Partial homework solutions for §3.9, 3.10, 4.2

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1. (§3.9 #21) For g the acceleration due to gravity, the period, T, of a pendulum of length ℓ is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$.

(a) We must show that if the length of the pendulum changes by $\Delta\ell$ then the period changes by $\Delta T \approx \frac{T}{2\ell}\Delta\ell$. We use the fact that for small $\Delta\ell$ we have $\frac{\Delta T}{\Delta\ell} \approx \frac{dT}{d\ell}$. Thus,

$$\frac{\Delta T}{\Delta\ell} \approx \frac{dT}{d\ell} = \frac{d}{d\ell}\left(2\pi\sqrt{\frac{\ell}{g}}\right) = \frac{\pi}{\sqrt{g\ell}} = \frac{\pi}{\sqrt{g\ell}} \cdot \frac{2\sqrt{\ell}}{2\sqrt{\ell}} = \frac{T}{2\ell}.$$

We conclude that $\frac{\Delta T}{\Delta\ell} \approx \frac{T}{2\ell}$, so indeed $\Delta T \approx \frac{T}{2\ell}\Delta\ell$.

(b) If the length increase 2%, we must determine by what percent the period changes. The length increasing by 2% means that $\Delta\ell = 0.02\ell$. Using our result from the previous part, we see that

$$\Delta T \approx \frac{T}{2\ell}\Delta\ell = \frac{T}{2\ell}0.02\ell = 0.01T.$$

We conclude that a 2% increase in the length causes a 1% increase in the period.

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2. (§3.10 #13) We are told that $p(x)$ is a degree 7 polynomial with 7 distinct zeros, and we must determine how many zeros $p'(x)$ has. Since $p'(x)$ will be a degree 6 polynomial, $p'(x)$ will have **at most** 6 zeros. We will show that in fact $p'(x)$ has exactly six zeros. Keep in mind that an arbitrary degree 6 polynomial need not have 6 (real) zeros: $x^6 + 1$ has none!

We now show that $p'(x)$ has **at least** 6 zeros; thus it will have exactly 6 zeros since it cannot have any more. Let $a_1, a_2, \ldots, a_7$ be the distinct zeros of $p(x)$ listed in increasing order, i.e. $a_1 < a_2 < \cdots < a_7$. Our intuition is that between every consecutive pair of zeros of $p(x)$ it must be the case that $p(x)$ has a local maximum or minimum. Each each maximum or minimum corresponds to a zero of $p'(x)$, so $p'(x)$ should have at least 6 zeros. This intuition is correct, and we will use the Mean Value Theorem (MVT) to prove it.

Remember that each a_i is a zero of $p(x)$, so in particular, $p(a_1) = 0 = p(a_2)$. Also, $p(x)$ is polynomial, so it is everywhere continuous and differentiable. Hence, the MVT applies, and it tells us that there is a c_1 such that $a_1 < c_1 < a_2$ and

$$p'(c_1) = \frac{p(a_2) - p(a_1)}{a_2 - a_1} = \frac{0 - 0}{a_2 - a_1} = 0.$$

Thus, c_1 is a zero of $p'(x)$. We can repeat this process to find a zero c_i of $p'(x)$ with $a_i < c_i < a_i + 1$ for each i from 1 to 6. This shows that $p'(x)$ has **at least** 6 zeros, so we are done.

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3. (§4.2 #30) We are given the relationship $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ where R, r_1, and r_2 are positive and r_2 is constant.

(a) We are asked to show that R is an increasing function of r_1. We will do this be showing that $\frac{dR}{dr_1}$ is always positive. There are multiple ways to calculate $\frac{dR}{dr_1}$. Since we do not have R as an explicit function of r_1, we will use implicit differentiation. Observe that

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} \implies \frac{d}{dr_1}\left(\frac{1}{R}\right) = \frac{d}{dr_1}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \implies \frac{-1}{R^2}\frac{dR}{dr_1} = \frac{-1}{r_1^2} + 0 \implies \frac{dR}{dr_1} = \frac{R^2}{r_1^2}.$$

Thus, $\frac{dR}{dr_1}$ is always positive since it is a nonzero square.

(b) Since R is an increasing function of r_1, the maximum of R on any interval $a \leq r_1 \leq b$ will always occur at the right endpoint of the interval, i.e. when $r_1 = b$.
