frac{du}{\sqrt{u}} = \lim_{a\to 0^+}\left((21)^{1/2} - a^{1/2}\right) = \sqrt{21}.$$

4. Suppose that $\int_0^1 f(t)dt = 5$. Calculate the following:

   (a) $\int_0^{0.5} f(2t)dt$ $5/2$

   (b) $\int_0^1 f(1-t)dt$ 5

   (c) $\int_1^{1.5}(3-2t)dt$ I think it was supposed to say $f(3-2t)$, not just $(3-2t)$. If so, the answer is $5/2$.

   (a) $\displaystyle\int 2x\cos(x^2)\, dx$ $\sin\left(x^2\right) + C$

(b) $\int e^{2x}\sin(2x)\,dx$ $\frac{1}{4}e^{2x}(\sin(2x)-\cos(2x))+C$

(c) $\int \cos^2\theta\,d\theta$ $\frac{\theta}{2}+\frac{1}{4}\sin(2\theta)\frac{\cos(\theta)\sin(\theta)+\theta}{2}+C$

(d) $\int x^2\sin(x)dx$ $-x^2\cos x+2x\sin x+2\cos x+C$

(e) $\int \frac{1}{\sqrt{x^2-16}}\,dx\ln\left(\sqrt{x^2-16}+x\right)+C$

(f) $\int \frac{x}{\sqrt{6x-x^2}}\,dx$ $-3\sqrt{9-(x-3)^2}+3\sin^{-1}\left(\frac{x-3}{3}\right)=-3\sqrt{6x-x^2}+3\sin^{-1}\left(\frac{x-3}{3}\right)+C$

(g) $\int \frac{3x^2+6}{x^2(x^2+3)}\,dx$ $-\frac{2}{x}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$ (The partial fractions decomposition is $\frac{2}{x^2}+\frac{1}{x^2+3}$.)

(h) $\int \sqrt{25-x^2}\,dx$ $\frac{1}{2}\left(x\sqrt{25-x^2}+25\sin^{-1}\left(\frac{x}{5}\right)\right)+C$

(i) $\int \frac{3x-1}{x^2-5x+6}\,dx$ $8\ln|x-3|-5\ln|x-2|+C$

(j) $\int \sin^3(5x)\cos(5x)\,dx$ $\frac{1}{20}\sin^4(5x)+C$

(k) $\int_2^3 \frac{x^2}{1+x^3}\,dx$ $\frac{\ln(28)-\ln(9)}{3}=\frac{1}{3}\ln\left(\frac{28}{9}\right)$

(l) $\int_0^3 x\,e^{x^2}\,dx$ $\frac{1}{2}\left(e^9-1\right)$

(m) $\int x^7e^{x^4}\,dx$ $\frac{1}{4}e^{x^4}\left(x^4-1\right)+C$

(n) $\int(\ln(x))^2\,dx$ $2x+x\ln^2(x)-2x\ln(x)+C$

5. Evaluate the following integrals, using the substitutions provided.

(a) $\int y\sqrt{y^2+1}\,dy$; $w=y^2+1$. $\frac{1}{3}\left(y^2+1\right)^{3/2}+C$

(b) $\int y\sqrt{y+1}\,dy$; $w=y+1$. $\frac{2(y+1)^{\frac{5}{2}}}{5}-\frac{2(y+1)^{\frac{3}{2}}}{3}+C$

6. (a) Calculate $\int_2^4 \frac{dx}{(x-3)^2}$, if it exists. does not converge

(b) Find $\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\,dx$, if it converges. $\frac{\pi}{2}$ (the integral turns into $\int_0^{\infty}\frac{du}{u^2+1}$.)

7. Estimate $\int_1^2 \ln x\,dx$, by subdividing the interval $[1,2]$ into eight equal parts, and using:

(a) A left-hand Riemann sum: 0.342322

(b) A right-hand Riemann sum: 0.428965

(c) The midpoint rule: 0.386619

(d) The trapezoid rule: 0.385643

8. Estimate $\int_0^2 e^{x^2}\,dx$ by subdividing the interval $[0,2]$ into four equal parts, and using:

   (a) A left-hand Riemann sum: 7.245

   (b) A right-hand Riemann sum: 34.045

   (c) The midpoint rule: 14.486

   (d) The trapezoid rule: 20.645

9. Using the table, estimate the total distance traveled from time $t = 0$ to time $t = 6$ using the trapezoidal rule and the midpoint rule. Divide the interval $[0,6]$ into three equal parts.

