WORKSHEET: Techniques of Integration **ANSWERS IN RED** **MATH 2300**

The purpose of this worksheet is to get some practice in recognizing which integration technique to use. Recall that so far we have learned the following integration techniques: Integration by Substitution, Integration by Parts, Integration Tables, Method of Partial Fractions.

1. Evaluate the following integrals. Which techniques of integration can you use on each integral? (Hint: factoring $9 - x^2$ will help for some of the integrals.)

a)

$$\int \frac{4}{9-x^2}\,dx = -\int \frac{4}{(x-3)(x+3)}\,dx = -\frac{2}{3}\log\left|\frac{x+3}{x-3}\right| + C.$$

We've used the Table of Indefinite Integrals here, but you could also do a trig sub: put $u = 3\sin\theta$. Then $du = 3\cos\theta\,d\theta$, and you end up with an integral that, except for a constant out front, looks like $\int \sec\theta\,d\theta$, which equals $\ln|\sec\theta + \tan\theta| + C$. Plugging back in for θ in terms of x (use a reference triangle) will ultimately lead you to the same answer.

b)

$$\begin{aligned}\int \frac{4x}{9-x^2}\,dx &= -\frac{4}{2}\int \frac{1}{u}\,du \\ &= -2\ln|u| + C \\ &= -2\ln|9-x^2| + C.\end{aligned} \quad \left| \begin{aligned} u &= 9-x^2 \\ du &= -2x\,dx \\ x\,dx &= -\frac{du}{2}\end{aligned}\right.$$

You could also do this one using the Table of Indefinite Integrals, or a trig sub.

c)

$$\begin{aligned}\int \frac{4}{\sqrt{9-x^2}}dx &= \frac{4\cdot 3}{3}\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta \\ &= 4\int d\theta = 4\theta + C \\ &= 4\arcsin\left(\frac{x}{3}\right) + C.\end{aligned} \quad \left| \begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\,d\theta\end{aligned}\right.$$

d)

$$\begin{aligned}\int \frac{4x}{\sqrt{9-x^2}}dx &= -\frac{4}{2}\int \frac{1}{\sqrt{u}}\,du \\ &= -2\int u^{-1/2}\,d\theta = 4u^{1/2} + C \\ &= 4\sqrt{9-x^2} + C.\end{aligned} \quad \left| \begin{aligned} u &= 9-x^2 \\ du &= -2x\,dx \\ x\,dx &= -\frac{du}{2}\end{aligned}\right.$$

2. Evaluate the following integrals. Which method of integration works best for each integral?

a) $\int \frac{10+4x}{5x+x^2}dx$

This can be done using the Table. Or, by partial fractions:

$$\begin{aligned}\frac{10+4x}{5x+x^2} &= \frac{10+4x}{x(5+x)} = \frac{A}{x} + \frac{B}{5+x},\\ 10+4x &= A(5+x) + Bx = (A+B)x + 5A,\\ 10 &= 5A \text{ and } 4 = A+B,\\ A &= 2 \text{ and } B = 2.\end{aligned}$$

So

$$\int \frac{10+4x}{5x+x^2}dx = \int \left(\frac{2}{x} + \frac{2}{5+x}\right) dx = 2(\ln|x| + \ln|5+x|) + C.$$

b) $\int \frac{x+3}{5x+x^3}dx$

$$\begin{aligned}\frac{x+3}{5x+x^3} &= \frac{x+3}{x(5+x^2)} = \frac{A}{x} + \frac{Bx+C}{5+x^2},\\ x+3 &= A(5+x^2) + (Bx+C)x = (A+B)x^2 + Cx + 5A,\\ 3 &= 5A,\ 1 = C, \text{ and } 0 = A+B,\\ A &= \frac{3}{5},\ B = -\frac{3}{5} \text{ and } C = 1.\end{aligned}$$

So

$$\int \frac{x+3}{5x+x^3}dx = \int \left(\frac{3}{5x} + \frac{\frac{3}{5}x+1}{5+x^2}\right) dx.$$

The second integral can be done using the Table, or by the trig sub $x = \sqrt{5}\tan\theta$. The final result is

$$\int \frac{x+3}{5x+x^3}dx = \frac{3\ln|x|}{5} - \frac{3\ln|5+x^2|}{10} + \frac{\arctan\left(\frac{x}{\sqrt{5}}\right)}{\sqrt{5}} + C.$$

3. Evaluate the following integrals. Can you use Integration by Substitution to evaluate either integral?

a) $\int \frac{1}{x^2+4}dx$ A u-substitution won't work, but the trig sub $x = 2\tan\theta$ will; it gives

$$\int \frac{1}{x^2+4}dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

b) $\int \frac{x}{x^2+4}dx$ Substitute $u = x^2+4$. You get

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln|x^2+4| + C.$$