Time, t	0	1	2	3	4	5	6
Velocity, v	4	5	6	8	9	5	3

Trapezoidal Rule:
$\Delta x = 2$
$(2)(\frac{v(0)+v(2)}{2} + \frac{v(2)+v(4)}{2} + \frac{v(4)+v(6)}{2}) = (4+6)+(6+9)+(9+3) = 37$ meters

Midpoint Rule:
$\Delta x = 2$
$(2)(v(1)+v(3)+v(5)) = (2)(5+8+5) = 36$ meters

10. Consider the function $f(x) = x^2 + 3$ on the interval $[0,1]$. Determine whether each of the following four methods of integral approximation will give an overestimate or underestimate of $\int_0^1 f(x)dx$. In each case, draw a picture to justify your answer.

    (a) the left Riemann sum underestimate

    (b) the right Riemann sum overestimate

    (c) the trapezoidal rule overestimate

    (d) the midpoint rule underestimate

11. Suppose $f(x)$ is concave up and decreasing on the interval $[0,1]$. Suppose the approximations LEFT(100), RIGHT(100), MID(100), and TRAP(100) yield the following estimates for $\int_0^1 f(x)\,dx$: 1.10, 1.25, 1.35, and 1.50, but *not necessarily in that order*. Which estimate do you think came from which method? Please explain your reasoning. 1.10: RIGHT(100), 1.25: MID(100), 1.35: TRAP(100), 1.50: LEFT(100)

12. Which of the following integrals can be integrated using partial fractions?

    (a) $\int \frac{1}{x^4-3x^2+2}\,dx$ yes; the denominator factors as $(x^2-2)(x^2-1) = (x-\sqrt{2})(x+\sqrt{2})(x-1)(x+1)$

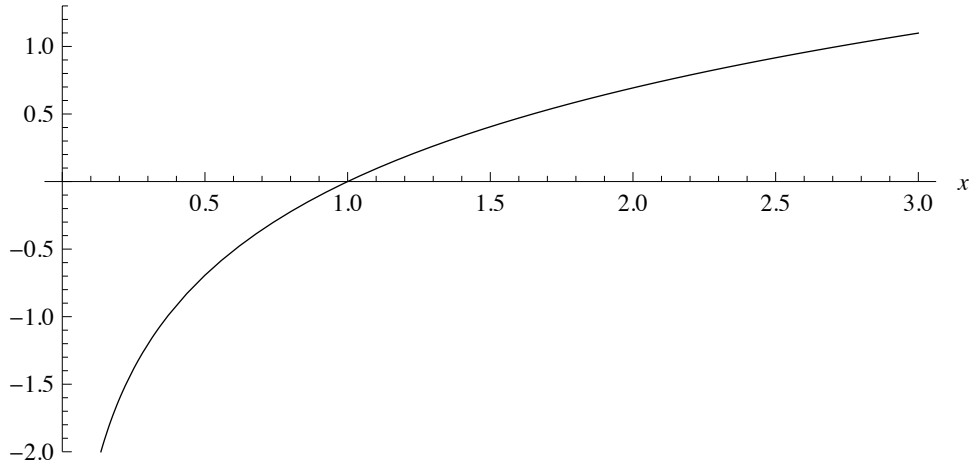
    (b) $\int \frac{1}{x^4+1}\,dx$ technically yes (but it's not easy)

    (c) $\int \frac{1}{x^3-8}\,dx$ yes; the denominator factors as $(x-2)(x^2+2x+4)$

    (d) $\int \frac{1}{x^4+2x^2+2}\,dx$ technically yes (but it's not easy)

    Make sure you can show the partial fraction decomposition.

13. Suppose $f(x)$ is a function whose graph looks like this:



Suppose the approximations LEFT(100), RIGHT(100), MID(100), and TRAP(100) yield the following estimates for $\int_1^2 f(x)\,dx$: 0.3423, 0.3857, 0.3866, 0.4920, but *not necessarily in that order*. Which estimate do you think came from which method? Between which two estimates do you think the true value of $\int_1^2 f(x)\,dx$ lies? Please explain your reasoning. 0.3423: LEFT(100), 0.3857: TRAP(100), 0.3866: MID(100), 0.4920: RIGHT(100)

14. For this set of problems, state which techniques are useful in evaluating the integral. You may choose from: integration by parts; partial fractions; long division; completing the square; trig substitution; or another substitution. There may be multiple answers.

(a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ parts, or trig sub

(b) $\displaystyle\int \frac{1}{\sqrt{6x-x^2-8}}\,dx$ completing the square; trig sub

(c) $\displaystyle\int x\sin x\,dx$ parts

(d) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$ substitution or trig sub

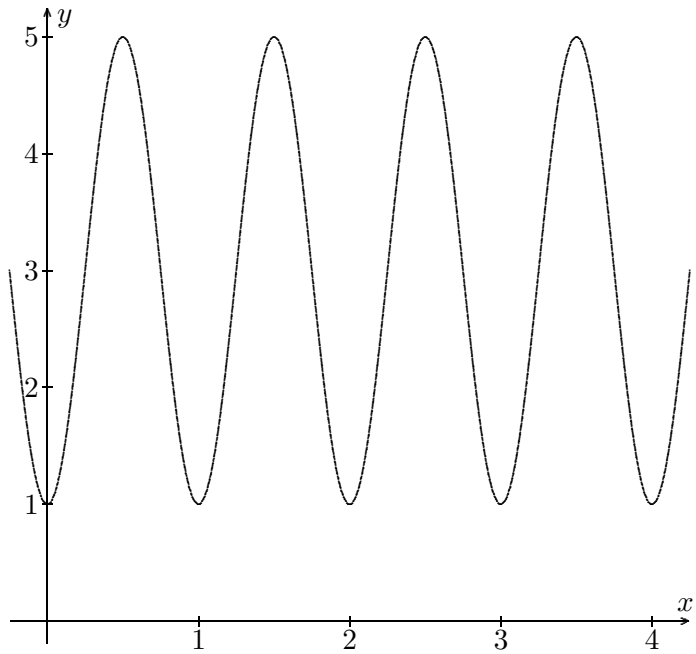
(e) $\displaystyle\int \frac{x^2}{1-x^2}\,dx$ long division; partial fractions

(f) $\displaystyle\int (1+x^2)^{-3/2}\,dx$ trig sub

(g) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$ substitution, followed by trig sub

(h) $\displaystyle\int \frac{1}{1-x^2}\,dx$ partial fractions

15. This problem relies on the following graph:



Using two subintervals, estimate $\int_0^4 f(x)\,dx$ with the left, right, midpoint, and trapezoid approximations. (If the picture was drawn correctly, you should find that they're all equal.) All methods give an estimate of 4.

(a) Both the left and right approximations with two rectangles underestimate the actual value integral. How is this possible (i.e., why isn't one of them an overestimate)? Because the function is sometimes increasing, sometimes decreasing on $[0, 4]$.

(b) Both the midpoint and trapezoid approximations with two rectangles underestimate the actual value of the integral. How is this possible (i.e., why isn't one of them an overestimate)? Because the function is sometimes concave up, sometimes concave down on $[0, 4]$.

16. A patient is given an injection of Imitrex, a migraine medicine, at a rate of $r(t) = 2te^{-2t}$ ml/sec, where t is the number of seconds since the injection started.

(a) By letting $t \to \infty$, estimate the total quantity of Imitrex injected. $\frac{1}{2}$ ml.

(b) What fraction of this dose has the patient received at the end of 5 seconds? About 99.95%

17. Let f be a differentiable function. Suppose that $f''(0) = 1$, $f''(1) = 2$, $f'(0) = 3$, $f'(1) = 4$, $f(0) = 5$, $f(1) = 6$. Compute $\int_0^1 f(x)f'(x)dx$. 11/2

18. For some constants A and B, the rate of production $R(t)$ of oil in a new oil well is modelled by:

$$R(t) = A + Be^{-t}\sin(2\pi t),$$

where t is the time in years, A is the equilibrium rate, and B is the "variable" coefficient.

(a) Find the total amount of oil produced in the first N years of operation.
$AN - \dfrac{Be^{-N}\left(-2\pi e^{N} + \sin(2\pi N) + 2\pi\cos(2\pi N)\right)}{1 + 4\pi^2}$ (this simplifies a lot if you assume N is a whole number)

(b) Find the average amount of oil produced per year over the first N years. it's just the above divided by N

(c) From your answer to part (b), find the average amount of oil produced per year as $N \to \infty$. A

(d) Looking at the function $R(t)$, explain how you might have predicted your answer to part (c) without doing any calculations. Because $Be^{-t}\sin(2\pi t) \to 0$ as $t \to \infty$

(e) Do you think it is reasonable to expect this model to hold over a very long period? nope

19. The rate, r, at which a population of bacteria grows can be modeled by $r = te^{3t}$, where t is time in days. Find the total population of bacteria after 20 days. $\frac{1+59e^{60}}{9} = 7.48649 \times 10^{26}$

20. Recall that the error in the tangent line approximation to $f(x)$ at $x = a$ is given by

$$E(x) = f(x) - f(a) - f'(a)(x - a).$$

(a) Show that $\int_a^x (x - t)f''(t)\mathrm{d}t = E(x)$. Hints: (a) an antiderivative of $f''(t)$ is $f'(t)$. (b) To integrate $tf''(t)$, put $u = t$ and $dv = f''(t)\,dt$.

(b) Compute TRAP(1) for $\int_a^x (x-t)f''(t), \mathrm{d}t$ and use it to explain why $E(x) \approx \dfrac{f''(a)}{2}(x-a)^2$. We compute that TRAP(1) $= \frac{f''(a)}{2}(x-a)^2$. Now use part (a).

21. Suppose $f(0) = 1$, $f(1) = e$, and $f'(x) = f(x)$ for all x. Find

$$\int_0^1 e^x f'(x)dx.$$

$\dfrac{1}{2}(e^2 - 1)$